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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 8

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RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1909—JUNE, 1910.

Thursday, November 11th, 1909.

ANNUAL GENERAL MEETING.

Sir W. D. NIVEN, President, in the Chair.

Present twenty-three members.

Mr. W. P. Milne was admitted into the Society.

The President presented the Treasurer's Report. On the motion of Mr. A. B. Basset, seconded by Mr. E. Cunningham, the Report was received.

Dr. J. G. Leathem was re-appointed Auditor.

Mr. Grace, as Secretary, reported the number of members at the beginning of the current Session as 286. During the Session 6 new members had been elected, and 1 member had died.

The Council and Officers for the ensuing Session were elected, as follows:—President, Sir W. D. Niven; Vice-Presidents, Mr. A. Berry, Prof. W. Burnside, Major P. A. MacMahon; Treasurer, Sir J. Larmor; Secretaries, Prof. A. E. H. Love, Mr. J. H. Grace; other members of the Council, Dr. H. F. Baker, Mr. G. T. Bennett, Dr. T. J. I'A. Bromwich, Mr. E. Cunningham, Mr. A. L. Dixon, Dr. L. N. G. Filon, Dr. E. W. Hobson, Mr. H. W. Richmond, Mr. A. E. Western.

The following papers were communicated:—

*(1) The Ordinal Relations of the Terms of a Convergent Sequence, *(2) The Application to Dirichlet's Series of Borel's Exponential Method of Summation, *(3) Theorems relating to the Summability and Convergence of Slowly Oscillating Series: Mr. G. H. Hardy.

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* Printed in this volume.

Notes on Synthetic Geometry: Prof. W. Esson.

*Kummer's Quartic Surface as a Wave Surface: Mr. H. Bateman.

*The Green's Function in a Wedge and other Problems in the Conduction of Heat: Prof. H. S. Carslaw.

The Envelope of a Line cut Harmonically by Two Conics: Mr. J. L. S. Hatton.

On a Case of q-Hypergeometric Series: Rev. F. H. Jackson.

Informal communications were made as follows:—

On the Extension of Abel's Theorem concerning the Sums of Power Series at Points on the Circle of Convergence to Cases in which the Series are Oscillatory: Dr. E. W. Hobson.

On the Effect of the Earth's Rotation upon the Observed Values of the Lunar Disturbance of Gravity: Prof. A. E. H. Love.

Thursday, December 9th, 1909.

Sir W. D. NIVEN, President, in the Chair.

Present eighteen members and two visitors.

Messrs. J. Mercer and H. T. H. Piaggio were elected members.

Prof. Love presented the Report of the Auditor (Dr. J. G. Leathem).

On the motion of Lieut.-Col. A. Cunningham, seconded by Dr. Eccles, the Treasurer's Report was adopted, and the thanks of the Society were voted to the Treasurer and to the Auditor.

Mr. T. H. Blakesley exhibited an instrument for the †"Kinematical Solution of Cubic Equations," and also models of various modified forms of the instrument.

The following paper was communicated:—

*The Eliminant of the Equations of Four Quadric Surfaces: Mr. A. L. Dixon.

Thursday, January 13th, 1910.

Sir W. D. NIVEN, President, in the Chair.

Present nineteen members and two visitors.

Mr. J. H. Woolston was elected a member.

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* Printed in this volume. † See "Notes and Corrections" in this volume.

The following papers were communicated :—

On the Theory of the Cubic Surface : Dr. H. F. Baker.

On an Example of the Expansion of a Function in a Series of Polynomials : Dr. H. F. Baker.

*The Harmonic Functions associated with the Parabolic Cylinder : Mr. G. N. Watson.

*The Transformations of Coordinates which can be used to Transform one Physical Problem into Another : Mr. H. Bateman.

†Note on a Former Paper on the Theory of Divergent Series : Mr. G. H. Hardy.

*On Homogeneous Oscillation of Successions of Functions : Dr. W. H. Young.

*On the Determination of a Semi-Continuous Function from a Countable Set of Values : Dr. W. H. Young and Mrs. Young.

*On Cyclic Congruences : Mr. J. E. Campbell.

*On the Representation of a Group of Finite Order as a Group of Linear Substitutions with Rational Coefficients : Prof. W. Burnside.

Dr. H. de S. Pittard made an informal communication "On the Theory of Sets in Probabilities."

Thursday, February 10*th*, 1910.

Sir W. D. NIVEN, President, in the Chair.

Present fifteen members.

The following papers were communicated :—

Note on Double-Sixes of Lines : Mr. H. W. Richmond.

*On the Diffraction of a Solitary Wave : Prof. H. Lamb.

Dr. H. F. Baker made an informal communication in the form of three Notes on the theory of functions : (1) "On a certain Logical Principle," (2) "On the Establishment of the Order of a Doubly Periodic Function," (3) "Two Queries."

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* Printed in this volume.

† The substance of this note is printed as a footnote, pp. 279–281, of the present volume.

Thursday, March 10th, 1910.

Sir W. D. NIVEN, President, in the Chair.

Present seventeen members and a visitor.
Messrs. L. Narayan and S. Brodetsky were elected members.

The following papers were communicated :—

*The Scattering of Light by a Large Conducting Sphere: Mr. J. W. Nicholson.

The 3–3 Birational Space Transformation: Miss H. P. Hudson.

The Expression of the Sum of the r-th Powers of the First n Natural Numbers and other Similar Functions of n in Terms of $n+\frac{1}{2}$, and Forms for the Remainder in the Euler-Maclaurin Sum-Formula: Dr. W. F. Sheppard.

Thursday, April 28th, 1910.

Sir W. D. NIVEN, President, in the Chair.

Present sixteen members and a visitor.
Mr. S. Brodetsky was admitted into the Society.

The following papers were communicated :—

The Accuracy of Interpolation by Finite Differences (Second Paper): Dr. W. F. Sheppard.

Theorems connected with Maclaurin's Test for the Convergence of Series: Mr. G. H. Hardy.

Lieut.-Col. A. Cunningham communicated two notes on the theory of numbers† : (i) "The Factorisation of $2^{77}+1$," (ii) "The Indivisibility of 2^n-2 by p^2, p being Prime."

Thursday, May 12th, 1910.

In consequence of the death of His Majesty King Edward the Seventh, the meeting announced for this date was cancelled.

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* Printed in this volume.

† See "Notes and Corrections" in this volume.

Thursday, June 9th, 1910.

Sir W. D. NIVEN, President, in the Chair.

Present twenty members and a visitor.

The following papers were communicated :—

The Composition of Finite Screw Displacements: Mr. G. T. Bennett.

On Differential Equations with Fixed Branch Points: Prof. M. J. M. Hill.

On Geiser's Method of Generating a Plane Quartic Curve: Miss M. Long.

The Generation of Cubic Curves by Apolar Pencils of Lines: Mr. W. P. Milne.

The Constitutive Equations of Material Media in Electrodynamics: Mr. E. Cunningham.

(1) A New Method in the Theory of Integration; (2) On Semi-Integrals and Oscillating Successions of Functions: Dr. W. H. Young.

The Transformation of the Equations of the Theory of Electrons for Quasi-Stationary Motion: Mr. H. R. Hassé.

LIBRARY

Presents.

BETWEEN November, 1909, and July, 1910, the following presents were made to the Library, from their respective Authors or Publishers :—

Ball, Sir Robert.—"Contributions to the Theory of Screws," 1910.
Ball, Sir Robert.—"Quaternion Expression for the Coordinates of a Screw Reciprocal to Five Given Screws," 1909.
Basset, A. B.—"Treatise on Geometry of Surfaces," 1910.
Cabreira, A.—"Les Mathématiques en Portugal," 2me défense des Travaux, 1910.
Huggins, Sir W.—"The Royal Society," 1910.
Kelvin, Baron.—"Mathematical and Physical Papers," vol. 4, arranged and revised by Sir Joseph Larmor, 1910.
Parris, Marion.—"Total Utility and the Economic Judgment," 1909.
Simon Newcomb.—"Memorial Addresses."
"Royal Society's Catalogue of Scientific Papers, 1800–1900, Subject Index," vol. 2, Mechanics, 1909.
"The Newhaven Mathematical Colloquium," Lectures delivered before Members of the American Mathematical Society, September, 1906; 1910.

Calcutta: Indian Engineering, vol. 46, nos. 18–26, 1909.
Cape Town: South African Journal of Science, vol. 4, nos. 1–4, 1909.
Coimbra: Accademia Polyt. de Porto, Ann. Scientificos, vol. 4, nos. 3, 4, 1909; vol. 5, no. 1, 1910.
Hamburg: Math. Gesellschaft Mittheilungen, bd. 4, heft 10, 1910.
London: Educational Times, vol. 62, nos. 583, 584, 1909; vol. 63, nos. 585–590, 1910.
London: Educational Times, Math. Questions and Solutions, vol. 16, 1909.
London: Mathematical Gazette, vol. 5, nos. 81–83, 1909.
London: Nautical Almanac for 1912.
Madras: Indian Mathematical Club, Journal, vol. 1, nos. 5, 6, 1909.
Paris: L'Enseignement Math., ann. 11, no. 6, 1909; ann. 12, nos. 1, 2, 1910.
Tokyo: Sūgaku-Butsurigaku, vol. 5, nos. 7–12, 1909–10.
Upsala: Reg. Soc. Scientiarum, Nova Acta, ser. 4, vol. 2, fasc. 1, 1907–9.
Warsaw: Wiadomości Matem., tom 13, zeszyt 5, 6, 1909.

Exchanges.

Between November, 1909, and July, 1910, the following exchanges were received :—

American Journal of Mathematics, vol. 31, no. 4, 1909; vol. 32, nos. 1, 2, 1910.
American Mathematical Society, Transactions, vol. 10, no. 4, 1909; vol. 11, nos. 1, 2, 1910.

American Mathematical Society, Bulletin, vol. 16, nos. 2–7, 1909–10.
American Philosophical Society, Proceedings, vol. 48, nos. 192, 193, 1909.
Amsterdam: Nieuw Archiev, deel 9, stuk 1, 2, 1909–10.
Amsterdam: Revue Semestrielle, tome 17, pt. 2, 1909.
Amsterdam: Wiskundige Opgaven, deel 10, stuk 4, 1909.
Belgique: Académie Royale des Sciences, Bulletin, 1909, nos. 9–12; 1910, nos. 1, 2.
Berlin: Jahrbuch über die Fortschritte, bd. 38, hefte 2, 3, 1909.
Berlin: Sitzungsberichte der K. Preuss. Akademie, 1909, nos. 30–53; 1910, nos. 1–23.
Boston: American Academy of Arts and Sciences, vol. 44, no. 26, 1909; vol. 45, nos. 1, 2, 1910.
Calcutta: Mathematical Society, Bulletin, vol. 1, no. 2, 1909.
Cambridge Philosophical Society, Proceedings, vol. 15, pt. 4, 1909.
Cambridge Philosophical Society, Transactions, vol. 21, pts. 9–11, 1909–10.
Cambridge, Mass.: Annals of Mathematics, vol. 11, nos. 1–3, 1909–10.
Catania: Accademia Gioenia, Atta, Ser. 5, vol. 2, 1909.
Catania: Accademia Gioenia, Bollettino, 1909, fasc. 1–10.
Dublin: Royal Irish Academy, Proceedings, vol. 27, no. 10, 1909.
Edinburgh: Mathematical Society, Proceedings, vol. 26, 1908.
Edinburgh: Royal Society, Proceedings, vol. 28, 1908; vol. 29, pts. 1, 2, 1908.
Edinburgh: Royal Society, Transactions, vol. 43, pt. 3, 1908.
Erlangen: Physik.-medicin. Societät, Sitzungsberichte, bd. 40, 1909.
France: Société Mathématique, Bulletin, tome 37, fasc. 4, 1909.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1909, hefte 2–4.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Geschaft. Mitth., 1909, heft 2.
Holland: Archives Néerlandaises, tome 14, liv. 5, 1909.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 33, hefte 19–24, 1909; bd. 34, hefte 1–8, 1910.
Leipzig: K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 30, no. 6, 1908; bd. 32, 1909; bd. 33, no. 1, 1910.
Livorno: Periodico di Matematica, anno 25, fasc. 2–5, 1909–10.
Livorno: Periodico di Matematica, Supplemento, anno 13, fasc. 1–6, 1909–10.
London: Institute of Actuaries, Journal, vol. 42, pts. 2–4, 1908; vol. 43, pt. 1, 1909.
London: Institution of Naval Architects, Transactions, vol. 49, 1909.
London: Nature, vol. 82, nos. 2083–2104, 1909–10; vol. 83, nos. 2105–2121, 1910.
London: Nat. Physical Laboratory, Collected Researches, vol. 6, 1910.
London: Royal Society, Proceedings, Series A, vol. 83, nos. 559–565, 1909–10.
London: Royal Society, Transactions, Series A, vol. 209, 1909.
London: Physical Society, Proceedings, vol. 21, pts. 5–7, 1909; vol. 22, pt. 1, 1910.
Manchester Literary and Philosophical Society, Memoirs, vol. 52, pts. 2, 3, 1908; vol. 53, pt. 1, 1909.
Milano: Reale Istituto, Rendiconti, vol. 41, fasc. 17–20, 1908; vol. 42, fasc. 1–16, 1909.
Napoli: Accademia delle Scienze, Rendiconti, vol. 15, fasc. 3–12, 1909.
Palermo: Rendiconti del Circolo Matematico, tome 28, fasc. 3, 1909; tome 29, 1910.
Paris: Bulletin des Sciences Mathématiques, tome 33, Aug.–Dec., 1909; tome 34, Jan.–Feb., 1910.
Paris: Journal de l'Ecole Polyt., cah. 13, 1909.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 18, sem. 2, fasc. 5–12, 1909; vol. 19, sem. 1, fasc. 1–8, 1910.
Stockholm: Acta Mathematica, bd. 32, pt. 4, 1909; bd. 33, pts. 1, 2, 1910.
Torino: R. Accademia delle Scienze, Atti, vol. 15, disp. 1–10, 1910.
Toulouse: Faculté des Sciences, Annales, tome 10, fasc. 3, 1908.
Warsaw: Prace Matematyczno-Fizyczne, tome 20, 1909.
Wien: Monatshefte für Mathematik, jahr 21, viertel. 1, 2, 1910.
Zurich: Vierteljahrsschrift, hefte 3, 4, 1909.

International Catalogue of Scientific Literature.

In the year April, 1909, to March, 1910 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature :—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXVII., 1909.

"Transactions of the Institution of Naval Architects," London, 1909.

"Journal of the Institute of Actuaries," Vol. XLIII, Pts. 2–4, and Vol. XLIV, Pt. 1 ; 1909–10.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. LIII, Pts. 2, 3, Vol. LIV, Pt. 1, 1909.

"Proceedings of the Royal Society of Edinburgh," Vol. XXIX, Pts. 3–8, Vol. XXX, Pts. 1–3, 1909–10.

"Proceedings of the Royal Irish Academy," Vol. XXVII, Section A, Nos. 11, 12, and Contents, and Vol. XXVIII, Section A, No. 1, 1909–10.

The following were also sent for the purposes of the Catalogue :—

"Mathematical Gazette," Nos. 78–83 ; London, 1909–10.

"Transactions of the Insurance and Actuarial Society of Glasgow," Ser. 6, Nos. 8, 9, 1908–9.

"Educational Times," Nos. 576–587 ; London, 1909–10.

NOTES AND CORRECTIONS.

The instruments and models shown by Mr. T. H. Blakesley at the meeting of the Society on December 9th, 1909, were constructed in accordance with the following figures:—

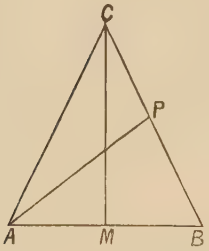


Fig. 1.

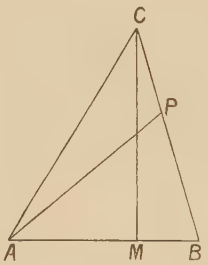


Fig. 2.

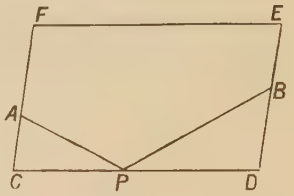


Fig. 3.

In Fig. 2, MC is a fixed groove at right angles to a fixed straight line AB; AP, PC represent an articulated pair of bars which can turn about pivots at A and C. The lengths AP, PC, AM, MB are denoted by a, b, c, e. As C moves along the groove, the joint P describes a curve, which intersects BC whenever the length BC, denoted by x, is a root of the equation

$$x^3-2bx^2+(c^2-e^2-a^2+b^2)\,x+2be\,(e+c)=0.$$

Here c and e need not have the same sign. Fig. 1 refers to the particular case in which $c=e$. In Fig. 3, CD is a side of an articulated parallelogram $CDEF$; A, B are fixed points on the two sides CF, DE; AP, PB represent an articulated pair of bars which can turn about pivots at A and B. The lengths AP, BP, AC, BD, CD are denoted by a, b, c, d, g. As the parallelogram is deformed, the joint P describes a curve, which intersects CD whenever the length CP, denoted by x, is a root of the equation

$$(d-c)\,x^3-g\,(d-2c)\,x^2+\left\{(b^2-d^2)\,c-(a^2-c^2)\,d-g^2c\right\}x+gd\,(a^2-c^2)=0.$$

Dr. W. H. Young sends the following corrections of his paper "On Term-by-Term Integration of Oscillating Series":—

P. 111, l. 14. Delete the words "since $w_n(x)$ is a bounded function of x."

l. 15. Delete the words "the lower bound of," "that of," and "and is therefore finite."

Lt.-Col. A. Cunningham reported at the April meeting the following new factorization :—

$$2^{77}+1 = (3.43.683)(617.78233.35532364099).$$

He further reported the result that (2^p-2) is not divisible by p^2 for any prime (p) less than 1000 ; this property is known to be connected with Fermat's " Last Theorem."

Mr. G. N. Watson sends the following corrections of his paper " The Solution of the Homogeneous Linear Difference Equation of the Second Order " :—

Pp. 150-152, *passim*. The coefficient of $\overline{\eta}'_{x+r}$ should be " 2 " instead of " 4."

P. 165, ll. 12, 13, and footnote. *For* " $\psi(\ldots)$ and $\psi_1(\ldots)$," *read* " $[\psi(\ldots)]^{-1}$ and $[\psi_1(\ldots)]^{-1}$," respectively.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

THE KINETIC IMAGE OF A CONVECTED ELECTRIC SYSTEM FORMED IN A CONDUCTING PLANE SHEET

By J. Larmor.

[Received March 8th, 1909.—Read March 11th, 1909.]

The mathematical problems associated with Arago's rotating disc led in Maxwell's hands* to his beautiful theory of a receding trail of images, as representing the reflection, from an infinite plane conducting sheet, of the disturbance due to a magnetic pole or other magnetic system travelling in front of it. In these days, when electrons at rest or in motion are recognised as the source of all electric and magnetic phenomena, it is natural to try to develop his solution in the direction of moving electric charges. It turns out that the train of images can in both the electric and magnetic cases, and indeed for any convected system, readily be reduced to a simple and direct synthetical form. The velocities of convection are not to reach the order of magnitude of the speed of light; otherwise radiation would become sensible, and the present "equilibrium theory" would not suffice.

The screening action due to motion of a conducting layer has been treated in a previous paper.† The direct synthetical analysis of that

* *Proc. Roy. Soc.*, February, 1872; *Elec. and Mag.*, §§ 656-669.

† "Electromagnetic Induction in Conducting Sheets and Solid Bodies," *Phil. Mag.*, January, 1884, cf. pp. 21-23.

problem, and of the one here treated, have recently acquired new physical interest from G. W. Hale's observations of the Zeeman magnetic effect in the spectrum at spots in the Sun. Observations appear at present to show that the magnetic field is confined to a thin layer in the solar atmosphere; and this would seem to require that the natural extension of the field due to the electric flow in the layer must be cut off by some kind of screening action from the regions above and below this layer.

If ϕ is the stream function of the currents $(u, v, 0)$ induced in the sheet, itself situated in the plane of xy, and

$$\chi = \int \phi r^{-1} dS,$$

then the magnetic potential due to them is $-\partial\chi/\partial z$, and its vector potential (F, G, H) is $(\partial\chi/\partial y, -\partial\chi/\partial x, 0)$. The equations of electric flow in the sheet, of specific resistance R, are

$$Ru = -\frac{\partial F}{\partial t} - \frac{\partial \psi}{\partial x}, \qquad Rv = -\frac{\partial G}{\partial t} - \frac{\partial \psi}{\partial y},$$

which are both satisfied by $\quad R\phi = -\partial\chi/\partial t,$

without requiring any potential ψ arising from accumulated free electricity.

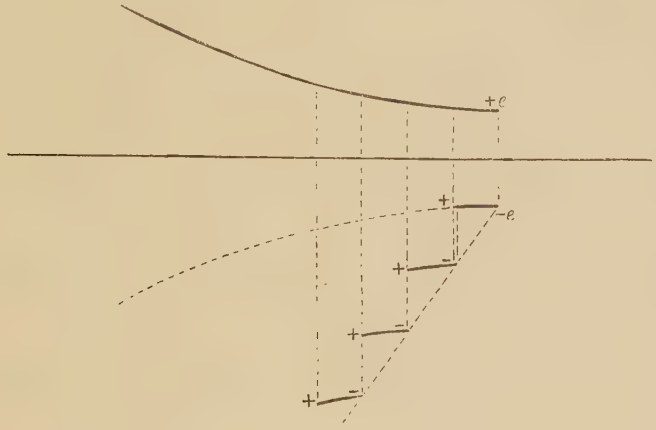
Also at the sheet $\qquad -\partial\chi/\partial z = 2\pi\phi.$

Thus at the sheet $\qquad \dfrac{R}{2\pi}\dfrac{\partial\chi}{\partial z} = \dfrac{\partial\chi}{\partial t}.$

This means, provided the sheet be *plane*, that the magnetic potential due to a decaying current sheet alters, just as if the system to which it belongs were receding, without other change, with velocity $R/2\pi$. For, on introducing the relative coordinate $z+Rt/2\pi$ in place of z, the relation becomes $\partial\chi/\partial t = 0$, showing that at the sheet and therefore beyond it χ is of the form $F(x, y, z+Rt/2\pi)$.

We first apply this principle of Maxwell's to the case of an electric point-charge e, moving with velocity v in front of the sheet. This continuous motion of the charge may be replaced by a series of instantaneous jerks between the positions which it occupies after successive infinitesimal intervals of time τ. Each such displacement of its position is equivalent to the creation of a doublet of moment $ev\tau$. The initial (magnetic) effect of the currents induced in the sheet is to annul the field of the doublet thus

instantaneously created, as regards the region beyond the sheet: thus the action of the currents in the sheet is on either side of it equivalent initially to that of this doublet with sign changed, supposed to be situated on the other side. The currents thus impulsively induced in succession gradually die away by resistance (R) in the sheet, and in so doing each of them exerts the same influence (Maxwell) as if the equivalent doublet on the other side of the sheet moved steadily away with velocity (U) equal to $R/2\pi$. The fields of force persisting from successive past intervals of time τ are thus expressed as due to the sources graphically represented in the diagram; in it the image system is inserted on the remote side of



the sheet, the conjugate image for the near side being, of course, its reflexion in the sheet, with sign changed if it is a magnetic, but not if it is an electric pole. Each step between a receding doublet image and the next is of length $U\tau$. At the instant when the moving charge $+e$ has reached the point marked as the end of the path, the aggregate image system consists of the doublets represented by the short continuous lines, each of which is a persisting travelling reflexion of the instantaneous effect of the jerk representing a previous element of the path of $+e$.

It is the change, occurring per unit time, in the conformation of this image system of doublets, that represents convection of electric charges and so produces magnetic effect. It is, in fact, only while complementary charges are being separated, so as to create a doublet, that a current element exists, of moment equal to charge multiplied by velocity: when the doublet has once been established, and merely continues to exist, the magnetic flux around it ceases, though there is a latent accumulation of such flux which persists without further electrodynamic effect, and would

be undone if the doublet were again absorbed by its poles moving into coincidence.

Thus the magnetic effect of the induced currents in the sheet is the same as that of the electric flux in the image system; and the aggregate flux in the time τ amounts to the distribution of vertical doublets, each of moment $eU\tau$, as marked by the + and − in the diagram, each receding from the sheet a distance $U\tau$, and in addition the creation of the fresh isolated image $-e$ at the end of the series. It is the limiting form assumed by this convection as τ vanishes with which we are concerned.

If the inducing charge e is at rest, this image system amounts to nothing, as of course it ought. If the inducing charge is moving directly towards the sheet with velocity u, the image system again assumes a simple form; for if each doublet $-e.U\tau$ is replaced by an equivalent but longer one $-eU/(U+u).(U+u)\tau$, the doublets will form here also a continuous line in which adjacent poles cancel, and will thus represent in the aggregate the removal in time τ of an image charge $-eU/(U+u)$ to infinite distance, or rather to the distance appropriate to the duration of the motion. Thus, of the instantaneous image $-e$, there remains in position, after subtracting the part thus removed, a residue $-eu/(U+u)$, moving towards the sheet with velocity u. This reduced geometrical image $-eu/(U+u)$ constitutes in the present case the entire image system, as regards magnetic field due to the disturbance excited in the sheet; but, as regards electric field, the effect of the sheet is the same as that of the complete electrostatic image $-e$. For a receding charge the sign of u would be changed.

This result may, of course, be generalized. When any rigid electric system is approaching directly with uniform velocity, from a distance towards an infinite sheet of specific resistance R, the magnetic effect on its own side of the sheet, due to the currents induced in the sheet, is the same as that of the moving optical image of the system with all charges altered in the ratio $-(1+R/2\pi u)^{-1}$.

Obviously the argument here applied to a moving charge applies equally to a moving magnet pole;* but in that case the image must be positive, instead of negative as here, in order to annul the normal component of the magnetic field at the sheet: thus the last result can be generalized so that the system may include magnets as well as electric charges. As circuital electric currents can be represented by magnetic polarity, the principle also extends so as to include such currents.

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* Maxwell, *loc. cit.*, § 655, gives these special results.

The representation by reduced geometrical images thus applies to any steady electro-magnetic system whatever, which is approaching directly to the conducting sheet or thin layer; and it is easy to extend it to a changing system.

When a moving charge e describes a closed orbit, the receding doublet images are distributed along curves drawn on a cylinder. Thus when it describes a circle parallel to the sheet with uniform velocity the image is a distribution of doublets, each of them parallel to the sheet, located along a spiral.

Here also we can arrange cases so that these doublets will fit together end to end. They will do so if a large number of equidistant discrete charges are attached along the circle so that each moves round it with the same velocity v. The image system is then a uniform distribution of tangential doublets spread around the image cylinder, each perpendicular to its axis. They can be enlarged in length so as to fit end to end, and thus form an area of continuous cyclic electric displacement around the cylinder, spreading downwards along it with velocity U. The intensity of this sheet of electric displacement is obtained by noting that, in the time of a revolution $2\pi r/v$, the images have spread down the cylinder a length $2\pi rU/v$, and that the whole charge E on the circle has in this time crossed a line of this length: thus the intensity of the sheet is $E/2\pi r \,.\, U/u$ or $\lambda u/U$, if λ is the electric line-density on the ring. The increment of this sheet of cyclic electric displacement, accruing at its remote end, is equivalent magnetically to a circular current there, of amount equal to the intensity of the sheet multiplied by U, that is, to λv, or to i, the measure of the current in the ring. This image current is too remote to have sensible effect, except at the very beginning of the steady motion. But, if the inducing ring is approaching the sheet with velocity w, the cylindrical image sheet of electric displacement will now be expanding at both ends; there would be an image current $iw/(U+w)$ in the position of the optical image of the ring, and another one $iw/(U-w)$ at the remote end of the cylindrical sheet. These results, generalized to a closed current system of any form, directly approaching the conducting sheet, may be at once verified from the magnetic image when the cyclic currents are replaced by equivalent magnetic shells.

We now revert to the case of a single electron $+e$ travelling in a curved path, as represented by the diagram. In the differential interval of time τ next before the time exhibited, each of the vertical doublets in the diagram will have receded a distance $U\tau$, while a new image $-e$ will have been instituted opposite to the charge, or rather transferred there from the further end of the image-system. To determine the magnetic

field due to convection of these parallel doublets, we make use of the vector potential, which for a current-element eu is parallel to it and equal to eu/r. The operation of moving an electric doublet of moment μ along its own direction (that of z) with velocity $-U$ thus produces a vector potential $U\partial/\partial z\,(\mu/r)$. In the present case $eU.\delta t$ is the aggregate moment of these doublets, all normal to the sheet, which correspond to an element δt of the time of motion. Thus the aggregate vector potential is parallel to z and equal to

$$U\int\frac{\partial}{\partial z}\left(\frac{eU}{r}\right)dt, \quad \text{or} \quad U^2\int\frac{\partial}{\partial z}\left(\frac{e}{v'r}\right)ds';$$

it is thus $(R/2\pi)^2$ multiplied by the component along z of static attraction (toward the sheet) of a line-density equal to e/v', distributed along the (elongated) image curve, where v' is the velocity along that curve corresponding to the motion of the inducing charge. If $(0, 0, H)$ denote this vector potential, the magnetic field is parallel to the sheet and equal to $(\partial H/\partial y, -\partial H/\partial x, 0)$; in fact, H is its stream function. The whole magnetic influence reflected from the conducting sheet is this magnetic field, together with the electric and magnetic fields of an image $-e$ existing and travelling in the position of the optical image of the inducing charge: as the latter is the instantaneous shielding image, the former part represents the trail due to imperfect conductance of the sheet, which prevents the currents from adapting themselves instantaneously into the shielding distribution.

If the inducing charge travels uniformly in an oblique straight line, the distribution of which H is the Newtonian potential is a uniform straight linear one; thus H is expressible in logarithmic form, and the solution is at once completed in simple finite terms, which it is needless to express at length.

The reaction of a conducting plane on any moving electric system can in this way be synthetically set out. Owing to the trail arising from imperfect conductance, each moving charge experiences a force transverse to its velocity equal to the component of the magnetic field in that direction multiplied by v, and a force towards the sheet equal to edH/dt, where H is determined as above; while, in addition, there is the forcive on the moving charge due to its moving reversed optical image. It is implied throughout that the velocity of convection does not approach within, say, one-tenth of that of radiation, as if it did so an "equilibrium" theory which neglects the propagation of waves in the æther could not effectively apply.

In the case of an inducing magnetic pole $+m$, it is the actual poles of

the image-system that produce the magnetic field, in contrast with the convection which alone operates in the electric case above. Thus, in addition to the direct instantaneous image $+m$ (and its complement $-m$ at the other end of the trail), there is a trail of decaying previous disturbance having a magnetic potential

$$-U\frac{\partial}{\partial z}\int\frac{m}{v'r}ds',$$

wherein the integral is the potential of a line density m/v' distributed along the receding image-path in the diagram.*

Similar conclusions apply if the sheet itself is in motion, everything being determined by the relative motion of the inducing system with regard to it.

Thus the specification of the disturbance reflected from the infinite sheet for any convected system containing charges, magnets, and currents is formally complete. The conducting sheet will cut off the direct action of the moving system from the other side, replacing it by a decaying trail represented by the receding images here investigated. But the advantages of this separation between shielding and decaying currents are in the main analytical, unless the velocity is large; it is only for velocities comparable with $R/2\pi$ that the sheet alters much the field of force on the other side, and even for copper 1 cm. thick, R is about 1600 cm./sec.

The Mode of Decay of a Plane Sheet of Currents.

It has been seen that, when an electromagnetic system is suddenly established in front of an unbounded plane sheet, of specific resistance R, it originates a current system induced in the sheet, which at first entirely cuts off the magnetic disturbance from the region beyond it; and that in time such a current system gradually dies away, so that its magnetic field is always the same as would be due to the inducing electromagnetic system, as thus suddenly established but with algebraic sign changed, itself moving away unchanged with velocity $R/2\pi$. Or, in other words, the magnetic field, belonging to an infinite plane current sheet, simply moves up into the sheet as it decays, without change of form or intensity, and with this velocity.

The character of the electric flow in this masking current sheet and the mode of its decay are amenable to simple graphical expression. They

* Cf. Maxwell, *loc. cit.*, § 664, where the case of uniform rectilinear motion is worked out into detail analytically, in rather different form.

are to be such as would produce, in the region beyond the sheet, the field of magnetic potential belonging to the receding reversed equivalent of the inducing system, as above specified. Now, if ϕ is this magnetic potential close to the sheet, the currents in the sheet are derived from a stream function $\phi/2\pi$; that is, the current crossing an element of arc δs, is

$$\frac{1}{2\pi}\frac{d\phi}{ds}\delta s.$$

Thus, if the magnetic equipotential surfaces of the inducing system are constructed, their sections by the plane of the conducting sheet are the initial lines of flow (with direction of course to be reversed) of the currents induced in it. The stages of subsequent decay of these currents in the sheet are determined similarly in terms of the sections of this initial system of equipotential surfaces by a parallel plane, moving away with velocity $R/2\pi$. In this representation, the current flowing in the strip of the sheet included between the same two equipotential surfaces remains the same throughout the decay, now represented by the receding of the sheet. Thus, on this mode of expression, the currents instantaneously established in the sheet may be said to decay (the sheet being infinite) simply by their lines of flow contracting in this manner without alteration of the flux along each, the central filaments of flow disappearing in succession by shrinking to nothing.

For example, the case may be taken of a current suddenly established in a circular wire in front of the sheet, whether its plane be parallel or oblique to the sheet. We have to draw the magnetic equipotentials of such a circular current. These are the orthogonals of the stream lines of a circular vortex filament as drawn by Maxwell in *Elec. and Mag.*, after Lord Kelvin, *Trans. R. S. Edin.* They are a system of surfaces abutting on the current, and if they are constructed so as to spring from it equally spaced at angles $2\pi/n$, the strip between two successive surfaces will, in the specification above, carry one n-th of the current flowing in the wire. The features of the current sheet and its mode of decay are thus open to direct inspection on Lord Kelvin's diagram aforesaid, or rather on a set of parallel sections of a solid model of it. Moreover, in cases of induction not merely instantaneous or impulsive, the lines of electric flow in the sheet at each instant are still the sections by the sheet of the magnetic equipotentials of the image system.

For the case of an electric system convected in front of the sheet, it has been seen (p. 6) that the effect of the currents induced is that of the instantaneous image (the same as if the sheet were a perfect conductor) together with a vector potential (0, 0, H) which is readily calcu-

lated, or graphically expressed, in the simpler cases. The value of H represents the effect of that part of the currents which is due to resistance in the sheet. This part, in fact, involves a magnetic field $(\partial H/\partial y, -\partial H/\partial x, 0)$ at the sheet; thus it belongs to a distribution of currents in the sheet expressed by $(2\pi)^{-1}(\partial H/\partial x, \partial H/\partial y, 0)$; and the part of the electric flow which arises from resistance, the only part which is not cancelled on the remote side of the sheet, is thus represented by the gradient of the function $H/2\pi$. For an inducing electron moving uniformly in a straight line, it is easy to see that these outstanding resistance currents flow along systems of ellipses which are parallel plane sections of the confocal level surfaces of a uniform line-distribution.

ON CHANGE OF THE VARIABLE IN A LEBESGUE INTEGRAL

By E. W. Hobson.

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The use of the Lebesgue integral, instead of the Riemann integral, has the advantage of imparting much greater generality to the results of investigations in which it is employed. It is therefore desirable that all the processes of the integral calculus should be extended to cases in which the Lebesgue integral takes the place of the Riemann integral. In the present communication I propose to consider the effect of a change of the independent variable in a Lebesgue integral, and to establish theorems relating to the equality of the integral in its original form with the new form obtained by a change of the independent variable, analogous to the known results relating to the Riemann integral.

1. Let the variable x denote points in a limited interval (a, b), and let the variable y denote corresponding points in the limited interval (α, β), the relation between the two variables being $x = \psi(y)$, where $\psi(y)$ is a continuous monotone function, with no lines of invariability. We shall suppose that $\psi(y)$ increases from a to b as y increases from α to β. To any set of points in the interval (a, b) of x there corresponds a set of points in the interval (α, β) of y; and if one of these sets be closed, so also is the other set. It is necessary for our purpose to consider under what circumstances a measurable set of points corresponds to a measurable set, and to express relations involving the measures of corresponding sets.

It is known* that, if $\phi(y)$ be a continuous function of y defined for the interval (α, β), and if $D\phi(y)$ denote one of its four derivatives, then, provided the set of points at which $D\phi(y)$ is infinite is a reducible set [or, in particular, if $D\phi(y)$ is everywhere finite, whether it be limited

* This theorem, which is due to Lebesgue, is stated and proved in this form by B. Levi; see the *Rendiconti della Reale Accad. dei Lincei*, Vol. xv, 2e Semestre, 1906, p. 358. The fundamental theorem of the integral calculus for Lebesgue integrals has been discussed by B. Levi and by Lebesgue in a series of Notes in the *Rendiconti*, Vols. xv and xvi.

in the interval or not], the necessary and sufficient condition that $\int_\alpha^\beta D\phi(y)\,dy$ should exist as a Lebesgue integral is that the function $\phi(y)$ should be of limited total fluctuation (*à variation bornée*) in (α, β). Further, the integral is then equal to $\phi(\beta)-\phi(\alpha)$. The condition that the function shall be of limited total fluctuation is satisfied by the function $\psi(y)$. It follows that, provided one of the four derivatives $D\psi(y)$ is either everywhere finite (even if it be not limited) or else, if it be infinite at a reducible set of points in the interval (α, β), the* integral $\int_\alpha^\beta D\psi(y)\,dy$ exists, and is equal to $b-a$. Also $\int D\psi(y)\,dy$ taken through any interval δ contained in (α, β) exists, and is equal to the length of the corresponding interval in (a, b).

It follows that, if Δ denote any set of intervals in (α, β), and Δ_1 denote the corresponding set of intervals in (a, b), then

$$\int_{(\Delta)} D\psi(y)\,dy = m(\Delta_1),$$

where $m(\Delta_1)$ denotes the measure of the set of intervals Δ_1, and the Lebesgue integral is taken through the corresponding set Δ.

If η be a prescribed arbitrarily small positive number, a positive number N can be determined such that, if E_N denotes the set of all points in (α, β) at which $D\psi(y) > N$, then

$$\int_{(E_N)} D\psi(y)\,dy < \tfrac{1}{2}\eta;$$

this follows from the existence of $\int_\alpha^\beta D\psi(y)\,dy$.

If, now, $m(\Delta) < \xi$, where ξ is a prescribed arbitrarily small positive number, we have

$$\int_{(\Delta)} D\psi(y)\,dy < N\xi + \tfrac{1}{2}\eta < \eta,$$

if ξ be chosen to be not greater than $\frac{\eta}{2N}$. It follows that, if η be arbitrarily chosen, $m(\Delta_1) < \eta$, provided $m(\Delta)$ be sufficiently small. Hence $m(\Delta_1)$ converges to zero as $m(\Delta)$ does so, and uniformly for all sets Δ in (α, β).

If e be a set of points of zero measure in (α, β), the corresponding set e_1 of points in (a, b) is measurable, and has zero measure. For the

* The integral is to be taken for the set of points which consists of the points of the interval (α, β) with the exception of the points at which $D\psi(y)$ is infinite.

set e can be enclosed in intervals of a set Δ such that $m(\Delta)$ is arbitrarily small, and it then follows that the set e_1 can be enclosed in intervals of a set Δ_1, where $m(\Delta_1)$ is arbitrarily small; and therefore the result follows.

Let a set of points E in (α, β) be defined as that set which is contained in every set of intervals of a sequence of sets of intervals $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}, \ldots$, such that all the intervals of each set are contained in the intervals of the preceding set. To the set E in (α, β) then corresponds a set E_1 in (a, b), which consists of that set of points which is contained in all the sets of intervals $\Delta_1^{(1)}, \Delta_1^{(2)}, \ldots, \Delta_1^{(n)}, \ldots$ of the sequence which corresponds to $\Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n)}, \ldots$. The sets E, E_1 are measurable, and

$$m(E_1) = \lim_{n=\infty} m(\Delta_1^{(n)}) = \lim_{n=\infty} \int_{(\Delta^{(n)})} D\psi(y)\,dy.$$

Now
$$\lim_{n=\infty} \int_{(\Delta^{(n)})} D\psi(y)\,dy = \int_{(E)} D\psi(y)\,dy;$$

for n_1 may be chosen so great that

$$\int_{(\Delta^{(n)})} D\psi(y)\,dy - \int_{(E)} D\psi(y)\,dy < \eta,$$

for $n \geqslant n_1$. To see this, we observe that the differences of the integrals is less than $N.m(\Delta^{(n)}-E)+\frac{1}{2}\eta$, provided N is chosen sufficiently great; we have then only to choose n_1 so that

$$m(\Delta^{(n)}-E) < \frac{\eta}{2N},$$

for $n \geqslant n_1$. It has now been shewn that

$$m(E_1) = \int_{(E)} D\psi(y)\,dy.$$

Next, let G be any measurable set of points in the interval (α, β), and let G_1 be the corresponding set of points in the interval (a, b). It will be shewn that G_1 is a measurable set.

The set G is contained* in a set $E^{(1)}$, and contains another set $E^{(2)}$, where $E^{(1)}$, $E^{(2)}$ are of the type of the set E above, and

$$m(G) = m(E^{(1)}) = m(E^{(2)}).$$

Let $E_1^{(1)}$, $E_1^{(2)}$ be the sets of points in (a, b) which correspond to $E^{(1)}$, $E^{(2)}$, respectively. If we denote the exterior and interior measures of G_1 by

* See *Theory of Functions of a Real Variable*, p. 107.

$m_e(G_1)$, $m_i(G_1)$ respectively, we have

$$m_e(G_1) = m(E_1^{(1)}) = \int_{(E^{(1)})} D\psi(y)\,dy$$

and

$$m_i(G_1) = m(E_1^{(2)}) = \int_{(E^{(2)})} D\psi(y)\,dy.$$

The two integrals differ only by the integral taken through the set $E^{(1)} - E^{(2)}$, which is of measure zero, and thus the two integrals have the same value.

Therefore $m_e(G_1) = m_i(G_1)$, and thus the set G_1 is measurable, its measure being $\int_{(G)} D\psi(y)\,dy$.

The following result has now been established:—

Let points in the interval (α, β) be denoted by y, and points in the interval (a, b) by x. Let corresponding points be defined by the relation $x = \psi(y)$, where $\psi(y)$ is a continuous monotone function without lines of invariability, and such that one of the four derivatives $D\psi(y)$ of the function $\psi(y)$ is finite everywhere in (α, β), except at points belonging to a reducible set of points. If G be any measurable set of points in (α, β), the corresponding set of points G_1 in (a, b) is also measurable and of measure $\int_{(G)} D\psi(y)\,dy$.

It thus appears that it is possible that a non-measurable set G_1 may correspond to a measurable set G only in case $D\psi(y)$ is infinite at all points of an irreducible set in (α, β). Constructions of non-measurable sets of points have been given by Lebesgue* and by Van Vleck.† These constructions are of the kind described by Lebesgue as *idealistic*, in that they involve an infinite number of acts of choice which are not determined *a priori* by a norm. It is possible that non-measurable sets might be defined by means of a correspondence $x = \psi(y)$, where one of the derivatives $D\psi(y)$ of the monotone function $\psi(y)$ is infinite for an irreducible set of values of y, in case such a function can be defined.

The relation $x = \psi(y)$ may also be expressed in the form $y = \chi(x)$, where $\chi(x)$ is a continuous monotone function of x. The four derivatives of $\chi(x)$ at a point x are expressed in terms of the four derivatives of $\psi(y)$ at the corresponding point y by means of the relations

$$D^+\psi(y)\,D_+\chi(x) = 1, \qquad D_+\psi(y)\,D^+\chi(x) = 1,$$

$$D^-\psi(y)\,D_-\chi(x) = 1, \qquad D_-\psi(y)\,D^-\chi(x) = 1;$$

* *Bulletin de la Soc. mat. de France*, Vol. xxxv, 1907, p. 209.

† *Transactions of American Math. Soc.*, Vol. ix, 1908, p. 237.

it being assumed that y and x increase together. If we now assume that the set of values of y for which one of the derivatives $D\psi(y)$ is zero is a reducible set, then the set of values of x for which the corresponding derivative $D\chi(x)$ is infinite is also a reducible set. In that case, we see that, to any measurable set G_1 in (a, b), there corresponds a measurable set G in (α, β).

The following theorem has thus been established :—

The correspondence between x in (a, b), and y in (α, β), being defined by $x = \psi(y)$, where $\psi(y)$ is a continuous monotone function, such that one of its derivatives $D\psi(y)$ is infinite or zero only at points of a reducible set in (α, β), if G_1, G be corresponding sets, then, if either set is measurable, so also is the other one.

2. The relation between x and y being of the character prescribed in the last theorem of § 1, it will be shewn that, if a function $f(x) \equiv f\{\psi(y)\}$ be such that one of the two Lebesgue integrals

$$\int_a^b f(x)\,dx, \qquad \int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)\,dy$$

exists, then the other integral also exists and the two have the same value. The function $f(x)$ may be unlimited in the interval (a, b).

Let us assume that $\int_a^b f(x)\,dx$ exists. It then follows that $f\{\psi(y)\}$ is a summable function, since a measurable set of points in (α, β) corresponds to a measurable set of points in (a, b). As $D\psi(y)$ is* a summable function, it then follows that $f\{\psi(y)\}\, D\psi(y)$ is a summable function. It will be sufficient to consider the case in which $f(x) \geqslant 0$ for all values of x, since any function may be taken to be the difference of two such functions.

Let ϵ be a fixed positive number, and N a positive number ; and let $f_1(x)$ be defined by $f_1(x) = f(x)$, for all values of x such that $f(x) \leqslant N$, and by $f_1(x) = N$ for all values of x such that $f(x) > N$. The number N may be so chosen† that

$$\left| \int_a^b f(x)\,dx - \int_a^b f_1(x)\,dx \right| < \epsilon.$$

The interval $(0, N)$ may be divided into parts (c_0, c_1), (c_1, c_2), ..., (c_{n-1}, c_n), where $c_0 = 0$, $c_n = N$. Let η be the greatest length of these intervals

* See *Functions of a Real Variable*, p. 552.

† *Ibid.*, p. 397.

$c_{r+1}-c_r$, and let e_r denote the set of points x, for which $c_r \leqslant f_1(x) < c_{r+1}$; also, let e_n denote the set of points for which $f_1(x) = c_n$. If ϵ' be an arbitrarily chosen number, the number η may be so chosen that

$$\left| \int_a^b f_1(x)\,dx - \sum_{r=0}^{r=n} c_r m(e_r) \right| < \epsilon'.$$

The sum

$$\sum_{r=0}^{r=n} c_r m(e_r) = \sum_{r=0}^{r=n} c_r \int_{(E_r)} D\psi(y)\,dy,$$

where E_r is that measurable set of points in (α, β) which corresponds to the set e_r in (a, b). We have now

$$\left| \sum_{r=0}^{r=n} c_r m(e_r) - \int_\alpha^\beta f_1\{\psi(y)\}\, D\psi(y)\,dy \right| < \eta \sum_{r=0}^{r=n} \int_{(E_r)} D\psi(y)\,dy < \eta(b-a).$$

Combining the three inequalities which have been established, we have

$$\left| \int_a^b f(x)\,dx - \int_\alpha^\beta f_1\{\psi(y)\}\, D\psi(y)\,dy \right| < \epsilon + \epsilon' + \eta(b-a).$$

Keeping ϵ and N fixed, ϵ' and η are arbitrarily small and converge together to zero, and therefore

$$\left| \int_a^b f(x)\,dx - \int_\alpha^\beta f_1\{\psi(y)\}\, D\psi(y)\,dy \right| \leqslant \epsilon.$$

Since N depends only on ϵ, and ϵ converges to zero as N is indefinitely increased, we have therefore

$$\int_a^b f(x)\,dx = \lim_{N=\infty} \int_\alpha^\beta f_1\{\psi(y)\}\, D\psi(y)\,dy.$$

The existence of $\lim_{N=\infty} \int_\alpha^\beta f_1\{\psi(y)\}\, D\psi(y)\,dy$ having now been established, we shall shew that $\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)\,dy$ exists and is equal to this limit.

Let $f\{\psi(y)\}\, D\psi(y)$ be denoted by $F(y)$; and consider a sequence of values of N which increase without limit. Let the corresponding values of $f_1\{\psi(y)\}\, D\psi(y)$ be denoted by $F_1(y)$, $F_2(y)$, ..., $F_n(y)$, ...: we then have $\lim_{n=\infty} F_n(y) = F(y)$, and we have to shew that, if

$$\lim_{n=\infty} \int_\alpha^\beta F_n(y)\,dy \text{ exists and } = A,$$

then also

$$\int_\alpha^\beta F(y)\,dy \text{ exists and } = A.$$

The sequence $\{F_n(y)\}$ is monotone, in the sense that $F_n(y) \leqslant F_{n+1}(y)$ for all values of y and n. Let the unlimited interval $(0, \infty)$ be divided

into parts (a_0, a_1), (a_1, a_2), ..., (a_r, a_{r+1}), ..., where $a_{r+1}-a_r < \eta$, for every value of r; and $a_0 = 0$. Denote by $E_r^{(n)}$ that set of points for which

$$a_r \leqslant F_n(y) < a_{r+1};$$

and denote by E_r the set of points for which

$$a_r \leqslant F(y) < a_{r+1}.$$

We have
$$\sum_{r=0}^{r=s} a_r m(E_r^{(n)}) \leqslant \int_\alpha^\beta F_n(y)\,dy < A$$

for all integral values of s and n. As N is increased indefinitely, the measure $m(g_1)$ of that set g_1 of points for which $f(x) > f_1(x)$ diminishes indefinitely; hence $m(g)$ diminishes indefinitely, where g is the set of points in (α, β) corresponding to g_1 in (a, b). For

$$m(g) = \int_{(g_1)} D\chi(x)\,dx,$$

and this diminishes indefinitely with $m(g_1)$. Therefore the set of points for which $f\{\psi(y)\} > f_1\{\psi(y)\}$ has measure $< \xi$, if N is sufficiently great. It follows that all those points in which $F_n(y)$ has a different value from that of $F(y)$ form a set of which the measure is less than ξ, if n is sufficiently great. It then follows that $m(E_r^{(n)})$ differs from $m(E_r)$ by less than ξ, if n is sufficiently great; and for all the values 0, 1, ..., s of r. Hence

$$\sum_{r=0}^{r=s} a_r m(E_r) < \sum_{r=0}^{r=s} a_r m(E_r^{(n)}) + \xi \sum_{r=0}^{r=s} a_r$$

$$< A + \xi \sum_{r=0}^{r=s} a_r,$$

and, since ξ is arbitrarily small, we have, for any fixed value of s,

$$\sum_{r=0}^{r=s} a_r m(E_r) \leqslant A.$$

This holds for every value of s; therefore the series $\sum_{r=0}^{r=\infty} a_r m(E_r)$ is convergent and its sum is $\leqslant A$. Since this holds for every value of η, however small, it follows* that $\int_\alpha^\beta F(y)\,dy$ exists and is $\leqslant A$. It is impossible that

$$\int_\alpha^\beta F(y)\,dy < A,$$

* See the definition of the Lebesgue integral of an unlimited function given in *Theory of Functions of a Real Variable*, p. 396.

say $= A-\lambda$, for we may take n so great that

$$\int_\alpha^\beta F_n(y)\,dy > A-\lambda,$$

and thus

$$\int_\alpha^\beta F_n(y)\,dy > \int_\alpha^\beta F(y)\,dy,$$

which is clearly impossible. It follows that

$$\int_\alpha^\beta F(y)\,dy = A$$

or $$\int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy = \lim_{N=\infty}\int_\alpha^\beta f_1\{\psi(y)\}\,D\psi(y)\,dy = \int_a^b f(x)\,dx.$$

If, instead of assuming the existence of $\int_a^b f(x)\,dx$, we assume that of $\int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy$, we apply to the latter the transformation $y=\chi(x)$, and remember that $D\chi(x) = \dfrac{1}{D\psi(y)}$, where $D\chi(x)$ is that one of the four derivatives of $\chi(x)$ which corresponds to $D\psi(y)$. The above proof then establishes the existence of $\int_a^b f(x)\,dx$ and its equality with the other integral.

The following theorem has now been proved :—

If $\psi(y)$ be a continuous monotone function such that the points, if any, of the interval (α, β) at which one of its four derivatives $D\psi(y)$ is either zero or infinite, form a reducible set of points, and if $f(x)$ be any limited or unlimited function such that one of the two

$$\int_a^b f(x)\,dx, \qquad \int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy$$

exists as a Lebesgue integral, then the other exists, and the two integrals have the same value. The interval (α, β) of y corresponds, for the transformation $x=\psi(y)$, to the interval (a, b) of x.

If we make use of Lebesgue's theorem that every function with limited total fluctuation, and in particular every limited monotone function, has a finite differential coefficient except at the points of a set of which the measure is zero, we see that, in the above theorem, $\int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy$ may be replaced by $\int_\alpha^\beta f\{\psi(y)\}\,\psi'(y)\,dy$. For the set of points at which $\psi'(y)$ has no definite value forms a set of zero measure, and such a set may be disregarded in calculating the Lebesgue integral.

If we suppose that $f(x)$ has the value 0 at all points of (a, b) except at those which belong to a measurable set E_1, we see that, if E be that set in (α, β) which corresponds to E,

$$\int_{(E_1)} f(x)\,dx = \int_{(E)} f\{\psi(y)\}\,D\psi(y)\,dy = \int_{(E)} f\{\psi(y)\}\,\psi'(y)\,dy,$$

provided $f(x)$ is such that either of the integrals exists.

3. The theorem of § 2 can be extended to the case of non-absolutely convergent improper integrals. The definition of such an integral which will be adopted is that generalization* of Harnack's definition which arises when the sequence of integrals by which the improper integral is defined consists of a sequence of Lebesgue integrals instead of one of Riemann's integrals.

If $f(x)$ have points of infinite discontinuity at the points of a non-dense closed set G_1 of zero content, the function $f\{\psi(y)\}$ has points of infinite discontinuity at the points of the corresponding closed set G which, as has been shewn in § 1, is of zero content.

Let the points of G_1 be enclosed in a finite number of intervals of which the sum is ϵ_1, each interval containing at least one point of G. The complementary intervals $\{\delta\}$ form a finite set, of which the sum is $b-a-\epsilon_1$. The set of intervals $\{\delta\}$ corresponds to a finite set of intervals $\{\Delta\}$ in (a, β) of which the sum is $\beta-a-\epsilon$, where ϵ and ϵ_1 converge together to the limit zero.

$\int_a^b f(x)\,dx$ is the limit of $\int_{\{\delta\}} f(x)\,dx$ as ϵ_1 is indefinitely diminished, when that limit exists. By applying the theorem of § 2, we see that, when $D\psi(y)$ satisfies the condition of that theorem, if $\int_a^b f(x)\,dx$ exists it is equal to the limit of $\int_{\{\Delta\}} f\{\psi(y)\}\,D\psi(y)\,dy$ for $\epsilon = 0$. In order that this latter limit may necessarily be represented by

$$\int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy,$$

it is in general necessary that $f\{\psi(y)\}\,D\psi(y)$ should be infinitely discontinuous at all the points of the set G. This we have seen to be the case for $f\{\psi(y)\}$, but it is possible that the factor $D\psi(y)$ may

* See *Theory of Functions of a Real Variable*, p. 397; also *Proceedings*, Ser. 2, Vol. 7, p. 20. It may be remarked that the Lebesgue integrals employed in this definition may be themselves integrals of unlimited functions.

remove some of the infinite discontinuities. If it does so, the limit, as ϵ is indefinitely diminished, of $\int_{\{\Delta\}} f\{\psi(y)\}\, D\psi(y)\, dy$ is not necessarily equal to $\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)\, dy$; for it is a necessary postulation in Harnack's definition, for the case of a non-absolutely convergent integral, and also in the extension of that definition, that each of the intervals of the set complementary to $\{\Delta\}$ should contain at least one point of infinite discontinuity of the function.* However, in case the set G_1 is finite, the limit of $\int_{\{\Delta\}} f\{\psi(y)\}\, D\psi(y)\, dy$ is equal to $\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)\, dy$; for in this case the number of intervals in $\{\delta\}$ and in $\{\Delta\}$ remains finite when ϵ_1 and ϵ converge to zero.

More generally it will be sufficient to assume that the closed set G can be divided into two parts, H, K, both closed, and such that all the points of H are points of infinite discontinuity of $f\{\psi(y)\}\, D\psi(y)$, and that none of the points of K are so. Since the two sets H, K are closed, the distance between a point of the one set and a point of the other set has a minimum value d, which is greater than zero. We may assume the set $\{\Delta\}$ to be such that $\epsilon < d$; it is then clear that the set of intervals complementary to $\{\Delta\}$ consists of two parts, $\{\xi\}$ and $\{\xi'\}$, such that each point of H is enclosed in one of the intervals of $\{\xi\}$, and each point of K in one of the intervals $\{\xi'\}$. The value of

$$\int_{\{\Delta\}} f\{\psi(y)\}\, D\psi(y)\, dy$$

differs from the value of the integral taken through the sets $\{\Delta\}$ and $\{\xi'\}$ together, by an amount which converges to zero as ϵ does so, since $f\{\psi(y)\}\, D\psi(y)$ has an absolutely convergent integral in the part of (α, β) which is complementary to the set of intervals $\{\Delta\}$. It follows that the limit of $\int_{\{\Delta\}+\{\xi'\}} f\{\psi(y)\}\, D\psi(y)\, dy$ exists, and is equal to that of $\int_{\{\Delta\}} f\{\psi(y)\}\, D\psi(y)\, dy$; and this latter limit therefore defines the integral $\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)\, dy$.

The condition that the two sets H, K should be closed may be replaced by the simpler one that K should be closed. For, let L denote the closed set of all the points of infinite discontinuity of $f\{\psi(y)\}\, D\psi(y)$, then the

* See *Theory of Functions of a Real Variable*, p. 386.

set H consists of all the points common to the two closed sets G and L, and is therefore itself a closed set.

We have now established the following theorem :—

If $\int_a^b f(x)dx$ *exists as a non-absolutely convergent improper integral, in accordance with its definition as the limit of a sequence of Lebesgue integrals taken through a finite set of intervals complementary to a finite set, each interval of which contains at least one point of the closed set* G_1, *of zero content, consisting of the points of infinite discontinuity of* $f(x)$, *then, in case the set of points at which* $D\psi(y)$ *is zero or infinite is reducible, the integral* $\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)dy$ *exists and is equal to* $\int_a^b f(x)dx$, *provided those points (if any) of the set* G, *corresponding to* G_1, *which are not points of infinite discontinuity of* $f\{\psi(y)\}\, D\psi(y)$ *form a closed set.*

The last condition in the theorem is sufficient, but not necessary. It is satisfied, in particular, if G is a finite set, or if those points of G which are not points of infinite discontinuity of $f\{\psi(y)\}\, D\psi(y)$ form a finite set.

When the condition is not satisfied, it is conceivable that

$$\int_\alpha^\beta f\{\psi(y)\}\, D\psi(y)dy$$

may diverge or that it may have a value different from that of $\int_a^b f(x)dx$.

4. A Lebesgue integral taken through an unlimited interval may be defined as the limit of a Lebesgue integral taken through a finite interval. Thus

$$\int_a^\infty f(x)dx \equiv \lim_{h=\infty} \int_a^h f(x)dx, \qquad \int_{-\infty}^b f(x)dx \equiv \lim_{k=-\infty} \int_k^b f(x)dx,$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{h=\infty,\ k=-\infty} \int_k^h f(x)dx.$$

These integrals may exist without necessitating the existence of

$$\int_a^\infty |f(x)|dx, \qquad \int_{-\infty}^b |f(x)|dx, \qquad \int_{-\infty}^\infty |f(x)|dx.$$

It is clear how the theorems of §§ 2, 3 can be extended to cases in

which one of the four numbers a, b, α, β has the improper value ∞, or $-\infty$, or in which two or more of them have one of these values.

The following statement is clearly correct:—

The existence of one of the integrals

$$\int_a^b f(x)\,dx, \qquad \int_\alpha^\beta f\{\psi(y)\}\,D\psi(y)\,dy,$$

where any of the limits a, b, α, β has the improper value ∞ or $-\infty$, implies the existence of the other integral, and the equality of the two, provided the conditions of either the theorem of § 2 or of that of § 3 are satisfied for all pairs of corresponding finite intervals which are contained in the intervals of the above integrals.

ON SOME FUNDAMENTAL PROPERTIES OF LEBESGUE INTEGRALS IN A TWO-DIMENSIONAL DOMAIN

By E. W. Hobson.

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It has been shewn* by Lebesgue that, if $f(x, y)$ be a limited function defined for a measurable and bounded domain E, and the function be summable, the integral of $f(x, y)$ in E is equal to either of the two expressions

$$\int_{\sup} dx \int_{\sup} f(x, y)\, dy, \qquad \int_{\inf} dy \int_{\inf} f(x, y)\, dy,$$

in accordance with his definition of the superior and inferior integrals† of a function of one variable, which is not assumed to be summable. From this he deduced that, if $\int dx \int f(x, y)\, dy$ have a meaning, it is equal to the Lebesgue integral of $f(x, y)$ in the domain E.

From the first result it has been deduced by‡ Fubini that, whether $f(x, y)$ be limited or unlimited in E, then provided $\int_{(E)} f(x, y)(dx\, dy)$ exists, the repeated integrals $\int dx \int f(x, y)\, dy$, $\int dy \int f(x, y)\, dx$ necessarily exist, and have the same value as $\int_{(E)} f(x, y)(dx\, dy)$.

In the first part of the present communication I have given a direct proof of this last result which does not require the use of Lebesgue's superior and inferior integrals. I have further shewn that, if one of the repeated integrals of $|f(x, y)|$ exists, and the limited or unlimited function be summable in E, then the Lebesgue integral of $f(x, y)$ through

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* See my work, *Theory of Functions of a Real Variable*, pp. 576–581.

† Lebesgue's superior and inferior integrals are not to be confused with Darboux's upper and lower integrals.

‡ See his paper, "Sugli integrali multipli," *Rendiconti della Reale Accad. dei Lincei*, Vol. xvi, 1907.

E exists, and is then in accordance with the former theorem equal to the corresponding repeated integrals; this requires no further proof when the function is limited.

In the second part of the paper, I have employed the results established in the first part to obtain theorems relating to the transformation of a Lebesgue integral in a plane domain, when the variables are changed. The results obtained in my paper,* "On Change of the Variable in a Lebesgue Integral" for the case of integrals through linear sets of points have been here utilized.

There is no difficulty in the extension of the results here obtained to the case of functions of any number of variables.

1. Let f denote a function defined for all points of a measurable set, of any number of dimensions, and let U, L denote the upper and lower limits of f in the set. If f is unlimited, we have $U = \infty$, or $L = -\infty$, or both $U = \infty$ and $L = -\infty$.

Let $a_1, a_2, \ldots, a_n, \ldots$ be an enumerable set of numbers everywhere dense in the limited or unlimited interval (L, U). It will be shewn that, G_n denoting that set of points for which $f \geqslant a_n$, if the sets $G_1, G_2, \ldots, G_n, \ldots$ are all measurable, then the function f is summable in the domain for which it is defined.

Let A be any number in the interval (L, U), then a sequence $a_{p_1}, a_{p_2}, \ldots$, of which A is the limit, can be selected from the sequence $\{a_n\}$; moreover, the sequence can be so chosen as to be monotone and diminishing. The set of points for which $f > A$, is such that each point belongs to all except a finite number of those sets for which

$$f \geqslant a_{p_1}, \quad f \geqslant a_{p_2}, \quad \ldots .$$

By a known theorem,† it follows that the set of points for which $f > A$ is a measurable set. If the set of numbers $a_{p_1}, a_{p_2}, \ldots$ were taken to be monotone and increasing, that set of points which consists of all points common to all the sets for which $f \geqslant a_{p_1}, f \geqslant a_{p_2}, \ldots$ is the set of points for which $f \geqslant A$; hence this set is measurable. It has been shewn that the sets for which $f > A$ and $f \geqslant A$ are both measurable; it then follows that the sets for which $f \leqslant A$, $f < A$ are also measurable. It is then seen that, if A, B are any two numbers in the interval (L, U), the set of points for which $A < f < B$ is measurable, where $B > A$. It has thus

* *Proceedings, ante*, p. 10.

† See *Functions of a Real Variable*, p. 106.

been shewn that f is a summable function. We have obtained the following theorem :—

If a function f defined for all points of a measurable set be such that the set of points for which $f \geqslant a_n$ is, for each value of n, measurable, where $\{a_n\}$ is an enumerable set of numbers everywhere dense in the interval (L, U), then the function f is summable. The numbers L, U denote the lower and upper limits of f in its domain, and either of them may be infinite, or both may be infinite.

2. Let E denote a set of points in the plane of (x, y), and let the set be bounded and measurable in the plane. Denote by $E(x)$ the linear section of the set made by a line parallel to the y-axis. It will be shewn that the linear set of points $E(x)$ is linearly measurable for every value of x except for a set of linear measure zero. It will further be shewn that, if $m_l E(x)$ denote the linear measure of $E(x)$, then the function $m_l E(x)$ is linearly summable. The fact that this function is undefined for a set of values of x of zero measure is irrelevant.

Let the set E be contained in the rectangle bounded by $x = a$, $x = b$, $y = c$, $y = d$. There exists a set of points E_2 which consists of the points common to all the sets of a sequence of sets of rectangles, such that the rectangles have the sides parallel to the axes, and such that E_2 contains E, and that $m(E_2) = m(E)$.

Let $D_{\iota 1}, D_{\iota 2}, \ldots, D_{\iota\gamma}, \ldots$ denote the rectangles of the ι-th set. Then denoting the set of points consisting of all the points of all these rectangles by D_ι, we have

$$m(D_\iota) = \sum_{\gamma=1}^{\infty} \int_a^b m_l \{D_{\iota\gamma}(x)\}\, dx,$$

where $D_{\iota\gamma}(x)$ is the section of the rectangle $D_{\iota\gamma}$. We have then, by employing a known theorem,

$$m(D_\iota) = \int_a^b m_l \{D_\iota(x)\}\, dx.$$

Hence
$$m(E) = m(E_2) = \lim_{n=\infty} m(D_\iota) = \lim_{l=\infty} \int_a^b m_l \{D_\iota(x)\}\, dx$$
$$= \int_a^b m_l \{E_2(x)\}\, dx.$$

Also the set E contains a set E_1 of the same type as E_2, and such that

$m(E) = m(E_1)$; in a similar manner to the above proof we find that

$$m(E) = m(E_1) = \int_a^b m_l \{E_1(x)\}\, dx.$$

It follows that $$\int_a^b m_l \{E_2(x) - E_1(x)\}\, dx = 0;$$

and since $E_2(x)$ contains $E(x)$ which contains $E_1(x)$, we see that for all values of x except a set of zero linear measure, we must have

$$m_l \{E_2(x) - E_1(x)\} = 0;$$

and therefore $$m_l E_2(x) = m_l E_1(x) = m_l E(x).$$

Moreover, we see that $$m(E) = \int_a^b m_l \{E(x)\}\, dx.$$

The following theorem has thus been established:—

If E be a measurable set of points in a plane, and be a bounded set, then its section $E(x)$ is linearly measurable for each value of x with the exception of those belonging to a set of linear measure zero. The function $E(x)$ is linearly summable, and its Lebesgue integral is equal to the plane measure of the set E.

3. The function $f(x, y)$ being summable in the two-dimensional domain for which it is defined, we shall consider whether it is linearly summable for a constant value of x.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ be a sequence of numbers everywhere dense in the finite or infinite interval (L, U). Those sets of points $E_1, E_2, \ldots, E_n, \ldots$, for which $f \geqslant \alpha_1$, $f \geqslant \alpha_2$, $\ldots, f \geqslant \alpha_n, \ldots$ being all measurable in the plane, are in accordance with the theorem of § 2, such that their sections $E_1(x), E_2(x), \ldots, E_n(x), \ldots$ are all measurable linearly, for each value of x which does not belong to one of certain sets $F_1, F_2, \ldots, F_n, \ldots$, each of linear measure zero. If $\bar{F}$ denote that set of numbers, of linear measure zero, which consists of $M(F_1, F_2, \ldots, F_n, \ldots)$; we see that, except for values of x belonging to this set $\bar{F}$, all the sets $E_1(x), E_2(x), \ldots, E_n(x), \ldots$ are linearly measurable.

By applying the theorem of § 1, we now see that, for each value of x which does not belong to the set $\bar{F}$, the function $f(x, y)$, regarded as a function of y, is linearly summable. In case the function f is limited, it then follows that $\int_c^d f(x, y)\, dy$ exists for each value of x not belonging to $\bar{F}$.

The following theorem has thus been established:—

If $f(x, y)$ be a summable function defined for a bounded domain contained

in the rectangle given by $x = a$, $x = b$, $y = c$, $y = d$, *then the function is linearly summable for each constant value of* x, *except those which belong to a set* $\bar{F}$ *of linear measure zero. The integral* $\int_c^d f(x, y)\,dy$ *exists, although it may diverge in case* f *is an unlimited function, when* x *does not belong to the set* $\bar{F}$. *We assume that* $f = 0$ *for values of* x, y *not belonging to the measurable set of points for which* f *was originally defined.*

4. Let the function $f(x, y)$ be summable and limited in its domain, and let the interval (L, U) be divided into parts (c_0, c_1), (c_1, c_2), ..., (c_{n-1}, c_n), where $c_0 = L$, $c_n = U$. Let e_0 denote the set of points for which $c_0 \leqslant f < c_1$, let e_r denote the set of points for which $c_r \leqslant f < c_{r+1}$, and e_n the set for which $f = c_n = U$. We may choose the intervals so that the greatest of them is a prescribed arbitrarily small number η_s. Let $f_s(x, y)$ be a function which has the value c_r at the points of e_r, for each value of r; thus f_s has only a finite number of distinct values. If x does not belong to a certain set G_s of values of linear measure zero, all the sections $e_0(x)$, $e_1(x)$, ..., $e_n(x)$ are linearly measurable, and therefore the integral $\int_c^d f_s(x, y)\,dy$ which is equal to $\sum_{r=0}^{r=n} c_r m_l\{e_r(x)\}$ exists; we shall denote this integral by $\phi_s(x)$.

Let s have the values 1, 2, 3, ..., the corresponding values of η_s, forming a diminishing sequence which converges to the limit zero. The linearly measurable sets $G_1, G_2, \ldots, G_s, \ldots$ have a set

$$G \equiv M(G_1, G_2, \ldots, G_s, \ldots),$$

consisting of all the points which belong to any of these sets, and G has the linear measure zero.

For all values of x which do not belong to G, all the integrals $\phi_1(x), \phi_2(x), \ldots, \phi_s(x), \ldots$ exist.

Also

$$\int_a^b \phi_s(x)\,dx = \sum_{r=0}^{r=n} c_r \int_a^b m_l\{e_r(x)\}\,dx = \sum_{r=0}^{r=n} c_r\, m(e_r) = \int f_s(x, y)(dx\,dy),$$

those values of x which belong to G being disregarded.

Now $$f(x, y) = \lim_{s=\infty} f_s(x, y),$$

and the function $\phi(x)$ which denotes $\lim_{s=\infty} \phi_s(x)$ is summable. The sequence $\{\phi_s(x)\}$ is monotone and increasing, and is also limited for all values of s

and x; hence* $\phi(x)$ is summable, and

$$\int_a^b \phi(x)\,dx = \lim_{s=\infty} \int_a^b \phi_s(x)\,dx.$$

It then follows that $\phi(x) = \int_c^d f(x, y)\,dy,$

and that
$$\int_a^b dx \int_c^d f(x, y)\,dy = \lim_{s=\infty} \int f_s(x, y)(dx\,dy)$$
$$= \int_{(E)} f(x, y)(dx\,dy).$$

Therefore, when f is a limited function, the Lebesgue integral

$$\int_{(E)} f(x, y)(dx\,dy)$$

is equal to the repeated integral

$$\int_a^b dx \int_c^d f(x, y)\,dy\,;$$

the existence of the latter being a necessary consequence of the existence of the former integral. The integral $\int_c^d f(x, y)dy$ may fail to have a definite meaning for values of x belonging to a set of linear measure zero, but this does not affect the integrability with respect to x, in accordance with Lebesgue's definition.

5. With a view to the extension of the results of § 4 to the case of an unlimited function $f(x, y)$, we have need of the following theorem relating to a monotone sequence of functions :—

If† $\{s_n(x)\}$ *is a sequence of non-negative limited summable functions defined for a measureable set of points E, and the sequence be monotone and increasing, and such that* $\lim_{n=\infty} \int_{(E)} s_n(x)dx$ *has a finite value, then* (1) *the points of E at which the sequence* $\{s_n(x)\}$ *diverges form a set of*

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* See *Functions of a Real Variable*, pp. 542, 715 note.

† The first two parts of this theorem have been given substantially by Vitali, *Rendiconti di Circ. Mat. di Palermo*, Vol. XXIII, 1907, and by B. Levi, *Rendiconti dell' Istit. Lombardo*, 1906, respectively. The proofs there given differ from the above.

measure zero, and (2) *if* $s(x)$ *denotes the limit of the sequence* $\{s_n(x)\}$, $\int_{(E)} s(x)\,dx$ *exists and is equal to* $\lim_{n=\infty} \int_{(E)} s_n(x)\,dx$. *The set of points of* E *at which* $s(x) = \infty$ *may be disregarded in the integral. Conversely, if* $\int_{(E)} s(x)\,dx$ *has a finite value, then* $\lim_{n=\infty} \int_{(E)} s_n(x)\,dx$ *exists and has the same value as the integral.*

Let k denote a positive number, and let the functions $s_n^{(k)}(x)$ be defined by $s_n^{(k)}(x) = s_n(x)$, for values of x for which $s_n(x) \leqslant k$, and $s_n^{(k)}(x) = k$, for values of x for which $s_n(x) > k$. Let $s^{(k)}(x)$ be a function such that $s^{(k)}(x) = s(x)$, for values of x such that $s(x) \leqslant k$, and $s^{(k)}(x) = k$, where x is such that $s(x) > k$.

We have $s_n^{(k)}(x) \leqslant s_{n+1}^{(k)}(x)$, for every value of k and x; also

$$s_n^{(k)}(x) \leqslant s_n^{(k')}(x),$$

if $k < k'$, for every value of x.

If n have the values in the sequence 1, 2, 3, ..., and k have the values in any monotone increasing sequence of positive numbers, with no upper limit, the values of $\int_{(E)} s_n^{(k)}(x)\,dx$ form a double sequence, and a monotone one. From a well known property of such sequences, we infer that the two repeated limits

$$\lim_{k=\infty} \lim_{n=\infty} \int_{(E)} s_n^{(k)}(x)\,dx, \qquad \lim_{n=\infty} \lim_{k=\infty} \int_{(E)} s_n^{(k)}(x)\,dx,$$

are such that, if either of them is finite, the other one is also finite, and the two have the same value. It will be seen that this fact is sufficient to establish the theorem.

Since $s_n(x)$ is limited,

$$\lim_{k=} \int_{(E)} s_n^{(k)}(x)\,dx = \int_{(E)} s_n(x)\,dx;$$

and therefore $$\lim_{n=\infty} \lim_{k=\infty} \int_{(E)} s_n^{(k)}(x)\,dx = \lim_{n=\infty} \int_{(E)} s_n(x)\,dx,$$

provided that one of these limits is assumed to exist.

Also, by a known theorem,

$$\lim_{n=\infty} \int_{(E)} s_n^{(k)}(x)\,dx = \int_{(E)} s^{(k)}(x)\,dx,$$

since $s_n^{(k)}(x)$ is, for all values of n and x, not greater than k. Therefore

$$\lim_{k=\infty} \lim_{n=\infty} \int_{(E)} s_n^{(k)}(x)\,dx = \lim_{k=\infty} \int_{(E)} s^{(k)}(x)\,dx,$$

provided one of these limits exists.

By a known theorem, the function $s^{(k)}(x) \equiv \lim_{n=\infty} s_n^{(k)}(x)$ is summable, and also $s(x) = \lim_{k=\infty} s^{(k)}(x)$ is summable. In case $\lim_{k=\infty} \int_{(E)} s^{(k)}(x)\,dx$ is finite, its value defines $\int_{(E)} s(x)\,dx$.

First, let it be assumed that $\lim_{n=\infty} \int_{(E)} s_n(x)\,dx$ exists. If e_k be that set of points for which $s^{(k)}(x) = k$, we have

$$km(e_k) < \int_{(E)} s^{(k)}(x)\,dx < \lim_{n=\infty} \int_{(E)} s_n(x)\,dx.$$

The set of points at which $s(x) = \infty$ is contained in all the sets e_k, hence its measure is

$$\leqslant m(e_k) < \frac{1}{k} \lim_{n=\infty} \int_{(E)} s_n(x)\,dx\,;$$

hence, since k is arbitrarily great, the measure of the set of points of divergence of the sequence $\{s_n(x)\}$ is zero. Also

$$\int_{(E)} s(x)\,dx = \lim_{k=\infty} \int_{(E)} s^{(k)}(x) = \lim_{n=\infty} \int_{(E)} s_n(x)\,dx.$$

Next, if $\int_{(E)} s(x)\,dx$, or $\lim_{k=\infty} \lim_{n=\infty} \int_{(E)} s_n^{(k)}(x)\,dx$ is assumed to be finite, it is seen, from what has been proved above, that $\lim_{n=\infty} \int_{(E)} s_n(x)\,dx$ exists, and has the same value.

6. In order to extend the result of § 4 to the case of an unlimited function, it will be sufficient to consider the case in which $f(x, y)$ is never negative, since any function may be treated as the difference of two such functions.

Let then $f(x, y)$ be a non-negative function defined for the set E. Let $\{k_n\}$ be a sequence of positive numbers without upper limit, and let

$$f_n(x, y) = f(x, y),$$

for values of (x, y), such that $f(x, y) \leqslant k_n$, and let

$$f_n(x, y) = k_n,$$

for values of (x, y), such that $f(x, y) > k_n$.

We have then, from § 4,

$$\int_{(E)} f_n(x, y)(dx\,dy) = \int dx \int f_n(x, y)\,dy.$$

If, now, $\int_{(E)} f(x, y)(dx\,dy)$ exists, it is defined as

$$\lim_{n=\infty} \int_{(E)} f_n(x, y)(dx\,dy),$$

which is equal to $$\lim_{n=\infty} \int dx \int f_n(x, y)\,dy.$$

Denoting $\int f_n(x, y)\,dy$ by $s_n(x)$, we see that $\{s_n(x)\}$ is a monotone sequence, and on the assumption that $\int_{(E)} f(x, y)(dx\,dy)$ exists, we conclude that $\lim_{n=\infty} \int s_n(x)\,dx$ exists, and has the same value as the integral.

Applying the theorem of § 5, we see that $\{s_n(x)\}$ converges for all values of x, except those of a set of zero measure, and also that

$$\int s(x)\,dx = \int_{(E)} f(x, y)(dx\,dy),$$

where $$s(x) = \lim_{n=\infty} s_n(x) = \lim_{n=\infty} \int f_n(x, y)\,dy.$$

Applying again the theorem of § 5, we see that for each value of x for which $\lim_{n=\infty} \int f_n(x, y)\,dy$ is finite,

$$\int f(x, y)\,dy = \lim_{n=\infty} \int f_n(x, y)\,dy.$$

Therefore $\int dx \int f(x, y)\,dy$ exists, and is equal to $\int_{(E)} f(x, y)(dx\,dy)$.

The complete result obtained here and in § 4, may be stated as follows:—

If $f(x, y)$ be any function, limited or unlimited, defined for a measurable and bounded set E, and such that the Lebesgue integral $\int_{(E)} f(x, y)(dx\,dy)$ exists, then the repeated integrals $\int dx \int f(x, y)\,dy$, $\int dy \int f(x, y)\,dx$ exist and have the same value as the Lebesgue integral.

If it be assumed that $\int dx \int |f(x, y)|\, dy$ exists, and that $f(x, y)$ is summable in the set E, it can be shewn that $\int_{(E)} f(x, y)(dx\, dy)$ exists, and is therefore equal to $\int dx \int f(x, y)\, dy$. To see this consider the function $|f(x, y)|$, and apply the last part of the theorem of § 5; thus, if $\int s(x)\, dx$ exists, it follows that

$$\lim_{n=\infty} \int s_n(x)\, dx \text{ exists and } = \int s(x)\, dx.$$

Also $$\int s_n(x)\, dx = \int dx \int |f_n(x, y)|\, dy = \int_{(E)} |f_n(x, y)|\, (dx\, dy);$$

therefore $\int_{(E)} |f(x, y)|\, (dx\, dy)$ which is defined as $\lim_{n=\infty} \int_{(E)} |f_n(x, y)|\, (dx\, dy)$ or $\lim_{n=\infty} \int s_n(x)\, dx$, exists, and is equal to $\int s(x)\, dx$ or $\int dx \int |f(x, y)|\, dy$. It then follows that $\int_{(E)} f(x, y)\, (dx\, dy)$ exists.

It has thus been shewn that :—

If $f(x, y)$ be any function, limited or unlimited, which is summable in the measurable and bounded set E, and if $\int dx \int |f(x, y)|\, dy$ has a definite value, $\int_{(E)} f(x, y)(dx\, dy)$ exists and is equal to $\int dx \int f(x, y)\, dy$, and also to $\int dy \int f(x, y)\, dx$.*

In the case in which $f(x, y)$ is limited in E, the theorem is an immediate corollary from the preceding theorem.

It thus appears that the only conceivable case in which the repeated integral $\int dx \int |f(x, y)|\, dy$ could exist as a repeated Lebesgue integral in a measurable and bounded domain E, without $\int_{(E)} f(x, y)(dx\, dy)$ also existing and being equal to the corresponding repeated integrals, is when $f(x, y)$ is not summable in the two-dimensional domain E. It is conceivable that the function may be summable in linear sections of E made by parallels to the y-axis, and that the integral in such a section may be integrable with respect to x. It has, in fact, not been shewn that a set of points e is

* It may happen that the repeated integrals of $f(x, y)$ exist, but that $\int dx \int |f(x, y)|\, dy$ does not exist. In that case the Lebesgue integral of $f(x, y)$ does not exist, and the two repeated integrals of $f(x, y)$ may have unequal values. An example is the function $(x^2-y^2)/(x^2+y^2)^2$, taken over the square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, for which the repeated integrals are unequal.

necessarily measurable in the plane if all its sections are linearly measurable, even if the linear measure is an integrable function.

7.* The theorems established in § 6 will now be applied to the investigations of the transformation of a Lebesgue integral in a plane domain when the variables are changed.

The function $f(x, y)$ being defined for the measurable and bounded set E, it will be assumed that $\int_{(E)} f(x, y)(dx\,dy)$ exists.

Let a new variable ξ be defined by means of a relation $\xi = f_1(x, y)$, where f_1 is continuous with respect to x for each value of y, and with respect to y for each value of x, throughout the rectangle bounded by $x = a$, $x = b$, $y = c$, $y = d$, which contains the set E. It will also be assumed that, for each value of y, the function $f_1(x, y)$ regarded as a function of x is monotone (say increasing with x), and has no lines of invariability.

It follows that x may be expressed in the form $F_1(\xi, y)$, where F_1 is a monotone function of ξ, for each value of y. This transformation defines a (1, 1) correspondence of the points in a portion of the plane of (x, y) with the points of a corresponding portion of the plane of (ξ, y).

It will further be assumed that the function f_1 is such that, for each constant value of y, the points at which one of the derivatives, say $D_x^+ f_1(x, y)$, of f_1 with respect to x, is zero or infinite, form a reducible set of points. The corresponding derivative $D_+ F_1(\xi, y)$ will then have the same property, for each value of y, since $D^+f . D_+ F_1 = 1$.

It has been shewn in my paper,† "On Change of the Variable in a Lebesgue Integral," that, when the conditions here stated are satisfied, a linearly measurable set of points on a line parallel to the x-axis in either of the planes (x, y), (ξ, y), corresponds to a linearly measurable set of points on the same line in the other plane. If e_y, e'_y are these corresponding sets, for a fixed value of y, it has been there shewn that

$$m_l(e_y) = \int_{(e'_y)} D_{+\xi} F_1(\xi, y)\, d\xi,$$

and

$$m_l(e'_y) = \int_{(e_y)} D_x^+ f_1(x, y)\, dx.$$

It will now be shewn that, if G, G_1 are corresponding sets of points in the fundamental rectangle in the plane (x, y), and in the corresponding area on the plane (ξ, y), then, if G is measurable superficially, so also is G_1; and the converse.

* The remainder of the paper has been partially rewritten, May 4th, 1909, with a view to greater generality in the results.

† See *Proceedings*, *ante*, p. 10.

To any rectangle bounded by $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$ there corresponds in the plane of (ξ, y) an area bounded by $y = y_1$, $y = y_2$, and the two continuous curves $\xi = f_1(x_1, y)$, $\xi = f_1(x_2, y)$. This area is clearly measurable, being expressible as a Riemann integral taken over the rectangle.

From this it is clear that, if D_n denote an enumerable set of rectangles, each with its sides parallel to the axes, in the fundamental rectangle in the plane (x, y), there corresponds to D_n a measurable set of points Δ_n in the plane (ξ, y).

Assuming that G is a measurable set of points, there exists a sequence $\{D_n\}$ of sets of rectangles D_n, such that the set E_1 of points common to all the sets $\{D_n\}$ contains G, and is such that $m(E_1) = m(G)$. There also exists a set E_2 of points common to all the sets D'_n of a sequence $\{D'_n\}$ of sets of rectangles, such that E_2 is contained in G, and that

$$m(G) = m(E_2).$$

We may assume that D_n contains D'_n.

Let Δ_n, Δ'_n denote those measurable sets of points in the plane (ξ, y) which correspond to D_n, D'_n respectively. By a known principle in the theory of measure there exist measurable sets L_1, L_2 in the plane (ξ, y) which correspond to E_1, E_2 respectively; L_1 consists of those points common to all the sets Δ_n, and L_2 consists of those points common to all the sets Δ'_n. Moreover G_1 is contained in L_1, and contains L_2.

We have $$m(\Delta_n) = \int m_l \Delta_n(y)\, dy = \int dy \int D_x^+ f_1(x, y)\, dx,$$

where this repeated integral is taken over the set of points D_n.

It can be shewn* that the function $D_x^+ f_1(x, y)$ is superficially summable. It now follows, by employing the theorem of § 6, that

$$m(\Delta_n) = \int_{D_n} D_x^+ f_1(x, y)(dx\, dy).$$

Using the corresponding expression for $m(\Delta'_n)$, we have

$$m(\Delta_n) - m(\Delta'_n) = \int_{D_n - D'_n} D_x^+ f_1(x, y)(dx\, dy).$$

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* See *Theory of Functions of a Real Variable*, p. 552, where it is shewn that a derivative $D\phi(x)$ of a function of a single variable is linearly summable. The proof that $D_x \phi(x, y)$ is superficially summable is obtained from the proof there given, by making only slight modifications. The expression "measurable function" is there used as synonymous with "summable function."

It follows, since $m(D_n - D'_n)$ converges to zero as n is indefinitely increased, that $m(\Delta_n) - m(\Delta'_n)$ also converges to zero; and therefore that

$$m(L_1) = m(L_2).$$

Since G_1 is contained in L_1, and contains L_2, it follows that G_1 is measurable.

In a precisely similar manner it can be shewn that, if G_1 is assumed to be measurable, then G is also measurable.

The theorem will still hold if there be a finite number of values of y, for each of which either $D_{+\xi}F_1(\xi, y)$ is zero or infinite at points of an irreducible set or for which no (1, 1) correspondence between x and ξ is defined, or for which $F_1(\xi, y)$ is not everywhere continuous with respect to y. More generally, the theorem will still hold if there be an infinite number of such exceptional values of y, provided the linear measure of the set of such values of y is zero. For the set of such values of y may be disregarded in the Lebesgue integrals employed in the above proof. It is clear that the rectangles of the sets D_n may be so chosen that those of their sides which are parallel to the x-axis do not correspond to values of y which belong to the exceptional set. The set of points on the plane (ξ, y) which corresponds to such a rectangle is still measurable; its area being expressible by a Riemann integral, as the functions $f_1(x_1, y)$, $f_1(x_2, y)$ are continuous functions of y, except possibly for a set of values of linear measure zero.

The following theorem has been established :—

If the fundamental rectangle in the plane of (x, y) *be transformed into the plane* (ξ, y) *by a transformation* $\xi = f_1(x, y)$ *which satisfies the conditions* (1) *that the limited function* $f_1(x, y)$ *is, for each value of* y *not belonging to an exceptional set* K *of linear measure zero, continuous and monotone with respect to* x, *and without lines of invariability;* (2) *that* $f_1(x, y)$ *is everywhere continuous with respect to* y, *except that for values of* y *belonging to* K *there may be values of* x *for which this is not the case;* (3) *that one of the four derivatives* $D_x f_1(x, y)$ *is such that for each value of* y *not belonging to* K, *the set of values of* x *at which it is zero or infinite for an irreducible set, then a measurable set of points on either plane corresponds to a measurable set on the other plane.*

It will be remarked that the set K of values of y is such that for any such value, either no (1, 1) correspondence of ξ and x is defined, or the derivative $D_x f_1(x, y)$ is zero or infinite for an irreducible set of values of x, or $f_1(x, y)$ is not for every value of x continuous with respect to y.

8. Since a measurable set in one of the planes (x, y), (ξ, y) corresponds to a measurable set in the other plane, we see that, $f(x, y)$ being summable in the set E, the same function $f\{F_1(\xi, y), y\}$ is summable in the set E_1 corresponding to E. It follows, since $D_{+\xi}F_1(\xi, y)$ is a summable function, that the product $f\{F_1(\xi, y), y\} D_{+\xi}F_1(\xi, y)$ is a summable function.

We have then

$$\int_{(E)} f(x, y)(dx\,dy) = \int dy \int f(x, y)\,dx = \int dy \int f\{F_1(\xi, y), y\} D_{+\xi}F_1(\xi, y)\,d\xi;$$

and employing the second theorem of § 6, we see that

$$\int_{(E_1)} f\{F_1(\xi, y), y\} D_{+\xi}F_1(\xi, y)(d\xi\,dy)$$

exists, and is equal to $\int_{(E)} f(x, y)(dx\,dy)$.

Now, let a second transformation from the plane (ξ, y) to the plane (ξ, η) be made by means of a relation $y = F_2(\xi, \eta)$, where the function $F_2(\xi, \eta)$ satisfies corresponding conditions to those satisfied by the former function $F_1(\xi, y)$.

Let E' be the set of points in the plane (ξ, η) which corresponds to E_1 in the plane (ξ, y), or to E in the plane (x, y); we see then, by a repetition of the reasoning applied in the case of the first transformation, that

$$\int_{(E_1)} f\{F_1(\xi, y), y\} D_{\xi}F_1(\xi, y)(d\xi\,dy)$$

is equal to $\int_{(E')} \phi(\xi, \eta) D_{\xi}F_1(\xi, y) D_{\eta}F_2(\xi, \eta)(d\xi\,d\eta)$,

where $\phi(\xi, \eta)$ is what the function $f(x, y)$ becomes after the two transformations. It follows, then, that

$$\int_{(E)} f(x, y)(dx\,dy) = \int_{(E')} \phi(\xi, \eta) D_{\xi}F_1(\xi, y) D_{\eta}F_2(\xi, \eta)(d\xi\,d\eta),$$

the value of $D_{\xi}F_1(\xi, y)$ being expressed in terms of (ξ, η).

In case the monotone functions are decreasing instead of increasing functions, the negative derivatives must be replaced by their absolute values, in order that the measures of all sets of points may be reckoned positive.

The relation between a point (x, y) in the fundamental rectangle in the plane of (x, y) and the corresponding point (ξ, η) in a bounded area

in the plane of (ξ, η) may be expressed in the form

$$x = \psi_1(\xi, \eta), \quad y = \psi_2(\xi, \eta),$$

equivalent to the relations

$$x = F_1(\xi, y), \quad y = F_2(\xi, \eta) \equiv \psi_2(\xi, \eta).$$

If the function $F_1(\xi, y)$ is monotone for each value of y, and has no lines of invariability, and if further the function $\psi_2(\xi, \eta)$ is monotone for each value of ξ, and has no lines of invariability, then the relations

$$x = \psi_1(\xi, \eta), \quad y = \psi_2(\xi, \eta)$$

define a (1, 1) correspondence between the points (x, y) and (ξ, η). The converse statement that, if

$$x = \psi_1(\xi, \eta), \quad y = \psi_2(\xi, \eta)$$

define a (1, 1) correspondence between points in a fundamental rectangle in the plane (x, y), and a bounded area in the plane (ξ, η), then

$$x = F_1(\xi, y), \quad y = F_2(\xi, \eta),$$

where F_1, F_2 are monotone functions of ξ, η respectively, does not hold good.

In order to illustrate possible cases which may arise, let us consider as an example the transformation $x = \xi \cos \eta$, $y = \xi \sin \eta$, which represents in accordance with the method here adopted the ordinary polar coordinate transformation $x = r \cos \theta$, $y = r \sin \theta$.

The fundamental rectangle in the plane (x, y) contains, in general, portions which lie in all of the four quadrants into which the plane is divided by the two axes. We may assume ξ to be restricted to have no negative values, and η to have values such that $0 \leqslant \eta \leqslant 2\pi$. The function $\xi \sin \eta$ is not a monotone function of η for each value of ξ; we may, however, consider separately the four portions into which the fundamental rectangle in the plane of (x, y) is divided by the axes of x and y. These portions correspond to values of η in the four intervals $(0, \frac{1}{2}\pi)$, $(\frac{1}{2}\pi, \pi)$, $(\pi, \frac{3}{2}\pi)$, $(\frac{3}{2}\pi, 2\pi)$. Let us consider the portion in one quadrant, say the first. The successive transformations are given by

$$x = (\xi^2 - y^2)^{\frac{1}{2}}, \quad y = \xi \sin \eta;$$

the first of these functions $(\xi^2 - y^2)^{\frac{1}{2}}$ is monotone for each value of y, and the second is monotone for each value of $\xi (> 0)$. When, however, $\xi = 0$, the relation $y = \xi \sin \eta$ fails to define a unique relation between y and η. As has been shewn, this failure for the single value $\xi = 0$ will not affect the application of the method of transformation to that part of

the fundamental rectangle which is in the first quadrant. A similar argument applies to the portions in the other three quadrants, and thus the method of transformation can be extended to the whole rectangle.

Returning to the general case, a transformation

$$x = \psi_1(\xi, \eta), \qquad y = \psi_2(\xi, \eta)$$

applied to the fundamental rectangle in the plane of (x, y) will be designated a *normal transformation*, when it is such that the fundamental rectangle can be divided into a number of connex parts which correspond with the same number of connex parts of that area in the plane of (ξ, η) which corresponds to the fundamental rectangle, in such a manner that in each one of these parts the functions

$$x = F_1(\xi, y), \qquad y = F_2(\xi, \eta) \equiv \psi_2(\xi, \eta),$$

both satisfy the conditions of the theorem of § 7. Thus, in each portion $F_1(\xi, y)$ must be continuous and monotone with respect to ξ, and have a derivative $D_\xi F_1(\xi, y)$ which is zero or infinite only at a reducible set of points ξ, for each value of y; and it must further be continuous with respect to y; except that there may be an exceptional set of values of y, of linear measure zero, for which one or more of these conditions is not satisfied, or for which a (1, 1) relation between x and ξ is not defined. The function $y = F_2(\xi, \eta) \equiv \psi_2(\xi, \eta)$ must satisfy a similar set of conditions, there being possibly in this case an exceptional set of values of ξ, of measure zero.

Employing this definition of a normal transformation, it has now been established that—

If $x = \psi_1(\xi, \eta)$, $y = \psi_2(\xi, \eta)$ define a normal transformation of the fundamental rectangle in the plane of (x, y) to the plane of (ξ, η), then if either of the integrals

$$\int_{(E)} f(x, y)(dx\,dy), \qquad \int_{(E')} \phi(\xi, \eta)\,|\,D_\xi(x)\,D_\eta(y)\,|\,(d\xi\,d\eta)$$

exists as a Lebesgue integral, the other integral also exists, and the two have the same value. The functions $\phi(\xi, \eta)$, $f(x, y)$ have equal values at corresponding points.

If we employ Lebesgue's theorem, that every monotone function has a finite differential coefficient except at points of a set with linear measure zero, we see that in the second integral $|\,D_\xi(x)\,D_\eta(y)\,|$ may be replaced by

$$\left|\,\frac{\partial}{\partial\xi} F_1(\xi, y), \quad \frac{\partial}{\partial\eta}\,\psi_2(\xi, \eta)\,\right|.$$

9. If, at a point (ξ, η), the partial differential coefficients $\partial\psi_1/\partial\xi$, $\partial\psi_1/\partial\eta$, $\partial\psi_2/\partial\xi$, $\partial\psi_2/\partial\eta$ all exist, and are finite, then the relations

$$\delta x = \left(\frac{\partial\psi_1}{\partial\xi} + \theta_1\right)\delta\xi + \left(\frac{\partial\psi_1}{\partial\eta} + \theta_2\right)\delta\eta,$$

$$\delta y = \left(\frac{\partial\psi_2}{\partial\xi} + \theta_3\right)\delta\xi + \left(\frac{\partial\psi_2}{\partial\eta} + \theta_4\right)\delta\eta,$$

between increments δx, δy, $\delta\xi$, $\delta\eta$ of x, y, ξ, η hold, where θ_1, θ_2, θ_3, θ_4 converge to zero with $\delta\xi$, $\delta\eta$; provided a sufficient condition is satisfied. One such sufficient condition is that one at least of the two differential coefficients $\partial\psi_1/\partial\xi$, $\partial\psi_1/\partial\eta$, and also one at least of the two $\partial\psi_2/\partial\xi$, $\partial\psi_2/\partial\eta$ have definite values in a two-dimensional neighbourhood of the point (ξ, η) and are continuous at (ξ, η) with respect to the plane domain.

We find that, at a point (ξ, η) where these relations hold,

$$D_\xi(x) = J(x, y)\Big/\frac{\partial\psi_2}{\partial\eta}, \qquad D_\eta(y) = \frac{\partial\psi_2}{\partial\eta};$$

and thus the product $D_\xi(x)\, D_y(\eta)$ which occurs in the theorem of § 8, is equal to $J(x, y)$ the Jacobian of x, y with respect to ξ, η.

Assuming, as before, that a normal correspondence of the points of a fundamental rectangle in the plane (x, y) with the points of a bounded area in the plane of (ξ, η) are defined by means of the relations

$$x = \psi_1(\xi, \eta), \qquad y = \psi_2(\xi, \eta),$$

let it now be assumed that the four partial differential coefficients of ψ_1, ψ_2 have definite finite values everywhere in the whole area, and vanish only at points of a reducible set. Let it be further assumed that one at least of the two $\partial\psi_1/\partial\xi$, $\partial\psi_1/\partial\eta$ is continuous throughout the area, and that one at least of the two $\partial\psi_2/\partial\xi$, $\partial\psi_2/\partial\eta$ has the same property. Lastly, let it be assumed that the Jacobian $J(x, y)$ either nowhere vanishes, or vanishes only at the points of a reducible set. The conditions of the theorem of § 8 are then satisfied, and at every point we have

$$D_\xi(x)\, D_\eta(y) = J(x, y).$$

We thus obtain the following theorem:—

Let a normal correspondence be defined between the points of a fundamental rectangle in the plane of (x, y), *and the points of a corresponding bounded domain in the plane of* (ξ, η) *by means of relations*

$$x = \psi_1(\xi, \eta), \qquad y = \psi_2(\xi, \eta).$$

Let (1) *the four partial differential coefficients of* ψ_1, ψ_2 *have definite*

finite values everywhere in the domain of (ξ, η), *and vanish at most only at the points of a reducible set of points;* (2) *let one at least of the two partial differential coefficients of* ψ_1 *be continuous throughout the domain, and also one at least of the two partial differential coefficients of* ψ_2; *and* (3) *let the Jacobian of* x, y *with respect to* (ξ, η) *vanish at most at the points of a reducible set. Let* $f(x, y)$ *be a limited or unlimited function defined for a set* E *contained in the fundamental rectangle in the plane of* (x, y), *and* $\phi(\xi, \eta)$ *be the same function expressed in terms of* (ξ, η), *and thus defined for the set* E' *of points in* (ξ, η) *corresponding to* E *in* (x, y).

Then, if one of the two integrals

$$\int_{(E)} f(x, y)(dx\,dy), \qquad \int_{(E')} \phi(\xi, \eta)\,|\,J(\psi_1, \psi_2)\,|\,(d\xi\,d\eta)$$

exists as a Lebesgue integral, the other one also exists, and the two have the same value.

The theorem can be extended* to cases in which some of the conditions of validity of the theorem are violated at all or some of the points of a closed set L' of zero measure in the plane of (ξ, η), provided the corresponding closed set L in the plane of (x, y) has also zero measure. For example, at points of L', some or all of the partial differential coefficients of ψ_1, ψ_2 may not exist, or all may be discontinuous, or the Jacobian may vanish or be infinite.

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* See *Theory of Functions of a Real Variable*, pp. 449–451. The proof there given can be applied to make the extension here indicated.

ON INDETERMINATE FORMS

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1. The subject of indeterminate forms has frequently engaged the attention of mathematicians. Besides earlier papers, there are several memoirs of Stolz* which treat the matter at length from a more or less modern standpoint. Comparatively recently Osgood† has subjected the proofs of the usual rules to a careful scrutiny. It might therefore be supposed that the subject was now exhausted. The fact is, however, that the whole theory of limits is one which we have only just begun to treat with ease and security, thanks to the extension of knowledge as to possible modes of variation with which the theory of sets of points has supplied us.

Many of the earlier writers must have had a somewhat vague notion of what they meant when they spoke of the existence or non-existence of a limit, for otherwise the latter term would scarcely have been as fruitful of difficulties as it has proved. And even at the present day the prevalence, or rather the almost exclusive adoption, in the more subtle processes of analysis of what are known as ϵ-proofs points to a lack of familiarity with the theory of sets of points.

In the present paper ϵ-methods are entirely avoided, and the expression "non-existence of a limit" never occurs. It has always seemed to me that it is worth while to spend time and labour in getting rid of the ϵ-machinery, and I think I may say that I have almost invariably found the proof thus modified to be both shorter and more convincing.

The exclusion of the term "non-existence of a limit" naturally leads to the investigation of the plurality of limits, and at once shortens the enunciations, and, by removing a restriction, increases the generality of our theorems. Instead of saying that "If the limit of (2) exists, then the limit of (1) exists," we have the theorem that "The limits of (1) coincide

* *Math. Ann.*, Vols. xiv, xv, and xxxiii.

† *Ann. of Math.*, Vol. xii (1897).

with or are among or lie between, as the case may be, the limits of (2)," whence it, of course, follows that, if (2) has *an unique* limit, so has (1).

The result of the investigation of the subject from the point of view just explained has been, on the one hand, to shew that one or two of the older proofs have been somewhat too hastily abandoned as not rigid, when a slight change of wording removes all objections, and, on the other, to shew that one of the modern modes of dealing with the form ∞/∞ has not been put to all the uses of which it is capable. Modified in a suitable manner, it deals simultaneously with both of the two fundamental forms $0/0$ and ∞/∞, and that whether the limiting point considered is at a finite distance or at infinity, and whether the limits in question are finite or infinite. Further, it has been found that certain unnecessary restrictions, of a kind other than those above alluded to, have been introduced into the statements of theorems, and occasion has been taken to bring out clearly the fact that there are really two *alternative* conditions for the applicability of the usual rules.

These rules in the generalised form here given to them, taking the fraction to be $f(x)/F(x)$, and $x = a$ to be the point at which this fraction becomes indeterminate, are as follows:—

(A)—If the point a be not a limiting point either of common zeros or common infinities of $f'(x)$ and $F'(x)$, then the limits of $f(x)/F(x)$ are limits of $f'(x)/F'(x)$, and, in determining the latter, we may omit from consideration those points x at which the denominator vanishes.

(B)—If $F(x)$ is monotone and the point a be not a limiting point of common infinities of $f'(x)$ and $F'(x)$, then the limits of $f(x)/F(x)$ lie between the upper and lower limits of $f'(x)/F'(x)$, provided we include among the latter the infinite limit or limits with determinate sign which correspond to points, if any, at which $F'(x)$ is zero and $f'(x)$ is not zero.

If in the statement of the theorems we expressly assume that $f'(x)/F'(x)$ has an unique limit, the difference between these two enunciations becomes less pronounced, though they still do not coincide. It is then remarkable that $F(x)$ is necessarily monotone in the first case as well as in the second, this following from the conditions of (A) as stated, viz., from the fact that the point a is not a limiting point of common zeros.* But we cannot *replace* this condition by the monotony of $F(x)$.

In the very painstaking account Stolz has given of his own papers in his *Grundzüge*, he always requires that $F(x)$ should be monotone; but the sets of conditions that he gives are inferior to those given above, both in containing redundant elements and in being unnecessarily

* See below, § 28.

restricted. Thus he has not only failed to notice that the monotony of $F(x)$ follows on the hypothesis of an unique limit from the condition that the point a should not be a limiting point of common zeros, but throughout his investigations he explicitly assumes that $F'(x)$ is never infinite; and in that one of his theorems that corresponds to (B), that $f'(x)$ is also finite.*

As regards condition (B), which is characterised by the substitution of the monotoneness of $F(x)$ for the absence of common zeros of $f'(x)$ and $F'(x)$, this differs from the corresponding condition of Stolz in that he especially requires not only that the zeros of the latter should be zeros of the former, but also that $f'(x)$ and $F'(x)$ should both be finite—a condition which must be added if we wish to omit the points at which F' is zero.

After a short introduction on the theory of limits, in which the ideas and language of the theory of sets of points are necessarily freely employed, and which is required for the proper understanding of the sequel, the paper falls into three main divisions. In the first two the results of the hypothesis (A) are examined; in Part I by the use of a modernised form of the older mode of discussion, in Part II by means of a modification of the later machinery devised to deal with the form ∞ / ∞. In the third portion of the paper, the results of the hypothesis (B) are examined, the method adopted being, I believe, new, at least in the form given to it. One or two new results are obtained, *en passant*, and the paper terminates with some theorems which do not naturally belong to any one of the main divisions, and include one or two extensions to indeterminate forms involving more than one variable, as well as to the case when $f(x)$ and $F(x)$ are not in general differentiable.

It will be seen that the subject has been dealt with in some detail. Apart from its interest from a didactic point of view as constituting an important chapter in the differential calculus, this has seemed to me worth while, because the removal of a restriction in the case of an analytical result may, besides pleasing the mathematical sense, sometimes prove of real service. It corresponds to a sharpening of the analytical tool which does not in any way impair its strength. On the other hand, the subject appears to form such a good introduction to and exercise on the theory of limits, that few who have mastered it are likely to encounter serious difficulties in their subsequent use of processes involving limits.

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* Stolz has also evidently failed to realise that, if $F'(x)$ changes sign, *i.e.*, has in an interval two different signs, it necessarily has the value zero at some point of that interval. Compare the different enunciations of §§ 10 and 12, pp. 72 and 77, of the *Grundzüge der Differential- und Integralrechnung*. Note also that on p. 82 he refers to the case when $f'(x)/F'(x)$ has a plurality of limits.

Introduction on Limits.—§§ 2–7.

2. Any set of numbers determines the least number which is not less than any of them, and the greatest number which is not greater than any of them; the former of these is called *the upper bound* and the latter *the lower bound* of the set of numbers.

If the set of numbers consists of the values of a function y of a single variable x, or of any number of variables denoted in their ensemble by x, defined for the values of x belonging to some set S, and P be any limiting point of this set, we may take any interval or region containing P as internal point, and determine the upper and lower bounds of the function for values of x belonging to the set S in this region. If we take a second region inside the first, the upper bound will not increase and the lower bound will not diminish. Hence, taking any sequence of regions having P as sole common internal point, we get a monotone decreasing sequence of upper bounds U_1, U_2, ..., whose lower bound U is therefore unaltered if we omit any finite number of its constituents or if we interpolate the upper bound taken with respect to any intermediate region. Now, if we take any other such sequence of regions, there will certainly be one of them entirely internal to the first region of the first sequence, and we can then omit from the first sequence all the succeeding regions which are not internal to the first region of the second sequence. Proceeding thus we can select the two sequences of regions so that each region of the second sequence lies inside the corresponding region of the first sequence and contains the next region of the first sequence, without altering the lower bounds of the two sequences. Interpolating the one sequence in the other, we then do not alter the lower bound of either. Hence the two sequences had the same lower bound U.

Thus this lower bound U of the upper bounds is independent of the particular sequence of regions chosen, and depends only on the point P; it is called *the upper limit of the function y at the point P with respect to the set S.* Similarly, the upper bound of the lower bounds is called *the lower limit of the function y at the point P with respect to the set S*, and is independent of the particular sequence of regions chosen to determine it.

If the upper and lower limits coincide, the function is said to have their common value as unique limit at the point P with respect to the set S.

3. The concepts of upper and lower bound, and of upper and lower limit and unique limit at a point require no acquaintance with the theory

of sets of points. In the present paper we require, however, a further notion, that of limits at a point intermediate between the upper and lower limits. To define these clearly and concisely it is scarcely possible to avoid the language of the theory of sets of points. The notions involved are independent of the number of independent variables x or of the kind of set S may be. For simplicity, therefore, we shall take a single independent variable x, and represent both it and the dependent variable y in the usual manner by points on the coordinate axes; moreover, we shall take the set S to be initially an interval* (P, Q) open at its left-hand end, viz., at P.

As the function y is supposed to be any whatever, the mode of distribution of the set of points G_Q on the y-axis which represent the values of the function is any whatever. Certain of these points may be repeated an infinite number of times, that is, the corresponding value of the function may occur at an infinite number of values of x. If this be the case, we add all such points to the first derived set† of G_Q. The set H_Q so formed is necessarily a closed set, since the addition of points of a set to its first derived set does not introduce any new limiting points.

As Q is conceived to approach P along any sequence of points $Q_1, Q_2, \ldots$ (Fig. 1), the successive closed sets H_Q so obtained lie each inside all preceding sets, and therefore, by Cantor's *Theorem of Deduction*,‡ determine a closed set H common to all of them, which, by a similar argument to that used in discussing the upper limit at a point, is easily shewn to be the same for all modes of approach of Q to P.

The set of values of y corresponding to the points of this deduced set H are said to constitute *the limits of the function y for all modes of approach to P on the right.* It is easily verified that the upper and lower bounds of this set of limits are the upper and lower limits of the function at P with respect to the continuum on the right.

A similar definition interchanging left and right applies to the limits on the left at P.

It is clear from the above that the same argument would apply if, instead of considering all values of x in the half-open interval (P, Q), we omitted all values of y except those which corresponded to values of x belonging to some other chosen set S, having P for one of its limiting

* An interval which includes its end-points is called *a closed interval* or *a closed segment;* one from which one of the end-points is omitted, *a half-open;* and one from which both end-points are omitted, *a completely open interval* or *segment.*

† That is, the set consisting of all the limiting points of the set itself, viz., here of G_Q. A derived set is a closed set, that is, contains all its limiting points. See Young's *Theory of Sets of Points*, Cambridge Univ. Press, 1906, Chap. iii.

‡ *Loc. cit., supra*, p. 26.

points. The corresponding set H is now said to constitute *the limits of the function y at P for modes of approach to P along the set S on the right or left as the case may be.* If P is a limiting point of S on the left only, the limits at P for modes of approach along the set S will be limits on the right only, and, if P is a limiting point of S on the right only, the limits at P will be on the left only.

It is sometimes convenient to omit considerations of left and right; we then work with intervals containing P as internal point, instead of intervals with P as end-point. This is, in particular, the case if there is more than one independent variable x.

4. A great variety of cases may occur. That in which the set H consists of a single point has already been mentioned; in this case we say that the function has an unique limit at P along the set considered. The set H may, however, be any finite set, or any countably infinite closed set, or may have the potency c and be perfect, or the sum of a perfect and a countable set.*

So far we have supposed the function y to be any whatever. Next, suppose it to be continuous in a certain interval PQ open at P. Then it follows from the fact that y assumes all values between its upper and lower bounds (inclusive) that the set G_Q is a closed segment (which reduces to a point in the trivial case when y is constant) and that H_Q is identical with G_Q. Hence the set H is either a single point or a closed segment.

The property of a continuous function used in this proof is not confined to continuous functions. Other functions assume all values between their upper and lower bounds (inclusive). Moreover, it is sufficient for the purposes of the proof, if the function assumes only the intermediate values without actually assuming its upper and lower bounds as values; for then G_Q is an open segment and H_Q the segment got by closing it. The most important class of functions possessing this latter property is that constituted by all functions which are differential coefficients of other functions. It is especially with such functions, as well as with continuous functions, that we shall concern ourselves in the sequel.

5. We now proceed to shew that, *if L be any one of the limits of y at P, there is a sequence of values of y having L as unique limit and either as upper or lower bound, such that the corresponding values of x have P as unique limit.*

Let Y be the point of the closed set H which represents the limit L.

* *Theory of Sets of Points*, pp. 19, 20, 55, 56.

Take on the y-axis any sequence of points below Y, and, having Y as limiting point, $Z_1, Z_2, \ldots$ (Fig. 1).

On the x-axis we have, as before, the sequence $Q_1, Q_2, \ldots$ with P as limit, and we take the first interval PQ_i such that in it there is a value x_1 whose corresponding y_1 lies in YZ_1. We then take the first interval Q_iQ_j, where $j > i$, such that in it there is a value x_2 whose corresponding y lies in the first interval YZ_m which does not contain y_1. We continue this process, then, as before.

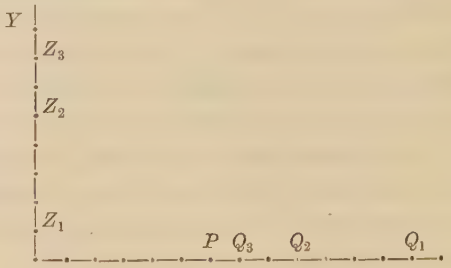

FIG. 1.

If at any stage this process comes to an end owing to there being no values of x within a certain distance of P whose correspondents lie below Y and sufficiently near to it, we perform the same process above Y, and it is bound to be endless, since Y belongs to every set H_Q and is therefore a limiting point of each of the sets G_Q.

We may therefore assume that, in our case, the process is endless when we work below Y. It is then clear that we get in $y_1, y_2, \ldots$ a sequence of values of y having Y for limiting point and lying all below it, the corresponding points $x_1, x_2, \ldots$ having P as unique limit. Thus the value L is at once the unique limit and the upper bound of a sequence of values of y whose corresponding x's have P as unique limit, as was to be shewn.

6. When a function is defined by means of a limiting process, it is sometimes convenient to regard the values of the function in connection with the mode in which they are obtained as limits. It is, for instance, almost essential to distinguish between $+\infty$ and $-\infty$ according as ∞ is approached by means of positive or by means of negative numbers. It should be noticed, however, that it is often equally convenient, if not equally necessary, to think of finite numbers, including zero, under these different aspects, to attribute to them, in fact, a *sense*. We shall denote by L_+, L_-, $L_\mp$ a limit, whose magnitude is L, and which is approached in the first case from above, in the second case from below, and in the third case indifferently from above and below.

This notation has the inconvenience that it makes

$$+\infty = \infty_-, \qquad -\infty = \infty_+.$$

On the other hand, it is more consonant with our notions to write 0_+ rather than 0_- when all the numbers with zero as limit are positive.

7. Thus each limit L will have a definite sense or both senses, except in the single case when it occurs *only* through the repetition of the value L, which must therefore be an *isolated* value of the function.

Now, in the case of a continuous function,* or any other function which assumes in every interval every value between its upper and lower bounds, an *isolated* value can only occur when it is the *only* value, that is, when the function is constant. Thus, unless such a function, whose limits at a point a, say on the right, are considered, is constant throughout some interval $(a < x)$, the limit L may be considered to be either L_+ or L_- or $L_\mp$. The case of $L_\mp$ may, moreover, if we please, by what has been shewn, be considered as the ensemble of the two limits L_+ and L_-, dividing up any sequence giving the limit $L_\mp$ into two sequences, one giving L_+ and the other L_-.

In other words, *the limits at a point a of a function f which is not constant in the neighbourhood of a and assumes all values between its upper and lower bounds in every interval, are precisely the limits of f along the various possible sequences on the right and on the left of a respectively, with a as limiting point, and such that along such a sequence $x_1, x_2, \ldots$, $f(x)$ has an unique limit L, and is always either $> L$ or always $< L$.*

The precise meaning to be attached to the term "limit," including its sense, has now been explained. In the present paper we shall, for convenience, usually consider only limits on the right, writing

$$\operatorname*{Lt}_{h=0} f(a+h)$$

for the unique limit, and

$$\operatorname*{Llt}_{h=0} f(a+h)$$

for a plurality of limits, at the point a on the right, h being positive. When we state that two functions have a common limit or that a limit of one function is a limit of a second function, we shall usually mean that that limit is approached in the same manner, from above or from below, by both functions.

PART I.—§§ 8–14.

8. We now proceed to the main subject of the paper. We shall require two lemmas.

* Whether finite or not, and whether the two infinities are distinguished from one another or not; in the latter case we cannot, of course, speak properly of upper and lower bounds, but in any interval the function assumes all values "between" the values at the extremities, that is, all values passing one way or the other, by 0 or by ∞.

LEMMA 1.—*If throughout a completely open interval $f(x)$ is a continuous differentiable function which, as x recedes away beyond any point of it on either side along a properly chosen sequence, has the unique limit zero, then, at some point of the completely open interval, $f'(x)$ has the value zero.**

[It is to be noticed that the completely open interval may be either a finite interval (a, b), not including the end-points a and b, or an infinite interval with one finite end-point, excluded from the interval, or the whole infinite straight line.

The word "differentiable" is to be understood to mean that the function has a differential coefficient at every point of the completely open interval, the value of the differential coefficient being finite or infinite with determinate sign.]

To prove the lemma, we remark that, if f has the value zero at every point, the lemma is obviously true. If not, there is a point Q at which f is not zero, so that we can assign a positive quantity e less than $|f(Q)|$. But, by the hypothesis, as x recedes away from Q on either side along a properly chosen sequence, $f(x)$ has the limit zero; hence we can find two points B and B', one on each side of Q in the completely open interval, such that

$$|f(B)| < e \quad \text{and} \quad |f(B')| < e.$$

In the closed interval (B, B') f is, by hypothesis continuous, and therefore $|f|$ assumes its upper bound. Since Q is internal to (B, B'), this upper bound is greater than e, and therefore the point P at which $|f|$ assumes its upper bound is neither B nor B'.

At P, $f(x)$ itself is either a maximum or a minimum, and therefore $f'(x)$ has there the value zero, which proves the lemma.

9. The conditions of Lemma 1 are, of course, satisfied if $f(x)$ exists and is continuous at the end-points of the interval; this special case, commonly called Rolle's theorem, is the basis of the proof of the fundamental property of a differential coefficient, that in any closed interval it assumes every value between its upper and lower bounds. From this we deduce, without any difficulty, a theorem which includes the above lemma as a special case, viz., that *at some point of a completely open interval*

* This theorem is included in that stated and proved in another manner in the paper "On Derivates," by W. H. Young and Grace Chisholm Young, in the *Quarterly Journal*, 1908, pp. 25, 26. The present proof serves as an alternative proof of the theorem there given, dropping the assumption that f is differentiable.

$f'(x)$ is zero, unless, however we recede beyond any point in both directions, we always get a unique limit on each side, and these limits are different from one another.

For, except in the excluded case, we are sure that the function is not monotone, so that $f'(x)$ has both positive and negative values, and therefore assumes the value zero.

10. From these remarks it follows that the lemma about to be stated and proved is only a special case of a much more general result. It is convenient, however, to state and prove it in the precise form in which we shall immediately require it.

We shall adopt the usual notation and denote by such expressions as

$$f(x+0), \quad F(y+0), \quad f(x-0), \quad F(y-0)$$

the limits, supposed to be unique or not as the case may require, of the the corresponding functions as we approach x or y from $+\infty$ and from $-\infty$ respectively. Here the points x and y may be either the proper ends of a finite interval or one or both may be improper points which close a completely open or half-open infinite interval.

LEMMA 2. — *If $f(x)$, $F(x)$ be functions which are continuous and differentiable throughout the completely open interval $(a < x < b)$, the differential coefficients having in the open interval no common zeros or infinities, then*

$$\frac{f(b-0)-f(a+0)}{F(b-0)-F(a+0)} = \frac{f'(x_1)}{F'(x_1)}$$

*for some internal point x_1 of the interval.**

CASE 1.—Let $f(b-0)-f(a+0)$ be not zero.

Put

$$g(x) = f(x)-f(a+0)-m\{F(x)-F(a+0)\},$$

where m denotes the left-hand side of the equality to be proved. Evidently

$$g(b-0) = g(a+0) = 0.$$

Moreover, since $f'(x)$ and $F'(x)$ are never infinite together, $g(x)$ is certainly differentiable, as well as continuous, throughout the open interval. Hence, by Lemma 1, for some value x_1 of the completely open interval

$$g'(x) = 0,$$

that is,

$$0 = f'(x_1)-mF'(x_1).$$

* The point x_1 is shewn in the proof to be neither a zero nor an infinity of f or of F if the numerator and denominator of the left-hand side of the preceding equation are both different from zero.

This equation would give us no information if f' and F' were known to have common zeros, as x_1 would presumably be one of these zeros. As such common zeros do not exist, it is evident from the equation that neither f' nor F' can be zero at the point $x = x_1$, and, as they have no common infinities, it is equally evident that their values when $x = x_1$ are not infinite. Dividing, then, the equation by $F'(x_1)$, we get the equality to be proved.

CASE 2.—Let $f(b-0)-f(a+0)$ be zero.

Then a point x_1 exists, by Lemma 1, at which $f'(x_1)$ is zero, and $F'(x_1)$, by our hypothesis, is not zero. Thus the theorem is still true.

11. In the preceding article we have not, in the statement of the lemma, expressly excluded the possibility of $F(b-0)-F(a+0)$ vanishing, because, with the ordinary symbolism, the fraction of which it is the denominator would not have any existence. It is perhaps worth while noting, however, that, if $f(b-0)-f(a+0)$ does not vanish, both sides of the equation to be proved assume the same form $c/0$, where c has in general a value necessarily different from zero and in general different on the two sides of the equation.

It is of importance also to remark that, *if* $f(b-0)-f(a+0)$ *and* $F(b-0)-F(a+0)$ *are both zero, then an internal point of the interval can be found such that*

$$\frac{f'(x)}{F'(x)} \tag{2}$$

has there any assigned value we please. In fact the argument of the previous article then applies if m, instead of the meaning there attached to it, is supposed to have the value in question.

It at once follows that, if a is a limiting point of common zeros of $f(x)$ and $F(x)$, the fraction (2) has all possible limits from $-\infty$ to $+\infty$ inclusive at a. It is remarkable, however, that the existence of this infinite set of common zeros so far deprives the fraction (1) of one of the fundamental properties of a continuous function that it no longer assumes all values between its upper and lower bounds. This is illustrated by the examples given at the beginning of Part III.

12. We now apply the preceding lemma to obtain a rule for the fundamental indeterminate form $0/0$; this rule is virtually contained in the following theorem:—

THEOREM 1.—*If* $f(x)$ *and* $F(x)$ *be continuous differentiable functions in an open interval* $(a < x)$, *where* a *may denote* $-\infty$, *and the differential*

coefficients f' and F' have no common zeros or infinities in the open interval, and if, as x approaches the value a, $f(x)$ and $F(x)$ have each the unique limit zero, then, as x approaches the value a, the limits of

$$\frac{f(x)}{F(x)} \tag{1}$$

are limits of

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

CASE 1.—Let a not be a limiting point of zeros of $F(x)$. We may then choose the neighbourhood so small that $F(x)$ be nowhere zero in the open interval. We then have, by the preceding lemma,

$$\frac{f(x)}{F(x)} = \frac{f'(y)}{F'(y)} \tag{3}$$

for some value of y in the completely open interval (a, x).

Now the limits of (1) are, as we have seen in the introduction, obtained by making x describe all possible sequences with a as limiting point along which (1) has an unique limit. Let x describe such a sequence, and let L be the corresponding limit of (1). Then, since the point y lies between a and x, y also describes a sequence with a as limit, and along this y-sequence (2) has, by virtue of the equation (3), the unique limit L.

This proves the theorem in Case 1.

The case when $F(x)$ is constant and zero throughout the interval requires no comment.

CASE 2.—Let a be a limiting point of common zeros of $f(x)$ and $F(x)$.

Then, as shewn in § 11, (2) has all possible limits from $-\infty$ to $+\infty$ inclusive, so that the theorem is certainly true.

CASE 3.—Let a be a limiting point of zeros of $F(x)$, but not of common zeros of $f(x)$ and $F(x)$.

The fraction (1) now has a definite finite value at each point except where $F(x)$ is zero, at which points it is infinite, and is at every point continuous on each side separately. Hence* (1) has no isolated values, so that, as pointed out in the introduction (§ 7), the limits of (1) can be all obtained from sequences of points along which the fraction (1) continually increases or continually decreases. Thus we certainly lose no limits by omitting, in determining the limits of (1), all points from the interval $(a < x)$ at which $F(x)$ vanishes. The argument used in Case 1, then, still applies, and hence the theorem holds in every case.

* See footnote to § 7, p. 47.

13. As on Stolz's hypothesis $F(x)$ is monotone, the case last treated in which the point a is a limit of zeros of $F(x)$ cannot then occur. Here one or two remarks are perhaps desirable.

First, it should be noticed that the monotoneness of F is involved in but does not involve the condition that $F'(x)$ should have no zeros. Secondly, notice the effect of allowing a to be a limiting point of zeros of $F(x)$.

One of two things may happen, F may change signs at zeros having a as limiting point or it may not. In the former case (1) is a continuous function of x in the extended sense in which the two infinities are identified with one another, and it will have values as near as we please to $-\infty$ and as near as we please to $+\infty$, but not necessarily all finite values. Thus (1) may have only the two limits $-\infty$ and $+\infty$, but must have these, so that there could be no question of a unique limit either for (1) or, by the reasoning of the preceding article, for (2). Thus, if, as in the application of the usual rule for evaluating an indeterminate form, we add to our assumptions the hypothesis that (2) has a unique limit, this case will exclude itself.

If, however, $F(x)$ never change sign in passing through the zeros considered, (1) is continuous in that extended sense in which $-\infty$ is distinguished from $+\infty$. If f is sometimes positive and sometimes negative at these zeros, (1) will have all finite limits as well as $-\infty$ and $+\infty$. If, however, $f(x)$ always has the same sign at these zeros, say the same sign as F in the whole neighbourhood, then (1) has values as near as we please to $+\infty$. In this case (1) may have $+\infty$ for unique limit.

It should be noticed that, if a is a limiting point of zeros of F, it is, by the Theorem of the Mean, necessarily a limiting point of zeros of F'. Moreover, such a zero of F' (unless $F = 0$ throughout the interval) is $0_{\mp}$, since it corresponds to a maximum or minimum of F.

The discussion of the behaviour of the fraction (2) is, however, more difficult than that of (1), owing to the fact that (2) is not continuous and the zeros of f' do not in general form a closed set. The present paper seems hardly the place to discuss the possibilities from the point of view of the Theory of Sets of Points.

14. We next proceed to deduce the corresponding theorem for the form ∞ / ∞. This was the usual course until comparatively recently, when it was too hastily assumed that it involved inevitably a *petitio principii*. This is not the case, if we are careful to word the proof suitably, bearing in mind that, if a unique limit does not exist, this can only be because there is more than one limit. The theorem is the following:—

THEOREM 2.—*If $f(x)$ and $F(x)$ be continuous differentiable functions throughout the open interval $(a < x)$, their differential coefficients having no common zeros or infinities, and if, as x approaches the value a, $f(x)$ and $F(x)$ have unique infinite limits (that is, each with determinate sign), then the limits of*

$$\frac{f(x)}{F(x)}, \tag{1}$$

as x approaches the value a, are also limits of

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

[Here, of course, as before, a may be $-\infty$, and a similar statement may be made on the left, in which case a may be $+\infty$.]

We may without loss of generality suppose that the limits of $f(x)$ and $F(x)$ are both $+\infty$, and, by taking the interval suitably small, we may ensure that both functions are positive, finite, and not zero.

Put, then,

$$f(x) = \frac{1}{g(x)}, \qquad F(x) = \frac{1}{G(x)},$$

so that $g(x)$ and $G(x)$ are also positive, finite and not zero, and each has the unique limit 0_+.

Hence we may write for (1),

$$\frac{G(x)}{g(x)},$$

and therefore the limits of (1) are among those of $\frac{G'(x)}{g'(x)}$, by the previous theorem.

But

$$\frac{G'(x)}{g'(x)} = \left(\frac{f(x)}{F(x)}\right)^2 \frac{F'(x)}{f'(x)}. \tag{3}$$

Now choose a sequence of values of x, having a for limit, along which (1) has the unique limit L_+. Then, by what we have just said, L_+ is also one of the limits of $\frac{G'(x)}{g'(x)}$. Hence, by (3), since the left-hand side has for the sequence of values of x in question the unique limit L_+, and the first factor of the right-hand side has the unique limit L_+^2, it follows that L_+ is, for the sequence in question, the unique limit of (2), provided L is not zero or infinite.

Next, suppose 0_+ to be one of the limits of (1), then 1_+ is one of the limits of

$$\frac{f(x)+F(x)}{F(x)},$$

and therefore of $$\frac{f'(x)+F'(x)}{F'(x)},$$

by what has just been proved. Hence 0_+ is a limit of (2).

Finally, suppose that $+\infty$ (*i.e.*, ∞_-) is one of the limits of (1).

Then 0_+ is one of the limits of $\frac{F(x)}{f(x)}$, and therefore, by the above, of $\frac{F'(x)}{f'(x)}$.

Hence, finally, $+\infty$ is one of the limits of (2).

Note.—In the present subject, the only functions which naturally occur are *finite* and continuous functions. A function which is only continuous in the extended sense in which infinite values with determinate signs are allowed, does not possess* a differential coefficient with determinate sign at a point at which it is itself infinite. It is for this reason that the present theorem is really a simpler one than Theorem 1, as it follows from Case 1 of that theorem alone.

PART II. §§ 15–18.

15. We now shew how the results so far obtained follow somewhat more simply from a line of argument identical for both the fundamental forms 0/0 and ∞/∞. The machinery employed consists of what may be called a modification of that given by Stolz for the case ∞/∞.†

THEOREMS 1 AND 2. — *If $f(x)$ and $F(x)$ be continuous differentiable functions throughout the open interval $(a < x)$, where the point a may be the point $-\infty$, the differential coefficients f' and F' having no common zeros or infinities, and if, as x approaches the value a, $f(x)$ and $F(x)$ have unique limits, which are either both zero or both infinite, then the limits of*

$$\frac{f(x)}{F(x)}, \tag{1}$$

as x approaches the value a, are limits of

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

CASE 1.—We proceed to treat first the fundamental form ∞/∞.

Let L be any one of the limits of (1), L being either finite, or zero, or

* For the proof of this property see Part II, Theorem 5, § 17.

† O. Stolz, *Grundzüge*, pp. 78, 79.

infinite, with determinate sense, and let $X_1, X_2, \ldots$ be a sequence of values of x with a as limit yielding L as the corresponding limit of (1).

We may, as usual, without loss of generality, assume that $+\infty$ is the limit of both $f(x)$ and $F(x)$. We can then certainly find an interval $a < x$, throughout which both

$$f(x) > 2f(x_1), \quad \text{and} \quad F(x) > 2F(x_1),$$

x_1 being any chosen point of the X-sequence.

Let x_2 be the first point X_i which falls in the interval so found such that

$$f(x_2) > 2f(x_1), \qquad F(x_2) > 2F(x_1).$$

Similarly, we determine the first point x_3 of the X-sequence such that

$$f(x_3) > 2^2 f(x_2), \qquad F(x_3) > 2^2 f(x_2).$$

Proceeding thus we obtain a sub-sequence of the X-sequence, $x_1, x_2, \ldots, x_n, \ldots$ such that, for all values of n,

$$f(x_{n+1}) > 2^n f(x_n), \qquad F(x_{n+1}) > 2^n f(x_n).$$

Along this sequence, therefore,

$$\frac{f(x_n)-f(x_{n+1})}{F(x_n)-F(x_{n+1})} = \frac{f(x_{n+1})}{F(x_{n+1})} \frac{1-f(x_n)/f(x_{n+1})}{1-F(x_n)/F(x_{n+1})}, \tag{3}$$

clearly has the unique limit L, since the first fraction on the right has the limit L, and the second the limit unity. But, by Lemma 2, there is a point y_n between x_n and x_{n+1}, at which

$$\frac{f'(y_n)}{F'(y_n)}$$

has the same value as the fraction on the left of the equation (3) last written down, and therefore has L as unique limit as n increases indefinitely; that is, since the point y_n lies between x_n and x_{n+1}, and therefore describes a sequence with a as limit, L is one of the limits of

$$\frac{f'(x)}{F'(x)}$$

as x approaches a as limit. This proves the theorem.

The above proof, as it stands, does not prove that, if L_+, for example, is a limit of (1), it is also a limit of (2), except in the case when L is zero, or infinity. When, however, L is zero, or infinity, since the second fraction on the right of (3) is essentially positive, the first factor has the same sign as the left-hand side, and therefore (2) has 0_+ or $+\infty$ for a limit when (1) has.

The case when the limit is L_+ and not zero or infinity is at once reduced to this by writing $f(x)-LF(x)$ for $f(x)$. Thus in every case of a finite limit the sense as well as the magnitude of the limit is the same for (1) and (2).

CASE 2.—Next, let f and F have each the unique limit zero at the point a.

If the point a is a limiting point of common zeros of $f(x)$ and $F(x)$, the truth of the theorem is almost self-evident: for

$$\frac{f'(x)}{F'(x)}, \tag{2}$$

will then have all limits from $-\infty$ to $+\infty$ inclusive, and therefore, whatever limit or limits

$$\frac{f(x)}{F(x)} \tag{1}$$

may have, they are certainly limits of (2). In fact, as proved at the end of § 11, if $f(x)$ and $F(x)$ both vanish at x_i and x_{i+1}, (2) assumes at intermediate points all finite values, and therefore has at the limiting point a of the sequence $x_1, x_2, \ldots$ every limit from $-\infty$ to $+\infty$, since the set of limits is a closed set. [The case when both f and F are zero throughout the interval considered is, of course, omitted from consideration, as in this case neither (1) nor (2) have any meaning.]

If the point a is not a limiting point of common zeros of $f(x)$ and $F(x)$, we may choose the neighbourhood so small that $f(x)$ and $F(x)$ have no common zeros: (1) has then a determinate finite value, and is continuous at every point which is not a zero of F, and at such a zero is infinite, and it has no isolated values. In this case, as explained in § 7, we may confine our attention to sequences of values of x along which both f and F have the zero limit with determinate sense, and we shall only discuss that case when these limits are both 0_+, the argument in the other cases being essentially the same.

Choosing the interval $(a < x)$ sufficiently small, we can then be sure that, at the points of the sequence considered, both $f(x)$ and $F(x)$ are finite and positive, not zero. The argument used in discussing the case of infinite limits of f and F now applies; we only have to change $>$ into $<$, and the multiplier 2 into $\frac{1}{2}$, and we obtain a sub-sequence $x_1, x_2, \ldots$ along which

$$f(x_{n+1}) < 2^{-n} f(x_n), \quad F(x_{n+1}) < 2^{-n} F(x_n).$$

Along this sequence, therefore

$$\frac{f(x_n)-f(x_{n+1})}{F(x_n)-F(x_{n+1})} = \frac{f(x_n)}{F(x_n)} \frac{1-\frac{f(x_{n+1})}{f(x_n)}}{1-\frac{F(x_{n+1})}{F(x_n)}}$$

has, as before, the same unique limit, with the same sense, as $\frac{f(x_n)}{F(x_n)}$, which, as before, proves the theorem.

16. We have thus proved, both by the older method suitably modified (§§ 12, 13), as well as by means of more modern machinery (§ 15), that the hypothesis (A) is sufficient to ensure that the limits of

$$\frac{f(x)}{F(x)} \tag{1}$$

are limits of

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

The converse has not been proved, and is not in general true. Thus on the assumption that the hypothesis (A) holds, it follows that, if (2) has a unique limit, so has (1), namely the limit of (2). In other words, *we have proved the truth of the rule for the evaluation of the indeterminate forms* $0/0$ *and* ∞/∞, *under the restriction involved in* (A).

It has not been proved, and it is not true, that if (1) has a unique limit so has (2). The following example sufficiently illustrates this point, and also the further point that this is not the case, even if besides (A) the further restriction involved in (B) be imposed.

Ex. 1.—Let $\quad f(x) = x^2 \sin\frac{1}{x}, \quad F(x) = \sin x,$

in the interval $(0 < x < \frac{1}{2}\pi)$. Here the limit of (1) at the origin is unique, namely zero; but the limits of (2) fill up the whole interval $(-1, +1)$, since

$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}, \quad F'(x) = \cos x.$$

Ex. 2.—Let $\quad f(x) = \sin^2 x \sin\frac{1}{x}, \quad F(x) = x,$

so that $F(x)$ is monotone. Here the limit of (1) is zero as before; and as before the limits of (2) fill up the whole interval $(-1, +1)$, since

$$f'(x) = \sin 2x \sin\frac{1}{x} - \frac{\sin^2 x}{x^2}\cos\frac{1}{x},$$

$$F'(x) = 1.$$

17. The reasoning of the preceding articles at once gives us the following generalisation of some theorems of Stolz.*

THEOREM 3.—*If, the other conditions of Theorems* 1 *and* 2 *remaining the same as before,* $f(x)$ *has a unique finite limit, and* $F(x)$ *a unique infinite limit, when* $x = a$, *then one of the limits of*

$$\frac{f'(x)}{F'(x)} \tag{2}$$

is zero, whose sense is opposite to that of $\frac{f(x)}{F(x)}$.

We first prove the theorem when the finite limit of $f(x)$ is zero. Whatever sequence of x's be taken

$$\frac{f(x)}{F(x)} \tag{1}$$

has the limit zero. Also we can choose such a sequence $x_1, x_2, \ldots$ that the limit of

$$\frac{f(x_{n+1})}{f(x_n)}$$

is zero, and that of

$$\frac{F(x_n)}{F(x_{n+1})}$$

is also zero. We then have

$$\frac{f(x_n)-f(x_{n+1})}{F(x_n)-F(x_{n+1})} = -\frac{f(x_n)}{F(x_{n+1})}\,\frac{1-\dfrac{f(x_{n+1})}{f(x_n)}}{1-\dfrac{F(x_n)}{F(x_{n+1})}}. \tag{3}$$

Hence, making use of Lemma 2, and bearing in mind that the first factor on the right-hand side of (3) necessarily has the unique limit zero, while the second factor has the unique limit unity, the result follows.

If the limit of $f(x)$ is L other than zero, we have merely to write $g(x)$ for $f(x)$ in the above reasoning, where

$$g(x) = f(x) - L.$$

Hence the result follows since

$$g'(x) = f'(x).$$

* O. Stolz, *Grundzüge*, pp. 79, 80; his statement of the last of these theorems is at the bottom of p. 80. It will be seen that our statement of all three theorems is considerably more general.

N.B.—In the above result we have evidently made no assumption that the mode of approach is on one side, rather than on the other of the point a. Thus one of the limits on the right of (2) is 0_+, if the unique limit of (1) on the right is 0_+, and one of the limits on the right of (2) is 0_-, if the unique limit of (1) on the right is 0_+. Exactly the same statement is true if we approach on the left.

We notice also that the reasoning of the preceding proof gives us at the same time the following theorem.

THEOREM 4.—*Under the same conditions as to the continuity and differentiability of* $f(x)$ *and* $F(x)$, *and supposing hypothesis* (A) *still to hold, then if* $f(x)$ *has an unique infinite limit with definite sense, and* $F(x)$ *an unique limit with definite sense, one of the limits of*

$$\frac{f'(x)}{F'(x)} \tag{2}$$

is $+\infty$ *or* $-\infty$ *according as the unique limit of*

$$\frac{f(x)}{F(x)} \tag{1}$$

is $-\infty$ *or* $+\infty$.

From Theorem 4 we can at once deduce the following important result.

THEOREM 5.—*Whatever kind of differentiable function* $f(x)$ *may be, provided only it has a unique infinite limit on the right* (*left*) *of the point* $x = a$ *with definite sign, one of the limits of* $f'(x)$ *on the right* (*left*) *is infinite with the opposite* (*same*) *sign.*

Put $$F(x) = x - a,$$

so that $F'(x)$ is unity, and all the conditions of Theorem 4 clearly hold. Then

$$F(a+0) = 0_+, \quad \text{and} \quad F(a-0) = 0_-.$$

Hence the result follows.

COR.—*If a function is, at the point* $x = a$, *continuous in the extended sense in which infinite values are allowed, but the two infinities are distinguished from one another, then* $f'(x)$ *can at most exist and be continuous at* a *in the extended sense in which the two infinities are confounded.*

18. We conclude this portion of the paper, dealing with the effects of making the assumption (A), by considering briefly the case in which one

at least of the two functions $f(x)$ and $F(x)$ has, as x approaches the value a, more than one limit.

If this be the case, $f(x)-mF(x)$ has a unique limit for at most one value of m. Supposing m not to have this value, $f(x)-mF(x)$ has necessarily a plurality of limits, and hence, by the theorem whose proof is sketched in § 9, we can find values of x as near as we please to a for which

$$f'(x)-mF'(x) = 0,$$

bearing in mind, as before, that since $f'(x)$ and $F'(x)$ have no common infinities $f(x)-mF(x)$ is differentiable.

Hence, since $f'(x)$ and $F'(x)$ have no common zeros, we thus get a sequence of values of x having the point a as limiting point, for which

$$\frac{f'(x)}{F'(x)} \tag{2}$$

has the limit m. Thus (2) has every possible finite limit, except possibly the excluded value of m, and therefore, since the limits necessarily form a closed set, has every possible limit from $-\infty$ to $+\infty$, both inclusive. It is plain, therefore, that whatever limits $\frac{f(x)}{F(x)}$ may have they will be limits of (2).

There is, of course, however, no reason why the limits of $\frac{f(x)}{F(x)}$ should, like those of (2), fill up the whole infinite interval. On the contrary, they may reduce to a single limit, even in the case before us. This is, for instance, the case if

$$F(x) = \sin\frac{1}{x},$$

and

$$f(x) = (x+1)\sin\frac{1}{x},$$

at the point $x = 0$.

PART III. §§ 19–28.

19. We now proceed to consider the effect of making the hypothesis (B), (§ 1), so that $F(x)$ is monotone, and for convenience we shall assume that it is a non-decreasing monotone function of x. It then follows that f may be regarded as a function of F, say $g(F)$. It might be supposed that this fact would materially simplify our work, and that, by changing

the independent variable from x to F, the results of §§ 12, 13, 15, 16 might be extended. It will be seen, however, from the following Lemmas, which are needed in the sequel, and the examples illustrating them that this is not the case. Condition (A) must still be hypothecated, and all that we have done is to superimpose condition (B). If we are to avoid hypothecating (A), we must devise a new mode of treating the subject. Such a method is given immediately after the proof of Lemma 5.

It should be noted, however, that in special instances the change of the independent variable from x to F *may* be of use.* Thus, if it should happen that f has throughout the open neighbourhood $(0 < F)$, corresponding to the neighbourhood $(a < x)$ a differential coefficient with respect to F, while f and F do not possess differential coefficients with respect to x, we obtain

$$\operatorname*{Llt}_{x=a} \frac{f(x)}{F(x)} = \operatorname*{Llt}_{F=0} \frac{g(F)}{F} = g'(0),$$

if this exists, and, if not, the limits in question are included among the limits, necessarily manifold, *i.e.*, at least twofold, of $g'(F)$.

Here we have supposed the form under consideration to be 0/0 ; similar remarks apply when the form is ∞/∞.

20. LEMMA 3.—*If, throughout any interval* (a, b), $f(x)$ *and* $F(x)$ *are continuous and differentiable and* $F(x)$ *also monotone, then provided the zeros of* $F'(x)$ *nowhere fill up a continuum,* f *is a continuous function of* F *throughout the interval* $[F(a), F(b)]$, *and is differentiable except at the common zeros and infinities of* $f'(x)$ *and* $F'(x)$, *where in general a differential coefficient will not exist.*

(Here, as is usual, the word "differentiable" is intended to include the possibility of a differential coefficient being infinite with determinate sign.)

Since the zeros of $F'(x)$ nowhere fill up a continuum, $F(x)$ is not constant in any interval. Hence to each value of F there corresponds a single value of x, and therefore a single value of $f(x)$, which we take to be the value of the function $g(F)$. Further, as F describes the interval $[F(a), F(b)]$, x describes the interval (a, b). Thus $g(F)$ is defined as a function throughout the interval specified.

* This corresponds to taking $\operatorname*{Lt}_{h=0} \frac{f(x+h)-f(x)}{F(x+h)-F(x)}$ instead of $\frac{f'(x)}{F'(x)}$. Stolz never explicitly mentions $\frac{df}{dF}$, but uses the former limit.

Again, the relative incrementary ratio

$$\frac{f(x)-f(x')}{F(x)-F(x')} = \frac{f(x)-f(x')}{x-x'} \cdot \frac{x-x'}{F(x)-F(x')},$$

has the unique limit $\frac{f'(x')}{F'(x')}$ as x approaches x', provided both numerator and denominator do not here vanish or become infinite. This proves that $g(F)$ has at the point $F(x')$ a differential coefficient whose value is $\frac{f'(x')}{F'(x')}$, provided this value is definite.

The following example shews that at a point where this value is indeterminate $g(F)$ may not possess a differential coefficient.

Ex. 3.—Let $$f(0) = 0, \quad f(x) = x^2 \sin\frac{\pi}{x}, \quad (x \neq 0),$$

$$F(0) = 0, \quad F(x) = x^2.$$

Here at the origin both f' and F' are zero, and the relative incrementary ratio

$$\frac{x^2 \sin\frac{\pi}{x}}{x^2} = \sin\frac{\pi}{x},$$

has all possible limits between -1 and $+1$, so that a differential coefficient of f with respect to F does not exist at the origin.

Ex. 4.—Let

$$f(0) = 0, \quad f(x) = x^{\frac{1}{2}} \sin^2\frac{\pi}{x}, \ (0 < x), \quad f(x) = -(-x)^{\frac{1}{2}} \sin^2\frac{\pi}{x}, \ (x < 0),$$

$$F(0) = 0, \quad F(x) = x^{\frac{1}{2}}, \ (0 < x), \qquad F(x) = -(-x)^{\frac{1}{2}}, \ (x < 0).$$

Here at the origin both f' and F' have the value $+\infty$, and the limits of the relative incrementary ratio at the origin are all values between 0 and 1, so that a relative differential coefficient at the origin does not exist.

21. From the preceding we at once deduce the following Lemma.

LEMMA 4.—*If in any interval (a, b), as before, f and F are continuous and differentiable, and F is monotone, and f' and F' have no common zeros or infinities, then f'/F' assumes all values between its upper and lower bounds in the interval, whether it is open or closed. This is, in general, not the case if f' and F' have one or more common zeros or infinities.*

In fact, except in the case excluded f is necessarily a differentiable function of F throughout the interval $[F(a), F(b)]$, and we have, with the notation of the previous article,

$$g'(F) = \frac{f'(x)}{F'(x)}.$$

Hence the first part of the statement follows by one of the properties of a differential coefficient, mentioned in the introduction (§ 9).

That the second part of the statement is true follows from Ex. 5 below.

Ex. 5.—Let

$$f(0) = 0, \quad f(x) = x^3 \sin\frac{\pi}{x} + \pi x^2, \ (0 < x), \quad f(x) = x^3 \sin\frac{\pi}{x} - \pi x^2, \ (x < 0),$$

$$F(0) = 0, \quad F(x) = x^2.$$

Here $f'(0) = 0$, $f'(x) = 3x^2 \sin\frac{\pi}{x} - \pi x \cos\frac{\pi}{x} + \pi 2x$, $(0 < x)$,

$$f'(x) = 3x^2 \sin\frac{\pi}{x} - \pi x \cos\frac{\pi}{x} - \pi 2x, \quad (x < 0),$$

so that both $f(x)$ and $f'(x)$, as well as $F(x)$ and $F'(x)$ are continuous everywhere.

Nevertheless the presence of the single common zero of f' and F', viz., the origin, is sufficient to prevent f'/F' from assuming a whole interval of values between its upper and lower bounds.

In fact, $$\frac{f'(x)}{F'(x)} = \tfrac{3}{2} x \sin\frac{\pi}{x} - \frac{\pi}{2} \cos\frac{\pi}{x} \pm \pi,$$

the upper or lower sign being taken according as x is positive or negative. Hence it follows that the limits of f'/F' at the origin on the right have all values between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$, and on the left between $-\frac{1}{2}\pi$ and $-\frac{3}{2}\pi$, leaving a gap $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ in which no limits at the origin lie.

Since, however, the expression for f'/F' shews that it is continuous for all positive and negative values of x, we can choose the interval containing the origin in which we are working so small that the upper and lower bounds of f'/F' in the positive and negative parts of the interval respectively differ by as little as we please from the upper and lower limits on the right and left of the origin respectively. Hence there will, if the interval be sufficiently small, *e.g.*, if it be the interval $(-\frac{1}{6}\pi, \frac{1}{6}\pi)$, be a whole interval of values between the upper and lower bounds of f'/F' in the interval, which are not assumed as values by f'/F'.

22. It may be noted *en passant*, that the example also shews that if two functions, in our case $f'(x)$ and $F'(x)$, are continuous throughout an interval, here, for example, the interval $(-\frac{1}{6}\pi, \frac{1}{6}\pi)$, it does not follow that their ratio assumes all values between its upper and lower bounds.

It is unnecessary to give an example in which f' and F' have a common infinity; such an example may easily be constructed. The principle is, however, sufficiently elucidated by the preceding example.

23. From the preceding Lemma the next one follows immediately, the final part of the statement being proved by the succeeding example.

Lemma 5.—*If, as before, f and F are continuous and differentiable throughout an open interval (a, x), and F is monotone, then provided the point a is not a limiting point of common zeros or infinities of f' and F', the limits of $\frac{f'(x)}{F'(x)}$ at a on the right fill up a continuum.*

A similar statement holds on the left. Moreover, the same is true of all the limits at a on both sides, provided the point a is not a common zero or infinity of f' and F', or a limiting point of such common zeros or infinities.

In the omitted cases the result no longer holds.

24. Ex. 6.—Let b_1, b_2, ... be any sequence of points having the point a as limiting point (Fig. 2).

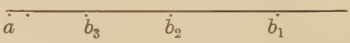

Fig. 2.

Define the functions f and F, so that in each of the intervals (b_2, b_1), (b_3, b_2), ..., $\frac{f'(x)}{F'(x)}$ does not assume any values between c and d, two values between its upper and lower bounds in each of these intervals. Example 5 shews that this is possible, provided in each of the intervals there is a common zero or infinity of f'/F', in which case the point a is a limiting point of such common zeros or infinities. In this case it is clear that f'/F' will have no limit between c and d, but will have limits both $\geqslant$ the greater of c and d, and $\leqslant$ the less.

Example 5 shews that the fact that the point a is itself a common zero or infinity of f'/F' is sufficient to prevent our being able to assert that *all* the limits at a fill up a continuum.

25. Theorem 6.—*If $f(x)$ and $F(x)$ are continuous differentiable functions of x whose differential coefficients have no common infinities, in*

some open interval $(a < x)$, $F(x)$ *being also monotone in that interval, and if* $f(x)$ *and* $F(x)$ *have both the unique limit zero as* x *approaches* a, *then the limits of*

$$\frac{f(x)}{F(x)}, \tag{1}$$

as x *approaches* a, *lie between the upper and lower limits of*

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

Since $F(x)$ is, by hypothesis, monotone, and we may evidently omit from consideration as trivial the case where, throughout some neighbourhood $(a < x)$, $F(x)$ is zero, we may, without loss of generality, suppose $F(x)$ to be always positive, and $F'(x)$ to be never negative throughout the given interval. Moreover it will be sufficient to prove that the upper limit of (1) is less than or equal to the upper limit of (2), for the same argument, *mutatis mutandis*, will prove that the lower limit of (1) is greater than or equal to the lower limit of (2). Finally, we may suppose the upper limit of (2) to be finite or $-\infty$, since, if it is $+\infty$, what is required to be proved is obvious.

Let then L be any quantity greater than the upper limit of (2), so that

$$\frac{f'(x)}{F'(x)} - L < 0,$$

for a sufficiently small neighbourhood $(a < x)$ at all points at which (2) has a definite value. At a point which is a common zero of $f'(x)$ and $F'(x)$ the sign of inequality becomes one of equality. At a point at which $F'(x)$ is infinite, $f'(x)$ is not infinite and therefore the above sign of inequality holds. At a point which is a zero of $F'(x)$ and not of $f'(x)$, the sign of inequality will have to be changed from less to greater, if $f'(x)$ is positive there. As, however, $F'(x)$ is always positive or zero, it is consistent with the procedure adopted in this paper, if we regard its value at a point where it vanishes to be 0_+, so that the fraction (2) has the value $+\infty$. Unless therefore we definitely exclude such fractions in forming the limits of (2), the upper limit of (2) is in this case $+\infty$, and no further discussion is required. Hence in all cases we have

$$f'(x) - LF'(x) \leqslant 0.$$

Now, since $f'(x)$ and $F'(x)$ are never infinite together,

$$f(x) - LF(x), \tag{3}$$

has throughout the interval

$$f'(x) - LF'(x),$$

for differential coefficient, and therefore, since the latter is never positive, the former never increases as x increases. But, as x approaches the value a, (3) has the limit zero; therefore it is never positive in the open neighbourhood $(a < x)$. Hence, as $F(x)$ is positive in the neighbourhood,

$$\frac{f(x)}{F(x)} - L$$

is throughout the open neighbourhood never positive. Therefore the upper limit of (1) as x approaches the value a is less than or equal to L. But L is any quantity greater than the upper limit of (2), hence the upper limit of (1) is less than or equal to any quantity greater than the upper limit of (2), and is therefore less than or equal to that upper limit.

Note.—If in considering the limits of (2), we disregard those fractions for which the numerator is zero, and the denominator not, the theorem as it stands will not be true, as the example appended shews. We should then have to require, as Stolz does, that the denominator should never be zero except when the numerator is so also; the detail of the preceding proof shews that this would suffice.

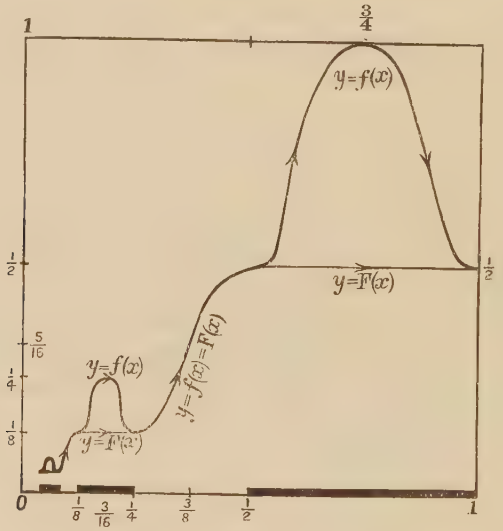

Fig. 3.

26. Ex. 7.—Take the sequence of points 1, $\frac{1}{2}$, $\frac{1}{4}$, ... with zero as limit,

and alternately blacken and leave white the abutting intervals between adjacent points of this sequence.

In each black interval let $F(x)$ be constant and equal to the abscissa of the left-hand end-point of the black interval, and let $f(x)$ be of the form

$$2a^3\left(X^2-\frac{1}{a^2}\right)^2+\frac{2}{a},$$

where $2/a$ is the length of the interval, and is also the abscissa of the left-hand end-point of the interval, and the coordinate X is measured from the middle point of the interval. The form of the curve

$$y=f(x)$$

in the black intervals is then that drawn in the intervals $(\frac{1}{2}, 1)$ and $(\frac{1}{8}, \frac{1}{4})$ in the figure, it touches

$$y=F(x)$$

at the ends of the black interval, has its maximum at the middle point of the interval, where the value of $f(x)$ is twice that of $F(x)$, and is monotone on each side of the maximum.

In each white interval $\quad f(x)=F(x),$

and is of the form $\quad X-\frac{1}{4}a^2X^3+\frac{5}{2a}.$

The form of the corresponding curve is given in the interval $(\frac{1}{4}, \frac{1}{2})$ in the figure. It is monotone ascending with an inflexion at the middle point of the white interval.

The origin is here the limiting point of the sequence $1, \frac{1}{2}, \frac{1}{4}, \ldots$ of common zeros of $f'(x)$ and $F'(x)$. Accordingly the anomaly presents itself that the fraction (2) has at the origin only two limits, unity and $+\infty$. The fraction (1), however, has all limits from 1 to 2 both inclusive.

Thus, if we exclude from consideration fractions (2) where the numerator is finite and the denominator zero, the theorem ceases to be true that the limits of (1) lie between the upper and lower limits of (2).

27 THEOREM 7.—*If $f(x)$ and $F(x)$ are continuous differentiable functions of x whose differential coefficients have no common infinities in some open interval $(a<x)$, $F(x)$ being also monotone in that interval, and if $f(x)$ and $F(x)$ have both as unique limit either $+\infty$ or $-\infty$, not necessarily*

the same for both, as x approaches a, then the limits of

$$\frac{f(x)}{F(x)}, \tag{1}$$

as x approaches a, lie between the upper and lower limits of

$$\frac{f'(x)}{F'(x)}. \tag{2}$$

We may clearly, without loss of generality, suppose the two infinite limits in question to be both $+\infty$, since otherwise we have only to change the sign of one or both the functions everywhere. Similar considerations to those adduced in the preceding proof allow us to suppose that $F(x)$ is always positive, and $F'(x)$ to be never positive, throughout the given interval, and to assume that the upper limit of (2) is finite, say L.

Then, as before, we shew that, for a sufficiently small neighbourhood $a < x$,

$$f(x) - LF(x), \tag{3}$$

never *decreases* as x increases, and therefore, as x decreases to the value a, has necessarily a definite limit, its lower bound.

Now, suppose it possible that (1) had a limit greater than L, say $L+2e$. Then

$$f(x) - (L+e)\,F(x),$$

that is,

$$f(x) - LF(x) - eF(x),$$

is positive for some set of values of x having a as limit.

But $F(x)$ has the limit $+\infty$ for all modes of approach to a; the same is therefore true of $eF(x)$, since e is a fixed quantity: the same is therefore true of

$$f(x) - LF(x), \tag{3}$$

for the set of values of x in question, and therefore, for all modes of approach to a, since it has, by the above, an unique limit.

But, by the above, as x increases from (a), (3) cannot decrease, and must therefore remain $+\infty$, which is absurd.

This proves the truth of the theorem.

Note.—A similar remark to that made at the end of the preceding theorem is also applicable here, and it would be easy to suitably modify the example there given to meet the new circumstances of the present theorem.

In the enunciations of the two preceding theorems we may, if we

please, omit the word "unique," and suppose that zero in the former and infinity in the latter theorem is only one of the limits of $f(x)$, without disturbing the truth of the theorem.

In fact, as $F(x)$ is monotone, it necessarily has an unique limit, and the argument shews that, if the upper limit of (2) is finite,

$$f(x)-LF(x)$$

has an unique limit, and therefore $f(x)$ has an unique limit, so that the argument applies, as it stands. If the upper limit of (2) is $+\infty$, the theorem is obvious, as already pointed out.

28. It is scarcely necessary to call attention to the great difference between the results just proved and those obtained in Parts I and II. It was there shewn that every limit of (1) was a limit of (2), sense as well as magnitude being taken into account. This is now far from being the case. There may be whole stretches of limits of (1) which are not limits of (2), and even when a certain limit of (1) is a limit of (2), their senses are by no means bound to be the same.

Moreover, as will be shewn, if $f'(x)/F'(x)$ has an unique limit, it can be proved from condition (A) that $F(x)$ must be monotone. If, then, it should further happen that $F'(x)$ should never be zero without $f'(x)$ also being zero, condition (A) becomes a less general statement of condition (B). If, on the other hand, the point a is not a limiting point of zeros of $F'(x)$ at all, the two conditions are exactly the same. Thus a simple form of rule is the following :—

If a is not a limiting point of zeros of $f'(x)$ and not a limiting point of common infinities of $F'(x)$ and $f'(x)$, the ordinary rule for the evaluation of the indeterminate form $f(x)/F(x)$ holds.

We now give the theorem on which this depends.

THEOREM 8.—*If a be not a limiting point of common zeros of $f'(x)$ and $F'(x)$, and if further, as x approaches a, $f'(x)/F'(x)$ has an unique limit, then $F(x)$ is necessarily monotone in some neighbourhood of a. Moreover this is true whether or not we exclude in calculating the limit of $f'(x)/F'(x)$ all points at which the denominator is zero.*

Suppose that $F(x)$ is not monotone, then $F'(x)$ has both positive and negative values in every sufficiently small neighbourhood of the point a. Hence, if the unique limit of $f'(x)/F'(x)$ be not zero, $f'(x)$ must have both positive and negative values in every such neighbourhood and therefore have zero values in every such neighbourhood. But these

zeros will not be zeros of $F'(x)$, and therefore the hypothesis that the unique limit of $f'(x)/F'(x)$ is not zero, leads to the conclusion that it is zero. But even if we suppose the unique limit of $f'(x)/F'(x)$ obtained by omitting all fractions of which the denominator is zero, this unique limit cannot be zero, for the same argument would shew that the unique limit of $g'(x)/F'(x)$ would be zero, where

$$g(x) = f(x)+F(x),$$

which is clearly inconsistent with the unique limit of $f'(x)/F'(x)$ being zero.

Hence the hypothesis that $F(x)$ is not monotone has been shewn to lead to an absurdity.

Appendix. §§ 29–33.

29. The following theorem is often now omitted in consequence of incorrect proof or formulation in some of the old text-books.

Theorem 9.—*If $f(x)$ and $F(x)$ themselves vanish when $x = a$, and have their first $(n-1)$ differential coefficients also zero there, while their n-th differential coefficients are finite and not both zero, then*

$$\frac{f(x)}{F(x)}$$

has an unique limit as x approaches a, and the value of the limit is

$$\frac{f^{(n)}(a)}{F^{(n)}(a)}.$$

It will be noticed that we merely require the existence at the point a of the n-th differential coefficients and their finiteness, not their continuity. Indeed they need not exist except at the point a.

Since $f^{(n)}(x)$ exists at $x = a$, all the preceding differential coefficients exist in a closed neighbourhood of $x = a$. Hence, by Theorem 1,

$$\operatorname*{Lt}_{h=0} \frac{f(a+h)}{h^r} = \operatorname*{Lt}_{h=0} \frac{f'(a+h)}{rh^{r-1}} = \ldots = \operatorname*{Lt}_{h=0} \frac{f^{(r-1)}(a+h)}{r!\ h}.$$

But, since $$f^{(r-1)}(a) = 0,$$

$$\frac{f^{(r-1)}(a+h)}{h} = \frac{f^{(r-1)}(a+h)-f^{(r-1)}(a)}{h},$$

so that, by the definition of $f^{(r)}(a)$, the last expression has the limit $f^{(r)}(a)$. Hence

$$\operatorname*{Lt}_{h=0} \frac{f(a+h)}{h^r} = \frac{1}{r!} f^{(r)}(a).$$

Similarly,

$$\operatorname*{Lt}_{h=0} \frac{F(a+h)}{h^s} = \frac{1}{s!} F^{(s)}(a).$$

But

$$\frac{f(a+h)}{F(a+h)} = h^{r-s} \frac{f(a+h)/h^r}{F(a+h)/h^s},$$

whence, remembering that, as usual, h is positive, the theorem at once follows, since h^{r-s} and the numerator and denominator of the fraction multiplying it have unique limits.

30. We now pass to one or two generalisations to more than one variable. It has commonly, but erroneously, been supposed, that such generalisations did not exist.

We shall require the following Lemma.

LEMMA 6.—*If $f(x, y)$ be a function of the ensemble (x, y) which vanishes at the four corners of a rectangle whose sides are parallel to the axes, and which possesses a repeated differential coefficient $\frac{d}{dx} \cdot \frac{df}{dy}$ throughout the entirely open rectangle, then there is an internal point of that rectangle at which this repeated differential coefficient vanishes.*

This follows by means of a repeated application of the Theorem of the Mean for a single variable.

31. THEOREM 10.—*If $f(x, y)$ and $F(x, y)$ vanish at all points of the axial cross*

$$x = a, \quad y = b,$$

in a certain $(++)$-neighbourhood, and have differential coefficients*

$$\frac{d}{dx} \cdot \frac{df}{dy} = f_{xy} \quad \text{and} \quad \frac{d}{dx} \cdot \frac{dF}{dy} = F_{xy},$$

at all points of the neighbourhood

$$a < x, \quad b < y,$$

then, provided the point (a, b) is not a limiting point either of common zeros or common infinities of these differential coefficients, the double

* That is, a rectangle with sides parallel to the coordinate axes, having the point in question as left-hand lower end-points.

limits of

$$\frac{f(a+h,\ b+k)}{F(a+h,\ b+k)} \tag{1}$$

are double limits of

$$\frac{f_{xy}}{F_{xy}}. \tag{2}$$

CASE 1.—Let the point (a, b) not be a limiting point of zeros of $F(x, y)$.

As F_x and F_y are both zero at the point (a, b), this case may easily occur. Put

$$g(x, y) = f(x, y) - mF(x, y),$$

where m denotes (1).

Then $g(x, y)$ has a differential coefficient

$$g_{xy} = \frac{d}{dx} \cdot \frac{dg}{dy}.$$

Also $\quad g(a, b) = 0, \quad g(a+h, b) = 0, \quad g(a, b+k) = 0,$

since f and F vanish at these points. Also

$$g(a+h, b+k) = 0,$$

identically. Therefore, by Lemma 6, a point (x, y) inside the rectangle $(a, b\,;\ a+h, b+k)$ exists at which

$$0 = g_{xy} = f_{xy} - mF_{xy}.$$

But $F(x, y)$ cannot be either zero or infinite, for f_{xy} is, by hypothesis, not zero or infinite at the same point as $F(x, y)$. Hence, dividing by $F(x, y)$ we obtain an equation which proves that, to each sequence giving a unique limit of (1), there corresponds a sequence giving the same unique limit of (2).

CASE 2.—Let the point (a, b) be a limiting point of zeros of $F(x, y)$. This will, for example, occur if F possesses a second differential at the point (a, b) and

$$F_{aa} F_{bb} < F_{ab}^2,$$

in which case there would be a curve of zeros of $F(x, y)$ passing through the point (a, b).

If (a, b) is a limit of common zeros of $f(x, y)$ and $F(x, y)$, we have only to take the point $(a+h, b+k)$ to be such a common zero, and, giving m any value we please, the argument used in Case 1 holds, so that (2) will have all finite limits, and therefore also $+\infty$ and $-\infty$, and the theorem is self-evident. If this is not the case, we may choose the neighbourhood so small that it contains no common zeros of $f(x, y)$ and $F(x, y)$. In this

case (1) has a definite finite value, and is continuous in the ordinary sense at every point which is not a zero of $F(x)$, and at these zeros is infinite and continuous when we identify the two infinities with one another. Hence (1) has no isolated values, so that, as pointed out in the introduction, (§ 7), and already used in the corresponding proof in § 12, we may omit all infinite values of (1) in obtaining its limits at the point a. The argument used in Case 1 then still holds, and hence the theorem is true in every case.

32. It scarcely seems worth while here to obtain all the possible extensions, to more than one dimension, of the theorems of the main portion of the paper. As an example, however, of another such extension, we notice the following.

THEOREM 11.—*Supposing f and F have the same properties as in the preceding theorem, except that, instead of the proviso respecting the common zeros and infinities, we hypothecate that throughout the open neighbourhood $F_{xy} > 0$, then the double limits of*

$$\frac{f(a+h,\ b+k)}{F(a+h,\ b+k)}, \tag{1}$$

lie between the upper and lower double limits of

$$\frac{f_{xy}}{F_{xy}}. \tag{2}$$

The proof is precisely analogous to that of the corresponding theorem for one dimension, except that two steps are required.

It at once follows from the equation

$$F(a+h,\ b+k) = hkF_{xy},$$

where $(x,\ y)$ is a certain point in the interior of the rectangle $(a,\ b\ ;\ a+h,\ b+k)$, that $F(x,\ y)$ is positive throughout the whole open neighbourhood.

Let, then, L be a quantity greater, if such exist, than the upper limit of (2). Then

$$f_{xy} - LF_{xy} < 0.$$

Therefore

$$f_y - LF_y$$

decreases for all values of y, as x increases from a, and therefore, since it is zero for all values of y when x is a, it is negative for all values of y when $x > a$, and therefore

$$f - LF,$$

for all values of x, decreases as y increases. But when y is b, its value is zero. Therefore it is negative for all values of $x > a$ and $y > b$. Hence, bearing in mind that F is positive,

$$\frac{f}{F} < L.$$

The remainder of the argument may be left to the reader.

33. There is one other way in which the results of the main part of the present paper may be extended, viz., to the case when $f(x)$ and $F(x)$ are not in general differentiable. The results are given below in Theorem 12. They result as will be seen from a theorem given in a short "Additional Note on Derivates and the Theorem of the Mean," in which certain theorems of the paper quoted in the footnote to § 8, p. 48, were completed by the authors. This theorem is as follows:—

If $f(x)$ and $F(x)$ are continuous throughout the interval (a, b), $f(x)$ being finite or quasi-finite, and $F(x)$ monotone and never constant, and if the ratios of corresponding derivates of f and F are definite at every point of the completely open interval (a, b), each of these ratios, viz.,*

$$\frac{f^-(x)}{F^-(x)}, \quad \frac{f^+(x)}{F^+(x)}, \quad \frac{f_-(x)}{F_-(x)}, \quad \frac{f_+(x)}{F_+(x)},$$

has in the interval the same upper and lower bounds as the relative incrementary ratio

$$\frac{f(x_1)-f(x_2)}{F(x_1)-F(x_2)},$$

where x_1 and x_2 are any two points of the interval; these bounds are, moreover, the same whether or not the end-points a and b of the interval are included.

We can now prove the following theorem:—

THEOREM 12.—*If $f(x)$ and $F(x)$ be continuous throughout the open interval $(a < x)$, $f(x)$ being finite or quasi-finite and $F(x)$ monotone and never constant, and provided the point a is not a limiting point of common zeros or infinities of corresponding derivates of $f(x)$ and $F(x)$, then, if, as x approaches the value a, $f(x)$ and $F(x)$ have unique limits, which are either both zero or both infinite, then the limits of*

$$\frac{f(x)}{F(x)}, \tag{1}$$

* That is, a function whose infinities are dense nowhere.

as x approaches a, lie between the upper and lower limits of the ratios of corresponding derivates of f and F at the point a.

For taking the point x_2 in the theorem just quoted to be the point a, and supposing the common limiting value to be in this case zero, we see, replacing x_1 by x, that (1) is less than or equal to the upper bound of the ratio of corresponding derivates in the interval (a, x). Hence as this interval diminishes indefinitely, all the limiting values of (1) are less than or equal to the upper limit of the ratio of corresponding derivates. Similarly, all the limiting values of (1) are greater than or equal to the lower limit of the ratio of corresponding derivates, which proves the theorem when the common limit of f and F is zero.

Next, let the common limit of f and F be $+\infty$. We proceed exactly as in § 15 down to equation (3),

$$\frac{f(x_n)-f(x_{n+1})}{F(x_n)-F(x_{n+1})} = \frac{f(x_{n+1})}{F(x_{n+1})} \frac{1-\dfrac{f(x_n)}{f(x_{n+1})}}{1-\dfrac{F(x_n)}{F(x_{n+1})}}, \tag{3}$$

where the first fraction on the right-hand side of this equation has, as n increases, the unique limit L, and the second fraction the unique limit unity.

But the left-hand side of (3) has, by the theorem just quoted, a value between the upper and lower bounds of

$$\frac{f^+(x)}{F^+(x)}, \tag{4}$$

in the interval (a, x_n). Hence, as x_n moves up to a, we see that the limit L lies for all values of n between the upper and lower bounds of (4) in the interval (a, x_n), and therefore lies between the upper and lower limits of (4) at the point a, which proves the theorem in this case.

COR. 1.—*If any one (and therefore all) of the ratios of corresponding derivates of f and F is continuous at the point a, then* (1) *has a unique limit, viz., the common value of the ratios of corresponding derivates at a.*

COR. 2.—*If $f(x)$ and $F(x)$ have each a right-hand (left-hand) differential coefficient throughout the open interval (a, x), the other conditions of Theorem* 12 *holding as before, then the limits of $\dfrac{f(x)}{F(x)}$ lie between the*

upper and lower limits (inclusive) of the ratio of these right-hand (left-hand) differential coefficients, and therefore $\frac{f(x)}{F(x)}$ *has a unique limit if the latter ratio has.*

COR. 3.—*If* $f(x)$ *has a left-hand (right-hand) differential coefficient with respect to* $F(x)$ *throughout the open interval, the other conditions of Theorem* 12 *existing as before, the limits of* $\frac{f(x)}{F(x)}$ *lie between the upper and lower limits of this left-hand (right-hand) differential coefficient, and therefore* $\frac{f(x)}{F(x)}$ *has a unique limit if this latter has.*

THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS AND AN EXTENSION THEREOF

By E. CUNNINGHAM.

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Introductory.

1. The absence, as far as experiment can detect, of any phenomenon arising from the Earth's motion relative to the electromagnetic æther has been fully accounted for by Lorentz† and Einstein,‡ provided the hypothesis of electromagnetism as the ultimate basis of matter be accepted, so that the only available means of estimating the distance between two points is the measuring of the time of propagation of effects between the bodies, such propagation taking place in accordance with the equations of the electron theory. It has been proved not only within the limits of experimental accuracy, but exactly, that any actual effect is completely obscured by the fact that the observer necessarily shares in the motion of the earth, and has therefore different measures of time and space from those which he would have if he did not do so. The foundation of this theory of relativity is the set of relations subsisting between the space and time measures of two observers having a uniform relative velocity. If this is v and the axis of x is taken in the direction of v, these relations are

$$T = \beta\left(t - \frac{vx}{c^2}\right), \qquad X = \beta(x - vt), \qquad Y = y, \qquad Z = z. \tag{1}$$

where $$\beta = (1 - v^2/c^2)^{-\frac{1}{2}}.$$

The analytical result obtained is that, if e, h, u are vectors and ρ a scalar satisfying the equations

$$\frac{1}{c}\left(\frac{\partial e}{\partial t} + \rho u\right) = \operatorname{curl} h,$$

$$-\frac{1}{c}\frac{\partial h}{\partial t} = \operatorname{curl} e,$$

$$\operatorname{div} e = \rho,$$

$$\operatorname{div} h = 0,$$

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* This paper contains in an abbreviated form the chief parts of the work contributed by the Author to a joint paper by Mr. Bateman and himself read at the meeting held on February 11th, 1909, and also the work of the paper by the author read at the meeting held on March 11th, 1909.

† Lorentz, *Amsterdam Proceedings*, 1903–4, p. 809.

‡ Einstein, *Ann. der Physik*, 17. Cf. also Larmor, *Aether and Matter*, Ch. xi.

so that e, h are the electric and magnetic intensities, ρ is the density of electricity, and u the velocity of convection in a sequence of electromagnetic phenomena in the æther, the observer being at rest relative to the æther, then the quantities E, H, U, P given by the equations

$$E_X = e_x, \qquad E_Y = \beta(e_y - vh_z/c), \qquad E_Z = \beta(e_z + vh_y/c);$$

$$H_X = h_x, \qquad H_Y = \beta(h_y + ve_z/c), \qquad H_Z = \beta(h_z - ve_y/c);$$

$$\mathrm{P} = \beta\rho(1 - vu_x/c^2),$$

$$U = B^{-1}(1 - vu_x/c^2)^{-1}\{\beta(u_x - v),\ u_y,\ u_z\},$$

of which the last is deduced without further assumption from (1), satisfy the same equations in (X, Y, Z, T).

Thus E, H, U, P are quantities which can consistently represent a sequence of phenomena in which the second observer is supposed to be at rest in the æther.

The relations given above are reciprocal, and it is proved that the charges in corresponding elements of volume at corresponding times are the same in magnitude. Thus it is impossible for either observer to deduce from any experimental observation that he is at rest in the true æther and that the other is moving or *vice versa*.

This being so, the relations connecting the values, as estimated by the two observers of all physical quantities, must be such as to leave the constitutive equations for material media invariant also. This aspect of the question has been discussed by Minkowski,* Frank,† and Mirimanoff.‡ The two former agree in stating that the Lorentz equations must be modified if the theorem of relativity is to be maintained in its entirety; but the last-named shews that this may be avoided, and that the relativity is complete and exact provided a change is made in the constitutive equations as ordinarily given. The modification required is of the second order of small quantities only in the equation

$$D = \epsilon E + (\epsilon - 1)[U, B]/c,$$

which has been verified to the first order by H. A. Wilson, and is of the first order in u/c in the equation $B = \mu H$. Thus the suggested equations are not contrary to experiment. The first part of the present paper is a verification of Mirimanoff's results by a process of averaging.§

* *Gött. Nachr.*, 1908, p. 53.

† *Ann. der Phys.*, 27, 1908, p. 1059.

‡ *Ann. der Phys.*, 28, 1909, p. 192.

§ Frank (*Ann. der Phys.*, 27, 1908) gives a verification of Minkowski's result by a similar process, but the investigation does not appear to take into account the Röntgen current-curl $[pu]$.

The second part deals with a new transformation or, rather, group of transformations, which shews that a uniform translation through space is not the only type of motion which it will be impossible to detect so long as our measures of time and space are of electrodynamic origin. It has been pointed out by Minkowski that in a space of four dimensions in which the coordinates are $(x, y, z, ct\sqrt{-1})$, the geometrical transformation employed by Einstein, is simply a finite rotational displacement of the whole space about $y = 0$, $z = 0$. The equation $\nabla^2 V = 0$, *i.e.*,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{p}{c^2}\frac{\partial^2}{\partial t^2}\right) V = 0,$$

is known to be invariant for such a transformation.

But this equation is invariant for a larger group of transformations than that of rotations, viz., for the group of conformal transformations in the four dimensional space, which, as is known, is built up out of inversions with respect to the hyperspheres of the space.* The question arises whether the theorem of relativity also holds for the types of motion of an electromagnetic system derived from one another by such a transformation. The following investigation shews that this is so, and develops a scheme of correlation between the physical quantities in the system and its transformation. The constitutive equations take exactly the same form as in the first part of the paper.

The motions that arise in these transformations are naturally a good deal more complicated than in that of Einstein. In that case, a fixed configuration transforms into one every point of which has the same velocity of translation. But, in the present case, under the simplest operation of the group a fixed system becomes one in which the whole is expanding or contracting radially about a point in a certain way, which, though analytically simple, is difficult to describe geometrically. But the important property of it is, that any sphere which is expanding with velocity equal to that of light transforms into a sphere expanding (or contracting) with equal velocity.

It may be remarked here that it appears to be impossible for a uniform velocity of rotation of an electromagnetic system to be obscured in the way in which the types of motion above mentioned are. For, without considering the electromagnetic equations at all, if a disturbance propagated with velocity c equally in all directions is to transform into the like, it follows that the space (x, y, z, ict) of the one system must be conformal

* This was pointed out to me by Mr. Bateman, a conversation with whom suggested the present investigation.

with the space $(X, Y, Z, \iota cT)$ of the other: inasmuch as

$$\delta X^2+\delta Y^2+\delta Z^2-c^2\delta T^2 = 0$$

must be a consequence of

$$\delta x^2+\delta y^2+\delta z^2-c^2\delta t^2 = 0.$$

We may, however, prove that no conformal transformation in this space exists which will transform every point at rest in the space (x, y, z) into a point moving with angular velocity about a fixed line in the (XYZ) space. Thus, as in material dynamics, the equations of electrodynamics will not be preserved in the same form for a set of axes rotating uniformly relatively to the space in which they are satisfied by actual phenomena.

I.

Some considerations concerning the Lorentz-Einstein Transformation.

2. The following relations arising immediately from the Lorentz-Einstein transformation are used in what follows.

The velocities of a moving point in the two systems are connected thus:

$$\left.\begin{aligned} W_X &= (w_x-v)\Big/\left(1-\frac{vw_x}{c^2}\right) \\ W_N &= w_n\Big/\left(1-\frac{vw_x}{c^2}\right) \end{aligned}\right\}, \tag{2}$$

the suffixes n and N denoting components in any direction perpendicular to v.

The relative coordinates of two moving points (x, y, z), (x', y', z')*:

$$\left.\begin{aligned} X'-X &= (x'-x)\Big/\beta\left(1-\frac{vw'_x}{c^2}\right) \\ N'-N &= (n'-n)+vw'_n(x'-x)\Big/c^2\left(1-\frac{vw'_x}{c^2}\right) \end{aligned}\right\}. \tag{3}$$

The relative velocity of two moving points:

$$\left.\begin{aligned} W'_X-W_X &= (w'_x-w_x)\Big/\beta^2\left(1-\frac{vw_x}{c^2}\right)\left(1-\frac{vw'_x}{c^2}\right) \\ W'_N-W_N &= (w'_n-w_n)\Big/\beta\left(1-\frac{vw_x}{c^2}\right) \\ &\qquad -vw'_n(w'_x-w_x)\Big/\beta c^2\left(1-\frac{vw_x}{c^2}\right)\left(1-\frac{vw'_x}{c^2}\right) \end{aligned}\right\}. \tag{4}$$

* Inasmuch as the relative coordinates are changing, it is necessary to specify exactly when they are measured. The expressions given are the values at the instant at which the second point is at $(x'y'z')$. $n, n', \ldots$, here stand for either coordinate perpendicular to $x, x', \ldots$.

If a denote a volume which is moving with velocity (w_x, w_y, w_z), the corresponding volume A is given by

$$A = a \Big/ \beta\left(1 - \frac{vw_x}{c^2}\right). \tag{5}$$

In applying the transformation to the consideration of the number of free electrons per unit volume, it must be remembered that they are moving with different velocities, and that this velocity affects the relative volumes which they occupy in the two systems of coordinates.

But the magnitude of the charge of each electron and the number of them are unchanged. If, therefore, we confine our attention to a group of electrons all having the same component of velocity w_x' in the one system and W_x' in the other, the volumes occupied, δa, δA, are connected by the relation (5).

Thus this group contributes to the density of the free electricity as measured in the two systems different amounts, $\delta\rho = \Sigma e/\delta a$, $\delta \mathrm{P} = \Sigma e/\delta A$. Using (5), we have

$$\delta \mathrm{P} = \frac{\Sigma e}{\delta A} = \frac{\beta \Sigma e}{\delta a}\left(1 - \frac{vw_x'}{c^2}\right).$$

Let w_x be the mass velocity of the moving medium as distinguished from w_x' the velocity of the individual electron.

Then

$$\begin{aligned}\delta \mathrm{P} &= \frac{\beta \Sigma e}{\delta a}\left\{\left(1 - \frac{vw_x}{c^2}\right) - \frac{v}{c^2}(w_x' - w_x)\right\} \\ &= \beta\left(1 - \frac{vw_x}{c^2}\right)\delta\rho - \frac{\beta v}{c^2}\,\frac{\Sigma e(w_x' - w_x)}{\delta a} \\ &= \beta\left(1 - \frac{vw_x}{c^2}\right)\delta\rho - \frac{\beta v}{c^2}\,\delta j_x,\end{aligned}$$

where δj_x is the contribution of the same group of electrons to the conduction current through the medium.

This last relation is true for each particular value of w_x', so that on summing for all electrons, we have

$$\mathrm{P} = \beta\left(1 - \frac{vw_x}{c^2}\right)\rho - \frac{\beta v}{c^2} j_x, \tag{6}$$

an equation identical with Minkowski's and Mirimanoff's.

Similarly the contributions of the same group of electrons to the

current density in the two systems are,

$$\delta j = \frac{\Sigma e(w'-w)}{\delta a},$$

$$\delta J = \frac{\Sigma e(W'-W)}{\delta A},$$

these being vector equations.

Using equations (3) and (5) these give

$$\delta J_x = \frac{\Sigma e\,(w_x'-w_x)}{\delta a\,\beta\left(1-\frac{vw_x}{c^2}\right)} = \frac{\delta j_x}{\beta\left(1-\frac{vw_x}{c^2}\right)} = \beta\left(1+\frac{vW_x}{c^2}\right)\delta j_x,$$

$$\delta J_y = \frac{\Sigma e}{\delta a}\left\{(w_y'-w_y) - \frac{vw_y(w_x'-w_x)}{c^2\left(1-\frac{vw_x}{c^2}\right)}\right\}$$

$$= \delta j_y - \frac{\beta v}{c^2}W_y\delta j_x,$$

$$\delta J_z = \delta j_z - \frac{\beta v}{c^2}W_z\delta j_x.$$

Summing for all electrons, we obtain Minkowski's equation,

$$\left.\begin{aligned} J_x &= \beta j_x\left(1+\frac{vW_x}{c^2}\right)\\ J_y &= j_y - \frac{\beta v W_y}{c^2}j_x\\ J_z &= j_z - \frac{\beta v W_z}{c^2}j_x \end{aligned}\right\}. \qquad (7)$$

The Polarization p and Magnetization m.

So far no distinction has been made between various types of motion of the electrons; but now they must be differentiated.

Consider first the so-called polarization electrons which move with the body, but have an electric moment. Taking δa, δA to be the volumes occupied by such a polarized element, its contributions to the polarization in the two systems are related as follows:

$$\delta P_x = \frac{\Sigma eX}{\delta A},$$

where X is measured relative to the centre of the element.

Using (4) and (5) this becomes

$$\left.\begin{aligned} \delta P_x &= \frac{\Sigma ex}{\delta a} = \delta p_x, \ \dots \\ \text{Similarly,} \quad \delta P_y &= \frac{\Sigma eY}{\delta A} \\ &= \frac{\Sigma e}{\delta a}\left\{\beta\left(1-\frac{vw_x}{c^2}\right)y+\frac{\beta v w_y x}{c^2}\right\} \\ &= \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_y+\frac{\beta v w_y}{c^2}\,\delta p_x, \ \dots \\ \text{and} \quad \delta P_z &= \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_z+\frac{\beta v w_z}{c^2}\,\delta p_x, \ \dots \end{aligned}\right\}. \qquad (a)$$

Since the polarization electrons have no velocity relative to the body, they contribute nothing to the magnetization in either system.

Consider next the *magnetization electrons*, that is, systems of electrons spinning round centres fixed in the body and having a magnetic moment; possibly also an electric moment in a moving system.

If these have an electric moment, consider the contribution to the polarization

$$\delta P_X = \frac{\Sigma eX}{\delta A} = \frac{\Sigma e}{\delta a}\beta\left(1-\frac{vw_x'}{c^2}\right)X = \frac{\Sigma ex}{\delta a},$$

where we consider in the first place only those electrons having a definite value of w_x' and afterwards effect a summation.

Similarly δP_Y, *i.e.*, $\Sigma eY/\delta A$, becomes, after reduction,

$$\beta\left(1-\frac{vw_x}{c^2}\right)\delta p_y+\frac{\beta v w_y}{c^2}\delta p_x+\frac{\beta v}{c}\,\delta m_z,$$

and likewise $$\delta P_z = \beta\left(1-\frac{vw_x}{c^2}\right)\delta p_z+\frac{\beta v w_z}{c^2}\,\delta p_x-\frac{\beta v}{c}\,\delta m_y.$$

In the same manner for the same group of electrons

$$\delta M_X = \frac{1}{2c}\,\Sigma e\,\{Y(W_Z'-W_Z)-Z(W_Y'-W_Y)\}/\delta A,$$

which, on reduction, becomes

$$\delta m_x-(\beta v W_Y \delta m_y+\beta v W_Z \delta m_z)/c^2,$$

and, similarly, $$\delta M_N = \beta\left(1+\frac{vW_x}{c^2}\right)\delta m_n.$$

From these, by addition, the formulæ of transformation are seen to be exactly those given by Mirimanoff, viz.,

$$\left.\begin{aligned} P_X &= p_x \\ P_Y &= \beta\left(1-\frac{vw_x}{c^2}\right)p_y+\frac{\beta v w_y}{c^2}p_x+\frac{\beta v}{c}m_z \\ P_Z &= \beta\left(1-\frac{vw_x}{c^2}\right)p_z+\frac{\beta v w_z}{c^2}p_x-\frac{\beta v}{c}m_y \end{aligned}\right\}, \tag{8}$$

$$\left.\begin{aligned} M_X &= m_x-\beta v(W_Y\delta m_y+W_Z\delta m_z)/c^2 \\ M_N &= \beta\left(1+\frac{vW_x}{c^2}\right)m_n \end{aligned}\right\}. \tag{9}$$

It is thus verified, by direct method, that the equations of the field as given by Lorentz for moving ponderable bodies are left unchanged by a transformation in which the pairs of vectors (e, b), $(d, h-[pw]/c)$ are correlated in the same manner as were (e, h) for the free æther.*

The constitutive equations now become

$$\left.\begin{aligned} D_X &= \epsilon E_X \\ \left(1-\frac{v^2}{c^2}\right)D_N &= \left(\epsilon-\frac{v^2}{c^2}\right)E_N+\frac{1}{c}\,[v,\ \epsilon B-H]_N \end{aligned}\right\}, \tag{10}$$

$$\mu H = B+(\mu-1)[v,\ E]/c, \tag{11}$$

$$\left.\begin{aligned} J_X &= \sigma E_X/\beta \\ J_N &= \beta\sigma(E_n+[v,\ B]_n/c) \end{aligned}\right\}. \tag{12}$$

The first and last of these appear to be the exact forms of the better known approximate equations

$$D = \epsilon E+(\epsilon-1)[w,\ B]/c,$$

$$J = \sigma(E+[w,\ B]),$$

and reduce to these, if $\mu = 1$ and if v^2/c^2 is neglected.

The middle equation has as yet no experimental corroboration.

*Cf. Mirimanoff, *loc. cit.*

3. *The Mechanical Action of Radiation on a Bounding Surface under the Lorentz-Einstein Transformation.*

M. Planck* in a recent paper has shewn, from thermodynamic considerations, that the pressure of radiation in equilibrium in an enclosure must be independent of the motion of that enclosure, and obtains relations between the energy per unit volume, and the temperature of the radiation when considered at rest and in motion in turn. Inasmuch as energy and pressure are independent of thermodynamics, it seems appropriate to shew here that Planck's results for those quantities are an immediate result of the foregoing transformation.

Taking the expression for the force per unit area on a moving surface,†

$$p' = \frac{1}{8\pi}\{2e'e_\nu + 2h'h_\nu - n(ee' + hh')\},$$

this being a vector equation, we adapt it to the case where the surface is moving in the direction of the axis of x with velocity v.

The equations of the transformation are

$$e'_x = E_X, \quad e'_y = E_Y/\beta, \quad e'_z = E_Z/\beta,$$
$$h'_x = H_X, \quad h'_y = H_Y/\beta, \quad h'_z = H_Z/\beta.$$

Let the element of area upon which p' acts be ds, the normal having direction cosines (l, m, n), and let the transformed area be dS, with normal (L, M, N).

Then $\quad lds = LdS, \quad \beta mds = MdS, \quad \beta nds = NdS.$

Making these substitutions in p' and reducing, we obtain

$$p'_x ds = P_X dS - \frac{v}{4\pi c}[EH]_N dS,$$

where P_X is the component pressure per unit area on dS at rest as given by the electric and magnetic intensities E, H.

Likewise
$$p'_y ds = \frac{1}{\beta} P_Y dS,$$
$$p'_z ds = \frac{1}{\beta} P_Z dS.$$

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* *Annalen der Physik*, XXVI, No. 6 (1908).

† Cf. Abraham, *Theorie der Elektricität*, II, p. 333. Notice that in this expression

$$e' = e + [wh]/c, \quad h' = h - [we]/c,$$

so that e', h' are *not* the electric and magnetic intensities in the Lorentz-Einstein transformation.

Now pass to the consideration of the equilibrium radiation within a cavity in a body. The pressure is normal to the surface and the vector $\frac{c}{4\pi}[EH]$ vanishes at all points.

Thus $$P_X = LP, \quad P_Y = MP, \quad P_Z = NP.$$

Hence $$p'_x ds = LPdS = Plds,$$

$$p'_y ds = \frac{1}{\beta} MPdS = Pmds,$$

$$p'_z ds = \frac{1}{\beta} NPdS = Pnds.$$

Thus the pressure in the moving cavity is in the direction (l, m, n), and is equal to P.

Let the expression for the energy of the field be now similarly treated.

$$\frac{1}{8\pi}(e^2+h^2) = \frac{1}{8\pi}\left\{ E_X^2+H_X^2+\beta^2\left(1+\frac{v^2}{c^2}\right)(E_Y^2+H_Y^2+E_Z^2+H_Z^2) + \frac{4\beta^2 v}{c}(E_Y H_Z - E_Z H_Y) \right\}.$$

In the cavity at rest when the radiation is in equilibrium, the mean values of $(E_X^2+H_X^2)$, $(E_Y^2+H_Y^2)$, $(E_Z^2+H_Z^2)$ are each equal to $\frac{1}{3}(E^2+H^2)$, and that of $(E_Y H_Z - E_Z H_Y)$ is zero, the average being taken over any interval of time very small, but large compared with the periods of the constituent radiation.

Hence, denoting mean values by a stroke,

$$\frac{1}{8\pi}\overline{(e^2+h^2)} = \frac{1}{8\pi}\overline{(E^2+H^2)}\left\{\tfrac{1}{3}+\frac{2(c^2+v^2)}{3(c^2-v^2)}\right\},$$

or $$\bar{\epsilon} = \frac{3c^2+v^2}{3(c^2-v^2)}\bar{E}.$$

In identical manner, Planck's equation

$$\bar{g} = \frac{4v}{3(c^2-v^2)}\bar{E}$$

is obtained.

Finally, $$\bar{p} = \bar{P} = \tfrac{1}{3}\bar{E} = \frac{c^2-v^2}{3c^2+v^2}\bar{\epsilon},$$

$$\bar{g} = \frac{4v}{3c^2+v^2}\bar{\epsilon}.$$

II.

4. The group of conformal transformations in four dimensions referred to in the introduction can be built up of transformations by inversion in the hyperspheres of the space.

To establish the new theorem of relativity it is sufficient to consider a single member of the group, *i.e.*, a single inversion. Let the centre of inversion be taken as origin.

The analysis is best carried out in spherical polar coordinates, so that the geometrical correlation of the two systems is

$$R = \frac{k^2 r}{r^2 - c^2 t^2}, \qquad T = \frac{k^2 t}{r^2 - c^2 t^2},$$

the angular coordinates remaining unaltered.

Geometrical relations arising immediately are first given.

If u, U are the velocities of a moving point and the corresponding point in the other system,

$$U_R = \frac{(r^2 + c^2 t^2)\, u_r - 2c^2 r t}{2rt u_r - (r^2 + c^2 t^2)},$$

$$U_N = -\frac{(r^2 - c^2 t^2)\, u_n}{2rt u_r - r^2 + c^2 t^2}.$$

In particular, if v is the velocity in either system of a point corresponding to a stationary point in the other,

$$v = \frac{2c^2 r t}{r^2 + c^2 t^2},$$

and is a radial velocity.

As in the former transformation,

$$\beta = (1 - v^2/c^2)^{-\frac{1}{2}} = \frac{r^2 + c^2 t^2}{r^2 - c^2 t^2};$$

and, using this, the above equations become

$$U_R = -\frac{u_r - v}{1 - u_r v/c^2},$$

$$\beta U_N = \frac{u_n}{1 - u_r v/c^2}.$$

If we write $k^2 \Lambda = (R^2 - c^2 T^2)$, and consider a *small* volume at rest in

the (r, t) system, we find that to an observer in the (R, T) system its dimensions radially are diminished in the ratio $-1 : \beta\Lambda$, while transversely they are diminished in the ratio $1 : \Lambda$.

Thus a minute sphere, in the one system at rest, is observed in the other to be an oblate spheroid in the other, the axis of revolution being in the direction of the radius R, and the ratio of the axes being $1 : \beta$.

Passing to the field equations of the electron theory, and putting them in terms of polar coordinates, it is found by direct transformation that the following transformation leaves them invariant, λ being written for $(r^2-c^2t^2)/k^2$:

$$E_R = \lambda^2 e_r,$$

$$E_\theta = \lambda^2\beta(-e_\theta+vh_\phi/c),$$

$$E_\phi = \lambda^2\beta(e_\phi-vh_\theta/c),$$

$$H_R = -\lambda^2 h_r,$$

$$H_\theta = \lambda^2\beta(h_\theta+ve_\phi/c),$$

$$H_\phi = \lambda^2\beta(h_\phi-ve_\theta/c),$$

$$\mathrm{P} = -\lambda^3\beta(1-u_r v/c^2)\rho,$$

together with the transformations in velocity and coordinates given above.

It is a consequence of this result that, if e, h, u, ρ are the electric and magnetic intensities, the velocity of convection, and the density of electricity in a given set of phenomena, then another set of phenomena consistent with the fundamental equations is expressed by the quantities E, H, U, P, so obtained if expressed in terms of the variables R and T. The relations between the two sets of quantities are completely reciprocal.

Further, there is exact correspondence of charges. For, if the ratio of a small element of volume $r^2 \sin\theta\delta r\delta\theta\delta\phi$ moving with velocity u, to the corresponding element $R^2 \sin\theta\delta R\delta\theta\delta\phi$ moving with velocity U be calculated, it is found to be

$$-\lambda^3(1-vu_r/c^2),$$

that is P/ρ.

Hence it is impossible for an observer to discriminate between the sequence of electromagnetic phenomena as he knows them and the sequence obtained therefrom by this transformation. So long as his measures of time and space are dependent on electrically constituted

apparatus, the change in the space coordinates will be obscured by the change in the measure of time which will be automatically made.

It may be worth while to give the following forms of the transformation:

$$E_X = k^{-4}(r^2-c^2t^2)\,[(r^2-c^2t^2)\,e_x - 2\,\{(y^2+z^2)\,e_x - xye_y - xze_z\} - 2ct\,(yh_z - zh_y)],\ \ldots,$$

$$H_X = k^{-4}(r^2-c^2t^2)\,[-(r^2-c^2t^2)\,h_x + 2\,\{(y^2+z^2)\,h_x - xyh_y - xzh_z\} - 2ct\,(ye_z - ze_y)],$$

or, in vector notation,

$$E = k^{-4}(r^2-c^2t^2)\,\{(r^2-c^2t^2)\,e + 2\,[r\,[re]] - 2ct\,[rh]\},$$

$$H = k^{-4}(r^2-c^2t^2)\,\{-(r^2-c^2t^2)\,h - 2\,[r\,[rh]] - 2ct\,[re]\}.*$$

Invariants and Constitutive Equations under the Transformation.

5. The theorem of relativity being established, it is possible to deduce from any known set of phenomena another which geometrically is the transformation of it. The values of physical quantities in the new phenomena will be obtained from those in the old by the equations obtained in the last section. The results so obtained are quite analogous to those in § 2, and will be given below. Two quantities that are invariant will be mentioned first.†

The following equation is immediately established:

$$[E_R^2+E_\theta^2+E_\phi^2-H_R^2-H_\theta^2-H_\phi^2] = \lambda^4[e_r^2+e_\theta^2+e_\phi^2-h_r^2-h_\theta^2-h_\phi^2].$$

Thus, if l, L denote the differences of the magnetic energies in the two systems per unit volume,

$$L = \lambda^4 l.$$

Now $$\frac{\partial(R,\,T)}{\partial(r,\,t)} = \frac{1}{\lambda^2} \quad \text{and} \quad \frac{R}{r} = \frac{1}{\lambda^2}.$$

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* Mr. Bateman has suggested to me that the formulæ of transformation for the scalar and vector potentials are:—

$$\Phi = \beta\lambda\,\{va_r/c - \phi\},$$

$$A_R = \beta\lambda\,\{a_r - v\phi/c\},$$

$$A_N = -\lambda a_n.$$

† These invariant equations hold equally in the case of the Lorentz-Einstein transformation. Cf. Planck, *loc. cit.*, in reference to the invariance of the kinetic potential.

Hence $$LR^2 \sin\theta\, dR\, d\theta\, d\phi\, dT = lr^2 \sin\theta\, dr\, d\theta\, d\phi\, dt;$$

and therefore for a closed system, if we integrate through the whole volumes and corresponding times,

$$\iint L\, dV\, dT = \iint l\, dv\, dt.$$

Thus, if the electrodynamic equations are based on the principle of least action, the action being the time integral of the difference of the magnetic aad electric energies of the system, we may say that the *action* is invariant under the transformation.

The other invariant is expressed in the equation

$$E_R H_R + E_\theta H_\theta + E_\phi H_\phi = -\lambda^4 (e_r h_r + e_\theta h_\theta + e_\phi h_\phi).$$

Thus, if the electric and magnetic vectors are at right angles at any point in the one system, they are also at right angles at the corresponding point and instant in the other.

Consider now the conditions which hold at a reflecting surface which is at rest in the (r, t) system, viz.,

$$\frac{e_r}{l} = \frac{e_\theta}{m} = \frac{e_\phi}{n},$$

$$l h_r + m h_\theta + n h_\phi = 0,$$

where (l, m, n) are the direction cosines of the normal to the surface referred to the directions of (r, θ, ϕ) respectively.

The mechanical force per unit charge moving with the surface in the transformed system is

$$F = \left(E_r,\ E_\theta - \frac{v}{c} H_\phi,\ E_\phi + \frac{v}{c} H_\theta\right) = \lambda^2 \left(e_r,\ -\frac{e_\theta}{\beta},\ -\frac{e_\phi}{\beta}\right).$$

Hence $$\frac{F_r}{l} = \frac{F_\theta}{-m/\beta} = \frac{F_\phi}{-n/\beta}.$$

But if we consider corresponding areas ds, dS and their projections, we obtain

$$\frac{l\, ds}{L\, dS} = \frac{r^2}{R^2} = \lambda^2,$$

$$\frac{m\, ds}{M\, dS} = \frac{n\, ds}{N\, dS} = \frac{r\, \delta r}{R\, \delta R},$$

δr being an element of length at rest.

Hence $$\frac{m\,ds}{M\,dS} = \frac{n\,ds}{N\,dS} = -\lambda^2\beta,$$

so that $$\frac{l}{L} = \frac{-m/\beta}{M} = \frac{-n/\beta}{N} = \frac{1}{\kappa} \text{ (say)};$$

and therefore $$\frac{F_R}{L} = \frac{F_\theta}{M} = \frac{F_\phi}{N}.$$

Again, $$LH_r + MH_\theta + NH_\phi = -\lambda^2\kappa(lh_r + mh_\theta + nh_\phi),$$

since $$me_\phi - ne_\theta = 0.$$

Hence $$LH_r + MH_\theta + NH_\phi = 0.$$

Thus the conditions for a moving reflecting surface are satisfied at the transformed surface.

Thus a perfect reflector transforms into a perfect reflector.

6. There is no difficulty in adapting the analysis of § 2 to the new transformation if the following form of it is noticed.

The differential elements in the transformation are connected by the equation

$$\delta R = \lambda^{-1}\beta(-\delta r + v\,\delta t),$$

$$\delta T = \lambda^{-1}\beta\left(\delta t - \frac{v}{c^2}\delta r\right),$$

where, as before,

$$\lambda = (r^2 - c^2t^2)/k^2, \qquad v = 2c^2rt/(r^2 + c^2t^2), \qquad \beta = (1 - v^2/c^2)^{-\frac{1}{2}}.$$

Thus within a small element of volume the space time coordinates are changed by a transformation of the same form as the Lorentz-Einstein, save for the magnification factor λ^{-1}, and a difference of sign, which itself disappears in a sequence of two such transformations, *e.g.*, in an infinitesimal transformation of the group.

The same is true of the fundamental equations connecting the magnetic and electric intensities in the two systems.

If, therefore, we confine our attention to a portion of matter contained within a volume which is small, but large enough to allow of the process of averaging commonly employed in molecular physics, we shall obtain results similar to those of the last section. It will be enough here to give the equations without the analysis, referring to the corresponding equations of previous sections.

The velocity of a moving point

$$\left.\begin{aligned} W_R &= -\frac{w_r - v}{1 - \frac{v w_r}{c^2}} \\ W_N &= \frac{w_n}{\beta\left(1 - \frac{v w_r}{c^2}\right)} \end{aligned}\right\}, \tag{2'}$$

where W_N is the velocity in any direction perpendicular to R.

The relative velocity of two moving points

$$\left.\begin{aligned} W'_R - W_R &= -\frac{w'_r - w_r}{\beta^2\left(1 - \frac{v w_r}{c^2}\right)\left(1 - \frac{v w'_r}{c^2}\right)} \\ W'_N - W_N &= \frac{w'_n - w_n}{\beta\left(1 - \frac{v w_r}{c^2}\right)} - \frac{w'_n v (w'_r - w_r)}{\beta c^2\left(1 - \frac{v w_r}{c^2}\right)\left(1 - \frac{v w'_r}{c^2}\right)} \end{aligned}\right\}. \tag{4'}$$

The distance between two points (r, t), (r', t') at the time T corresponding to (r, t),

$$\left.\begin{aligned} \lambda \delta R &= -\frac{\delta r}{\beta(1 - v w'_r/c^2)} \\ \lambda \delta N &= \delta n + \frac{v w'_n \delta r}{c^2(1 - v w'_r/c^2)} \end{aligned}\right\}. \tag{3'}$$

The element of volume $\lambda^3 \delta A = -\frac{\delta a}{\beta(1 - v w_r/c^2)}$. (5')

The density of free electricity

$$-\lambda^{-3}\rho = \beta(1 - v w_r/c^2)\,\rho - \beta v j_r/c^2. \tag{6'}$$

The current density

$$\left.\begin{aligned} \lambda^{-3} J_R &= \beta(1 + v W_R/c^2)\, j_r \\ -\lambda^{-3} J_N &= j_n - \beta v W_N j_r/c^2 \end{aligned}\right\}. \tag{7'}$$

The polarization

$$\left.\begin{aligned} \lambda^{-2} P_R &= p_r \\ \lambda^{-2} P_\theta &= -\beta(1 - v w_r/c^2)\, p_\theta - \beta v w_\theta p_r/c^2 - \beta v m_\phi/c \\ \lambda^{-2} P_\phi &= -\beta(1 - v w_r/c^2)\, p_\phi - \beta v w_\phi p_r/c^2 + \beta v m_\theta/c \end{aligned}\right\}. \tag{8'}$$

The magnetization

$$\left.\begin{aligned} -\lambda^{-2} M_R &= m_r - \beta v\,(W_\theta m_\theta + W_\phi m_\phi)/c^2 \\ \lambda^{-2} M_N &= \beta(1 + v W_R/c^2)\, m_n \end{aligned}\right\}. \qquad (9')$$

7. The question now arises as to whether in a *ponderable* body the transformations above given between (E, H) and (e, h) stand as they are, or whether they should be really between (E, B) and (e, b). Examination shews that relativity is fully maintained if the latter is adopted in accordance with Mirimanoff's paper on the Lorentz-Einstein transformation. We know, changing (H, h) into (B, b) that the transformation obtained leaves invariant the equations

$$-\frac{1}{c}\frac{\partial b}{\partial t} = \text{curl } e,$$

$$0 = \text{div } b.$$

The transformation being now

$$\left.\begin{aligned} E_R &= \lambda^2 e_r, & B_R &= -\lambda^2 b_r \\ E_\theta &= \lambda^2\beta(-e_\theta + v b_\phi/c), & B_\theta &= \lambda^2\beta(b_\theta + v e_\phi/c) \\ E_\phi &= \lambda^2\beta(-e_\phi - v b_\theta/c), & B_\phi &= \lambda^2\beta(b_\phi - v e_\theta/c) \end{aligned}\right\}, \qquad (10')$$

with the use of (8′), (9′), the following equations are deduced:

$$\left.\begin{aligned} D_R &= \lambda^2 d_r, & Q_R &= -\lambda^2 q_r \\ D_\theta &= \lambda^2\beta(-d_\theta + v q_\phi/c), & Q_\theta &= \lambda^2\beta(q_\theta + v d_\phi/c) \\ D_\phi &= \lambda^2\beta(d_\phi - v q_\theta/c), & Q_\phi &= \lambda^2\beta(q_\phi - v d_\theta/c) \end{aligned}\right\}, \qquad (11')$$

where the new quantities are defined by

$$D = E + P, \qquad d = e + p,$$

$$B = M + H, \qquad b = m + h,$$

$$Q = H - \frac{1}{c}[PU], \quad q = h - \frac{1}{c}[pu].$$

These together with (6′), (7′) will, by comparison with the transformation in its original form, leave unchanged the equations

$$\frac{1}{c}\left(\frac{\partial d}{\partial t} + \rho u\right) = \text{curl } q,$$

$$\rho = \text{div } d.$$

These equations expanded become the second and third of Lorentz' equations for ponderable bodies.

If, now, at a point $(xyzt)$ the velocity u vanishes, and we assume that consequently $d = \epsilon e$, $b = \mu h$, we at once obtain from (7′), (10′), (11′) equations identical with (10), (11), (12), § 2.

Thus, if we assume those equations together with Lorentz' fundamental equations, to be the correct scheme of the electrodynamics of moving bodies, we arrive at the conclusion that that scheme is invariant, not only under the extension of the transformation of Lorentz and Einstein given by Mirimanoff, but also under the transformation obtained by inversion in the four dimensional space here developed, and therefore also under a transformation obtained by combining any number of such operations.

The Transformation of the Conditions holding at a Surface of Discontinuity.

8. In a paper in the *Annalen der Physik*,* starting from the equation

$$\frac{1}{c}\left(\frac{\partial D}{\partial t} + S\right) = \operatorname{curl} H,$$

Einstein and Laub deduce that the conditions to be satisfied at a surface of discontinuity in the material media are that the following quantities must be continuous in crossing the surface :

$$D_n, \quad \left\{H + \frac{1}{c}[DW]\right\}_{\bar{n}}, \quad B_n, \quad \left\{E - \frac{1}{c}[BW]\right\}_{\bar{n}},$$

n denoting the direction of the normal, and $\bar{n}$ of a tangent line to the surface. S, as in Minkowski's paper, denotes the sum of the conduction and convection currents.

If we apply the extended transformation to these conditions, we find that they are not conserved. But if we take the Lorentz equation which is obtained from that above by the substitution of the vector

$$Q = H - \frac{1}{c}[PW] \quad \text{for} \quad Q,$$

the conditions are that

$$D_n, \quad \left\{Q + \frac{1}{c}[DW]\right\}_{\bar{n}}, \quad B_n, \quad \left\{E - \frac{1}{c}[BW]\right\}_{\bar{n}}$$

* A. Einstein und J. Laub, 1909, 2, p. 446.

shall be continuous, provided there is no true electricity on the surface of separation.

There is no difficulty in proving that these conditions are conserved under Mirimanoff's extended transformation.

It will be proved now that they are also conserved under the transformation by inversion, and therefore by the most general transformation of the whole group which leaves the fundamental equations invariant.

From the equations (13′), we obtain if we write

$$H' = H + \frac{1}{c}[EW] = Q + \frac{1}{c}[DW],$$

$$\left.\begin{aligned} H'_R &= -\frac{\lambda^2}{1-\frac{vw_r}{c^2}}\left\{h'_r - \frac{v}{c^2}(w_r h'_r + w_\theta h'_\theta + w_\phi h'_\phi)\right\} \\ H'_\theta &= \frac{\lambda^2}{\beta\left(1-\frac{vw_r}{c^2}\right)} h'_\theta \\ H'_\phi &= \frac{\lambda^2}{\beta\left(1-\frac{vw_r}{c^2}\right)} h'_\phi \end{aligned}\right\}. \qquad (14')$$

If we consider an element of length $(\delta s, \delta S)$ of which the components are $(\delta r, \delta n)$, $(\delta R, \delta N)$ in the two systems respectively, δn, δN being the components perpendicular to r, R respectively, we have equations analogous to (4),

$$\delta R = -\frac{\lambda^{-1}\delta r}{\beta\left(1-\frac{vw_r}{c^2}\right)},$$

$$\delta N = \lambda^{-1}\left\{\delta n + \frac{vw_n\delta_r}{c^2\left(1-\frac{vw_r}{c^2}\right)}\right\}.$$

Hence

$$\begin{aligned} &H'_R\delta R + H'_N\delta N \\ &= \frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)}\left\{\frac{h'_r\delta_r}{1-\frac{vw_r}{c^2}} + h'_n\delta_n - \frac{v(w_r h'_r + w_n h'_n)}{c^2\left(1-\frac{vw_r}{c^2}\right)} + \frac{vw_n h'_n}{c^2\left(1-\frac{vw_r}{c^2}\right)}\right\} \\ &= \frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)}\{h'_r\delta_r + h'_r\delta_n\}, \end{aligned}$$

or
$$H'_S \delta S = \frac{\lambda}{\beta\left(1-\frac{vw_r}{c^2}\right)} h'_s \delta s.$$

Thus, since $\delta S/\delta s$ does not depend on the physical properties of the medium, and is consequently continuous in crossing the surface of discontinuity, H' will be continuous in those directions in which h' is continuous, and discontinuous in those directions in which h' is discontinuous.

Exactly the same is to be proved of $E' = E - \frac{1}{c}[BW]$ by means of (10′).

Taking the component of D perpendicular to the same element, we have

$$D_R \delta N - D_N \delta R = \lambda d_r \left\{ \delta_n + \frac{v w_n \delta_r}{c^2\left(1-\frac{vw_r}{c^2}\right)} \right\}$$
$$- \frac{\lambda \delta r}{1-\frac{vw_r}{c^2}} \left\{ d_n\left(1-\frac{vw_r}{c^2}\right) + \frac{v}{c^2} d_r w_n - \frac{v}{c} h'_{n_1} \right\},$$

where h'_{n_1} denotes the component of h' in the direction perpendicular to r and n.

Thus, on simplifying,

$$D_R \delta N - D_N \delta R = \lambda(d_r \delta_n - d_n \delta r) - \frac{\lambda v \delta r\, h'_{n_1}}{c\left(1-\frac{vw_r}{c^2}\right)}.$$

Thus D will be continuous in any direction in which d is continuous, provided the component of h' perpendicular to that direction is continuous.

Hence, with what has just been proved as to the continuity of h', it follows that the condition of the continuity of the normal component of d is conserved; and in the same manner it may be shewn of the normal component of b.

The Pressure, Energy, and Momentum, under the Generalized Conformal Transformation.

9. Let the physical quantities considered in § 3 be now treated in the light of the inversion in four-dimensional space.

If the elements ds, dS are corresponding elements of area of which dS is at rest in the (R, T) system, and (l, m, n), (L, M, N) are the direction cosines of the respective normals, by considering the projections of the

areas, we have

$$\frac{l\,ds}{L\,dS} = \frac{r^2}{R^2} = \frac{1}{\Lambda^2},$$

where

$$\Lambda = \frac{r^2 - c^2t^2}{k^2},$$

$$\frac{n\,ds}{N\,dS} = \frac{m\,ds}{M\,dS} = \frac{r\,\delta r}{R\,\delta R},$$

in which δr, δR are corresponding elements of a radius, δR being at rest in the (R, T) system.

Hence

$$\frac{\delta r}{\delta R} = -\frac{1}{\Lambda\beta},$$

and

$$\frac{m\,ds}{M\,dS} = \frac{n\,ds}{N\,dS} = -\frac{1}{\Lambda^2\beta}.$$

Using these and the final form of the transformation of the electric vectors in § 3, the following equations are obtained exactly as in the last section:

$$p_r\,ds = \Lambda^2 \left\{ P_r - \frac{v}{4\pi c}[EH]_v \right\} dS,$$

$$p_\theta\,ds = -\Lambda^2 P_\theta\,dS/\beta,$$

$$p_\phi\,ds = -\Lambda^2 P_\phi\,dS/\beta,$$

p and P being the mechanical forces per unit area according as the element is observed in the (r, t) or (R, T) system.

If the element dS belongs to the wall of a region in which there is radiation in equilibrium $[EH]_v$ is zero as before if the time-average be taken, and P is normal to dS.

Thus

$$p_r\,ds = \Lambda^2 PL\,dS = \Lambda^4 Pl\,ds,$$

$$p_\theta\,ds = -\frac{\Lambda^2}{\beta} PM\,dS = \Lambda^4 Pm\,ds,$$

$$p_\phi\,ds = -\frac{\Lambda^2}{\beta} PN\,dS = \Lambda^4 Pn\,ds.$$

Thus p is normal to ds and is equal to $\Lambda^4 P$.

For the energy the transformation gives

$$\epsilon = \frac{1}{8\pi}\{e^2 + h^2\}$$

$$= \frac{\Lambda^4}{8\pi}\left\{ E_R^2 + H_R^2 + \beta^2\left(1 + \frac{v^2}{c^2}\right)(E_\theta^2 + E_\phi^2 + H_\theta^2 + H_\phi^2) - \frac{4v\beta^2}{c}(E_\theta H_\phi - E_\phi H_\theta) \right\},$$

so that for the radiation in equilibrium

$$\epsilon = \frac{\Lambda^4}{3} E \left\{ 1+2\frac{c^2+v^2}{c^2-v^2} \right\} = \Lambda^4 \frac{3c^2+v^2}{3(c^2-v^2)} E.$$

Thus
$$p = \Lambda^4 P = \tfrac{1}{3}\Lambda^4 E = \frac{c^2-v^2}{3c^2+v^2}\epsilon.$$

This is the same equation as that obtained in § 3. Exactly in the same way for the momentum is found the result

$$g = \tfrac{4}{3}\frac{\Lambda^4 v}{c^2-v^2} E = \frac{4v}{3c^2+v^2}\epsilon.$$

ON TERM-BY-TERM INTEGRATION OF OSCILLATING SERIES

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1. In a previous paper published in these *Proceedings** I introduced a direct method of considering oscillating series of functions. The success of the method there adopted depended essentially on the remarkable way in which certain of the properties of a continuous function are distributed among the two wider classes of upper and lower semi-continuous functions respectively. It was this fact that rendered it possible to enunciate and prove theorems about the upper and lower functions of an oscillating series precisely corresponding to known ones for the sum-function of a convergent series. Moreover, with the introduction of the peak and chasm functions, by myself, in an earlier paper† on non-uniform convergence, it became possible to carry over not only the notion of non-uniformity of convergence, but the analytical machinery itself, to the discussion of the modes of oscillation of non-converging series.

One great advantage of the introduction of the peak and chasm functions, even when employed for the purpose of discussing converging series only, is that they enable us to distinguish both what goes on on the right and on the left, both above and below. This advantage was not exploited to the full in either of the two papers above referred to. In the present paper it will be found that these distinctions are fundamental. As regards what goes on above and below, we have interesting results with respect to each independently of the other, results therefore which from their very nature are no longer mere generalizations of theorems already known for converging series. As regards the distinction of right and left, everything turns on this in the present paper: it is more especially with the right- and left-handed peak and chasm functions that we work, and not the peak

* "On Oscillating Successions of Continuous Functions," *Proc. London Math. Soc.* (1908), Ser. 2, Vol. 6, pp. 298–328.

† "On Uniform and non-Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," *Proc. London Math. Soc.* (1907), Ser. 2, Vol. 6, pp. 29–52.

and chasm functions *par excellence*, with functions, in short, whose behaviour is known on one side only.*

For the rest the importance of the distinction of right and left appears in the fundamental theorem with which the paper begins. Just as the property of a continuous function that it assumes its upper and lower bounds is divided between the upper and lower semi-continuous functions, so in the fundamental theorem in question we see that the property a continuous function has of assuming in every interval every value between its values at the ends of the interval is passed on to the functions which are lower semi-continuous on the left and upper semi-continuous on the right, when the value at the left-hand end-point is less than that at the right-hand end-point and *vice versa*, while, on the other hand, the points at which such a function is $\leqslant k$, and $\geqslant k$, form in the first class of functions a set closed on the right and closed on the left respectively, and *vice versa* for the second class of functions.

It will be found that these functions, which are at every point lower semi-continuous on one side and upper semi-continuous on the other, naturally arise in the theory of integration of the series we are considering.

The results obtained take, in the first instance, the form of inequalities, not of equalities as in the case of converging series, and thus are not directly available for purposes of calculation. In estimating the value, however, of results of this kind from the point of view of their utility in other parts of analysis, it should be borne in mind that it frequently happens that we are presented with, or confronted by, a series or an integral in the course of our work which we may, or may not, know to be converging, and which may, or may not, happen to be known to be summable in the Cesàro or Borel sense. In such a case we should be landed in an *impasse*, if our only machinery consisted of theorems which assumed the convergence of the series and integrals. That the mathematician who uses series as a mere tool will not, in the ordinary course, come across series or integrals which do not converge, is as much and as little pertinent to the question of their utility as the fact that a man who computes builder's quantities does not come across complex numbers, bears on the question of the value of these numbers for mathematical purposes.

In this connexion it should be noted that *we also adopt as a matter of course the generalised definition of integral, first given by Lebesgue.* We

* It follows from a result of mine that their behaviour is also known on the other side, except possibly at a countable set of points, see footnote to § 2.

do not in this way add to the complexity of the theory, but the reverse. On the other hand *we do not specially consider the possibility of the series oscillating more than finitely.* No fresh ideas would present themselves did we do so, and there is no difficulty in following out the consequences of allowing this possibility to present itself.

The first main result of the present paper refers to *the nature of the oscillation of the series obtained by term-by-term integration of an oscillating series of functions, and to the nature of the upper and lower functions.*

Denoting by U and L the upper and lower functions of the integrated series, and by $s_n(x)$ the sum of the first n terms of the original series, we have:

If the point P is such that in its right-hand neighbourhood $s_n(x)$, regarded as a function of x and and n, is bounded above,	then the integral series oscillates uniformly above on the right at P, so that, in particular, $U(x)$ is upper semi-continuous on the right at P;
if P is such that in its left-hand neighbourhood $s_n(x)$ is bounded above,	then the integral series oscillates uniformly below on the left at P, so that $L(x)$ is lower semi-continuous on the left at P;
if P is such that in its right-hand neighbourhood $s_n(x)$ is bounded below,	then the integral series oscillates uniformly below on the right at P, so that $L(x)$ is lower semi-continuous on the right at P;
if P is such that in its left-hand neighbourhood $s_n(x)$ is bounded below,	then the integral series oscillates uniformly above on the left at P, so that $U(x)$ is upper semi-continuous on the left at P.

The second main result of the paper refers to *the error made in taking the upper (lower) function of the integral series as representing the integral of the upper (lower) function of the original series.* Denoting these latter integrals by H and G, we have

$$U \leqslant H,$$

provided $s_n(x)$ is bounded above, while if $s_n(x)$ is bounded below,

$$G \leqslant L,$$

the error in each case increasing as x increases.

Moreover, the results still hold when points in whose neighbourhood respectively $s_n(x)$ is unbounded above or below, exist, provided they are countable in number, and provided further L and U have the characteristics which they would have if such points did not exist, viz., if U is upper semi-continuous on the right and lower semi-continuous on the left, and L is lower semi-continuous on the right and upper semi-continuous on the left.

This extended result is obtained by applying the following extension of Scheeffer's theorem, which is here proved :—

If a function which is

lower semi-continuous on the left,

and upper semi-continuous on the right,

have one of its derivates everywhere less than or equal to the corresponding derivate of a function which is

upper semi-continuous on the left,

and lower semi-continuous on the right,

except possibly at a countable set of points, the excess of the first function over the second function is a monotone non-increasing function of x throughout the interval.

It will be noted that the results given have been stated only for series, but corresponding results are, of course, true for integrals.

2. The special case *when the given series converges* is, of course, of special interest: in this case G and H coincide, but L and U do not necessarily do so, so that *the integral series need not converge.*

Whether or not the original series converge, if its partial summations are bounded above (below), the integral series, if it converges, represents a function which is upper semi-continuous on one side and lower semi-continuous on the other, and therefore is continuous except possibly at a countable set of points ; while, if the infinite peaks of the original series are all on the right (left), and the infinite chasms on the left (right), the integral series, if it converges, represents an upper (lower) semi-continuous function.

As regards the integration theorem, we may note that if both the original series and the integral series converge, provided the points of infinite non-uniform convergence of the original series are countable,

term-by-term integration is allowable only if the sum of the integral series is a continuous function. If, for example, it is only

lower semi-continuous on the right,

and upper semi-continuous on the left,

its value will, in general, be too great; while, if it is

upper semi-continuous on the right,

and lower semi-continuous on the left,

its value will be too small.

If, on the other hand, only the original series is known to converge, and not the integral series, we shall have, provided the points of infinite non-uniform convergence of the original series are countable, *the integral of the sum-function of the original series* $= L$, *if* L *is continuous,* $= U$, *if* U *is continuous;* so that, if L and U are both continuous, the integral series is bound to converge, and term-by-term integration is allowable. The same is true if U is known to be

lower semi-continuous on the left,

and upper semi-continuous on the right,

while L is known to be

upper semi-continuous on the left,

and lower semi-continuous on the right,

so that again L and U would be equal and continuous.

Other results follow if we suppose the original series not to converge, but the integral series to do so.

3. THEOREM 1.—*If a function* $f(x)$ *is, at every point of a closed interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if the value at the left-hand end-point a *is less than the value at the right-hand end-point* b, *then* $f(x)$ *assumes* ALL *values between the two values in question.*

Let q be any value between $f(a)$ and $f(b)$ non-inclusive.

Since $f(x)$ is lower semi-continuous on the left, the points at which

$$f(x) \leqslant q$$

form a set closed on the right. Therefore there is a point at which this

inequality holds, and which bounds the whole set on the right, and this point is not b, since q is less than $f(b)$; let it therefore be denoted by x_q.

It follows that at every point to the right of x_q,

$$f(x) > q,$$

so that all the right-hand limits of $f(x)$ at q are $\geqslant q$. But $f(x)$ is upper semi-continuous on the right, hence

$$f(x_q) \geqslant q;$$

but we already had $f(x_q) \leqslant q,$

hence $f(x_q) = q,$

which shews that the value q is assumed, and so proves the lemma.

Note.—Bearing in mind the fact that there is no distinction of right and left in the nature of the discontinuities of a function except at most at a countable set of points,* we see that the type of function here considered is continuous except at a countable set of points.

4. THEOREM 2.—*If $f(x)$ is, at every point of a closed interval (a, b),*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if the value at the left-hand end-point a is less than the value at the right-hand end-point b, then the points at which any one of the four derivates of $f(x)$ is positive (not zero) have the potency of the continuum.

Let A and B be the points of the locus

$$y = f(x),$$

corresponding to the values a and b of x.

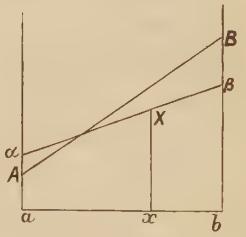

FIG. 1.

Take any point α on the ordinate of A, higher than A and lower than B, and any point β on the ordinate of B, higher than α but lower than B.

Then the intercept between this line and the locus is positive at $x = b$, and negative at $x = a$; also, since it only differs from $f(x)$ by a linear function of x, it is a function of the same type as f itself. Hence, by the preceding theorem, it assumes the value zero between a and b, and the set of these zeros

* Cf. W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," *Quar. Jour.*, Vol. XXXIX, 1907, pp. 67–83; cf. also W. H. Young, "Sulle due funzioni a più valori costituite dai limiti d'una funzione d'una variabile reale a destra e a sinistra di ciascun punto," *Rend. d. R. Acc. dei Lincei*, 1908. Vol. XVII, Serie 5, pp. 582–7.

possesses an extreme right-hand point x, which also bounds on the right the points at which the intercept is $\leqslant 0$. Thus the locus cuts the line $\alpha\beta$ at the corresponding point X, and to the right of this point lies above that line. It follows that the limiting directions of chords of the locus through X and on the right of it are at least as steep as $\alpha\beta$, so that the right-hand derivates of $f(x)$ at this point x are positive.

Now it is evident that, corresponding to each different choice of the point β, we get a different point x, since X cannot lie on more than one line through α. Since the point β was any point on the ordinate of B, higher than α and lower than B, this shews that the potency of the points x, and therefore of the points at which either right-hand derivate is positive, is that of the continuum.

On the other hand, the zeros in question possess an extreme left-hand point x', which also bounds on the left the points at which the intercept is $\geqslant 0$. It follows that, at the corresponding point X' of the locus, the limiting directions on the left are at least as steep as $\alpha\beta$, so that the left-hand derivates of $f(x)$ at this point are positive. As before, the potency of those points x' is that of the continuum, so that the same is true of the points at which either left-hand derivate is positive.

COR.—*If $f(x)$ is, at every point of a closed interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if one of the four derivates is known to be everywhere $\leqslant 0$, except at a countable set of points, then $f(x)$ is a monotone non-increasing function of x throughout the interval, so that at the exceptional points also the derivates are $\leqslant 0$.

5. The two preceding theorems have, of course, their correlatives interchanging greater and less. These theorems, which are stated below, result most simply from those already proved by putting

$$f(x) = -F(x).$$

THEOREM 1′.—*If a function $f(x)$ is, at every point of a closed interval,*

upper semi-continuous on the left,

and lower semi-continuous on the right,

and if the value at the left-hand end-point is greater than the value at

the right-hand end-point, then $f(x)$ assumes all values between the two values in question.

THEOREM 2'.—*Under the same hypothesis as in the preceding theorem, the points at which any one of the four derivates of $f(x)$ is negative have the potency of the continuum.*

COR.—*If $f(x)$ is, at every point of a closed interval,*

upper semi-continuous on the left,

and lower semi-continuous on the right,

and if one of the four derivates is known to be everywhere $\geqslant 0$, except at a countable set of points, then $f(x)$ is a monotone non-decreasing function of x throughout the interval, so that at the exceptional points also the derivates are $\geqslant 0$.

6. An immediate consequence of the results of the preceding articles is the following :—

The set of points, if any, at which any one of the four derivates of a continuous function is $> a$, and the set of points, if any, at which it is $< a$, have for all values of a each of them the potency of the continuum.

It will be noted that this theorem does not demand that the derivates should be bounded or even finite, and that therefore in every case the upper and lower bounds of the derivates are, whether these bounds be finite or infinite, unaltered, if in determining them we omit the derivates at a countable set of points.

7. THEOREM 3 (Extension of Scheeffer's Theorem).—*If $g(x)$ and $h(x)$ be two functions, of which the first is, throughout a given interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and the latter is upper semi-continuous on the left,

and lower semi-continuous on the right,

and if one of the four derivates of $g(x)$ be everywhere less than the corresponding derivate of $h(x)$, or equal to it and finite, except possibly at a countable set of points, then

$$g(x) - h(x),$$

is a monotone non-increasing function of x throughout the interval.

Put $$f(x) = g(x) - h(x),$$

so that $f(x)$ is lower semi-continuous on the left,

and upper semi-continuous on the right.

Then, since denoting by $f^+(x)$ and $f_+(x)$ the upper and lower right-hand derivates of $f(x)$,

$$f_+(x) \leqslant g_+(x) - h_+(x),$$

and also $$\leqslant g^+(x) - h^+(x),$$

with similar inequalities on the left, we see that, whether it is the upper or lower derivates alluded to in the statement of the theorem, we have, in either case,

$$f_+(x) \leqslant 0,$$

except at a doubtful countable set, whence the result follows by Theorem 2, Cor.

8. LEMMA.—*If $S(x)$ denote the integral (generalised or Lebesgue integral) of $s(x)$, the upper and lower bounds of the derivates of $S(x)$ in any interval lie between the upper and lower bounds of $s(x)$ in the same interval*

For $$\frac{S(x+h) - S(x)}{h} = \frac{1}{h}\int_x^{x+h} s(x)\,dx,$$

so that, if B is the upper bound of $s(x)$ in the interval considered, the left-hand side of the preceding equation is less than or equal to the right-hand side when we replace $s(x)$ by B. That is,

$$\frac{S(x+h) - S(x)}{h} \leqslant B.$$

Hence the upper bound of the derivates of $S(x)$, being the same as that of the incrementary ratio on the left of the preceding inequality, is less than or equal to B. This proves the statement about the upper bounds, and similar reasoning proves the statement about the lower bounds.

COR.—*Hence, for each fixed value of n, the upper bound of $s_n(x)$ is greater than or equal to the upper bound of the derivates of $S_n(x)$.*

Hence also $s_n(x)$ regarded as a function of x and n has necessarily an upper bound which is not less than that of the derivates of $S_n(x)$, when these derivates are regarded as functions of x and n.

9. THEOREM 4.—*If a sequence of functions of a single variable x, $s_1, s_2, \ldots$ is such that $s_n(x)$ regarded as a function of n and x for all values*

of x in a certain closed interval (a, b), and all positive integral values of n has a finite upper bound, then the corresponding sequence of integrals, taken from a to x, viz., the continuous functions $S_1, S_2, \ldots$ oscillates uniformly above on the right and uniformly below on the left, so that in particular its upper function U is upper semi-continuous on the right, and its lower function L is lower semi-continuous on the left.

A corresponding statement, mutatis mutandis, holds, of course, when $s_n(x)$ has a finite lower bound.

By the definition of the right-hand peak function of $S_1, S_2, \ldots$, it follows that, if P were a point at which it were greater than $U(x)+A$, there would be a sequence of points to the right of P, $x_1, x_2, \ldots$ with P as limiting point, and a corresponding sequence of constantly increasing integers n, such that

$$S_{n_i}(x_i) > U(x)+A. \tag{1}$$

But, from the definition of the upper function $U(x)$, it follows that we can determine an integer m, such that, for all values of $n \geqslant m$,

$$U(x) > S_n(x)-\tfrac{1}{2}A,$$

so that

$$S_{n_i}(x_i) > S\ (x)+\tfrac{1}{2}A\,;$$

and therefore, by the Theorem of the Mean, if applicable*

$$(x_i-x)\frac{d}{dy}S_{n_i}(y) > \tfrac{1}{2}A, \tag{2}$$

at some point $y = y_i$ internal to the interval (x, x_i).

That is, since x_i-x is positive,

$$\frac{d}{dy_i}S_{n_i}(y_i) > \frac{\frac{1}{2}A}{x_i-x},$$

where $y_1, y_2, \ldots$, like $x_1, x_2, \ldots$, form a sequence on the right of x, having x as limit.

If the Theorem of the Mean is not applicable, we only have to interpret the symbol $\dfrac{d}{dy_i}S_{n_i}(y_i)$ to mean one of the derivates of S_{n_i}, and the same holds.†

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* For instance, if $s_n(x)$ is a continuous function of x.

† E. W. Hobson, *Functions of a Real Variable*, § 219, p. 290, Camb. Univ. Press, 1907. W. H. and G. Chisholm Young, "On Derivates and the Theory of the Mean," *Quar. Jour.*, 1908, p. 12, *et alt. loc.*

Hence, in the neighbourhood of the point P this differential coefficient, or derivate, $\frac{d}{dx} S_n(x)$, regarded as a function of n and x, has no finite upper bound, so that, by the lemma, $s_n(x)$, regarded as a function of n and x, has in the neighbourhood of the point P, no finite upper bound, contrary to the data.

Hence for no positive value of A does the inequality (1) hold, so that the right-hand peak function is not greater than the upper function. But the right-hand peak function is never less than the upper function, whence the right-hand peak function is equal to the upper function.

Q. E. D.

Since the right-hand peak function is upper semi-continuous on the right, it follows, in particular, that the same is true of the upper function $U(x)$. Q. E. D.

Similarly, if there were a point at which the chasm function were less than $L(x)-A$, where $L(x)$ is the lower function, we should at such a point have a sequence of n's for which

$$S_{n_i}(x_i) < L(x)-A,$$

whence, as before, we get

$$(x_i-x)\frac{d}{dy_i} S_{n_i}(y_i) < -\tfrac{1}{2}A ;$$

and therefore, bearing in mind that (x_i-x) is negative,

$$\frac{d}{dy_i} S_{n_i}(y_i) > \frac{\frac{1}{2}A}{x-x_i}.$$

Thus, as before, we get to a contradiction, and so prove the remainder of the statement of the theorem. Q. E. D.

COR.—*If the functions $s_n(x)$ are continuous functions of x, and the sequence has no points at which the peak (chasm) function is infinite, the statement in the theorem holds.*

For, by the definition of the peak function, it is the limit, when the interval PQ shrinks up to the point P, of M_Q, where M_Q is the upper limit of $M_{n,Q}$, the upper bound of $s_n(x)$ in the interval PQ. Hence it follows that round each point x we can, if the peak function is always finite, describe an interval throughout which $s_n(x)$ is less than B, if B is greater than the upper bound of the peak function, provided $n \geqslant m$, where m is a certain integer, varying in general with the point P.

By the Heine-Borel theorem a finite number of these intervals suffices to cover the whole closed continuum in which we are working. Choosing

M to be the largest of the corresponding values of m, $s_n(x)$ is less than B, for all integers $n \geqslant M$, and all points x of the closed continuum considered. Hence, since $s_1, s_2, \ldots, s_{M-1}$ are bounded functions, $s_n(x)$, regarded as a function of n and x, has a finite upper bound, and therefore the result of the theorem holds.

10. The preceding theorem deals with uniform oscillation and semi-continuity throughout an interval, but it is clear from the proof that we can enunciate a corresponding theorem which deals with these concepts at a point only.

THEOREM 5.—*If a sequence of functions of a single variable $s_1, s_2, \ldots$, be such that $s_n(x)$, regarded as a function of n and x, has a finite upper double limit at the point P on the right (left), then the corresponding sequence of integrals oscillates uniformly above (below) on the right (left) at the point P.*

COR.—*If the functions $s_n(x)$ are continuous functions of x, and the right (left) hand peak function is finite at P, the sequence of integrals oscillates uniformly above (below) on the right (left) at the point P.*

We have here two alternate statements, and we get two more by assuming at P a finite lower double limit, or chasm function, interchanging the words above and below, viz.,

THEOREM 5′.—*If $s_n(x)$ has a finite lower double limit at the point P on the right (left), then the corresponding sequence of integrals oscillates uniformly below (above) on the right (left) at the point P.*

COR.—*If the functions $s_n(x)$ are continuous, and the right (left) hand chasm function is finite at P, the sequence of integrals oscillates uniformly below (above) on the right (left) at the point P.*

Corresponding to these results, we have in particular the following table, giving the characteristics of the upper and lower functions of the integrals.

When there are no points at which

	$\pi_R = +\infty$,	$\Pi_R = U$ *is upper semi-continuous on right ;*		
,,	$\pi_L = +\infty$,	$X_L = L$,, *lower*	,,	*left ;*
,,	$X_R = -\infty$,	$X_R = L$,, *lower*	,,	*right ;*
,,	$X_L = -\infty$,	$\Pi_L = U$,, *upper*	,,	*left.*

11. THEOREM 6.—*If the partial summations $s_n(x)$, regarded as functions of n and x, have a finite upper (lower) bound, the limits of their integrals are less (greater) than or equal to the integral of the upper (lower) function.*

We proceed to give the proof of the first of these alternative theorems; the proof of the other merely requires the change of "lower" into "upper" and "less" into "greater."

For, denoting by w_n the function whose value at any point is the upper bound of all the functions $s_n(x)$, $s_{n+1}(x)$, ..., we clearly have

$$s_n(x) \leqslant w_n(x), \tag{1}$$

so that $$\int s_n(x)\,dx \leqslant \int w_n(x)\,dx\,;$$

and therefore $$\text{upper } \lim_{n=\infty} \text{ of } \int s_n(x)\,dx \leqslant \text{upper } \lim_{n=\infty} \text{ of } \int w_n(x)\,dx. \tag{2}$$

But the quantities $w_n(x)$ form a monotone decreasing sequence, and their integrals are finite, since $w_n(x)$ is a bounded function of x; for, by (1) the lower bound of w_n is not less than that of s_n and is therefore finite, and, since $s_n(x)$, regarded as a function of n and x, is, by hypothesis, less than a finite quantity, say B, it follows from the definition that $w_n(x)$ is also always less than or equal to B.

Now, the generalised integral of the limit of a monotone sequence is the unique limit of the integrals of the constituent functions, so that

$$\text{Lt}_{n=\infty} \int w_n(x)\,dx = \int u(x)\,dx,$$

since $u(x)$ is the limit of the sequence $w_1(x)$, $w_2(x)$, ...

Hence, by (2), $$\text{upper limit of } \int s_n(x)\,dx \leqslant \int u(x)\,dx,$$

which proves the theorem.*

COR. 1.—*If the functions $s_n(x)$ are continuous, and the sequence has no points at which the peak (chasm) function is infinite positively (negatively), the result of the theorem holds.*

* The following example shews that, without further assumptions, the theorem states the utmost that can be proved. Divide the interval (0, 1) by continued bisection and call the successive intervals d_1, d_2, ... Let $f_n(x) = 1$ at every point of the closed interval d_n, and be zero elsewhere. Here $u(x) = 1$, but $U(x) = L(x) = 0$. Here $f^n(x)$ is discontinuous, but can easily be made continuous by rounding off the end-points of d_n suitably.

Cor. 2.—*If both the peak and the chasm functions are bounded, the limits of the integrals lie between the integrals of the lower and upper functions.*

This statement, which is an immediate result of the above theorem, can be proved independently by a method already employed to prove the corresponding result* in the case of convergent series, using one of my theorems in the theory of sets of points.†

12. In Theorems 4 and 5 we have shewn that the upper function of the integrals $S_n(x)$ is, under the conditions there enumerated, upper semi-continuous on the right. If we suppose further that the upper function of the original sequence has a finite integral,‡ we can go a step further and assert that the upper function U of the integrals is not only upper semi-continuous on the right, but also lower semi-continuous on the left.

We have, in fact, the following theorem :—

Theorem 7.—*If in addition to the condition of Theorem* 4, *that* $s_n(x)$ *has a finite upper bound, we assume that the upper function* $u(x)$ *has a finite integral, then the upper function of the integrals,* $U(x)$, *is lower semi-continuous on the left as well as upper semi-continuous on the right, and the same is true of the lower function* $L(x)$.

If on the other hand the lower function $l(x)$ *has a finite integral, while* $s_n(x)$ *has a finite lower bound, the lower function of the integrals* $L(x)$ *is upper semi-continuous on the left and lower semi-continuous on the left, and the same is true of the upper function.*

If therefore $s_n(x)$ *is bounded both above and below, both* $U(x)$ *and* $L(x)$ *are continuous functions.*

For

$$\text{lower limit of } (A-B) \leqslant \text{upper limit of } A - \text{upper limit of } B \leqslant \text{upper limit of } (A-B),$$

since we can get a limit of $(A-B)$ which is greater (less) than the middle member of this inequality, by proceeding along a sequence which gives for A (or B) its upper limit.

* E. W. Hobson, *Theory of Functions of a Real Variable*, pp. 539 *et seq.*

† *Idem*, § 93, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 25.

‡ If the upper function u had infinite negative discontinuities, as may well be the case even when the sequence oscillates finitely everywhere, and $s_n(x)$ has a finite upper bound, u might not have a finite integral. As to this restriction see, however, a forthcoming paper by the author "On Homogeneous Oscillation of Successions of Functions," where also further results in the theory of the integration of oscillating series will be found.

Hence

$$U(x)-U(x-h) \leqslant \text{upper limit of } \int_{x-h}^{x} s_n(x)\,dx \leqslant \int_{x-h}^{x} u(x)\,dx,$$

the last inequality resulting from the above theorem.

This shews that, as h decreases towards zero, $U(x)-U(x-h)$ has no positive limit, so that $U(x)$ is, as stated, lower semi-continuous on the left.

Again,

$$\text{lower limit of } (A-B) \leqslant \text{lower limit of } A - \text{lower limit of } B$$
$$\leqslant \text{upper limit of } (A-B).$$

Whence

$$L(x+h)-L(x) \leqslant \text{upper limit of } \int_{x}^{x+h} s_n(x)\,dx \leqslant \int_{x}^{x+h} u(x)\,dx,$$

which shews that $L(x)$ is upper semi-continuous on the right. Q.E.D.

Similarly the alternative statements may be proved.

13. In the case *when the functions $s_n(x)$ are continuous*, the points at which there is non-uniform oscillation either above or below form a set of the first category, so that *the points at which the peak and chasm functions are infinite,* which form a closed set, are dense nowhere.*

Hence the closed intervals throughout which the peak and chasm functions are bounded fill up a set of open intervals dense everywhere. It follows at once from the preceding article that *throughout the interior of this set of intervals dense everywhere both L and U are continuous, and the integral series oscillates uniformly both above and below.*

14. THEOREM 8.—*If there is at most a countably infinite number of points in the neighbourhood of which $s_n(x)$, regarded as a function of x and n, has no finite upper bound,† and if further $U(x)$ is the same type of function as it would be if there were no such points,‡ viz., if $U(x)$ be lower semi-continuous on the left and upper semi-continuous on the right everywhere, then*

$$\int u(x)\,dx - U(x)$$

is a positive non-decreasing function of x.

---

* Or $\geqslant k$, or $\leqslant k$.

† In particular this condition is fulfilled if we know that $s_n(x)$, regarded as a function of n and x, has at each point a finite upper double limit on the right or on the left; in other words, if there are no points in the neighbourhood of which $s_n(x)$ is unbounded above on *both* sides.

‡ Supposing $\int u(x)\,dx$ finite.

It is obvious that the countable set of exceptional points form a closed set. Hence, taking any point x not belonging to that set, it is internal to an interval throughout which the conditions of Theorem 7 hold. As in the proof of Theorem 7, we obtain the inequalities

$$\frac{U(x)-U(x-h)}{h} \leqslant \frac{1}{h}\int_{x-h}^{x} u(x)\,dx,$$

and

$$\frac{U(x+h)-U(x)}{h} \leqslant \frac{1}{h}\int_{x}^{x+h} u(x)\,dx,$$

the point $x-h$ and the point $x+h$ both lying in this interval.

Hence at such a point x each derivate of $U(x)$ is less than or equal to the corresponding derivate of $\int u(x)\,dx$.

Hence, by Theorem 3, the required result follows.

The alternative theorem, in which, instead of making the above assumption about $U(x)$, we assume that $L(x)$ is the same type of function as it would be if there were no such exceptional points, states, of course, that*

$$\int u(x)\,dx - L(x)$$

is a positive non-decreasing function of x.

If, on the other hand, we assume that the lower, instead of the upper, bound of $s_n(x)$, regarded as a function of n and x, is finite except at points not having the potency of the continuum, then according as $U(x)$ or $L(x)$ is the same type of function as it would be if there were no exceptional points,† viz., upper semi-continuous on the left and lower semi-continuous on the right everywhere, we get

$$U(x)-\int l(x)\,dx,$$

or

$$L(x)-\int l(x)\,dx$$

is a positive non-decreasing function of x.

Both these possibilities may, of course, take place simultaneously.

* Supposing $\int u(x)\,dx$ finite.

† Supposing $\int l(x)\,dx$ finite.

Combining these theorems, we obtain the following corollary :—

If there is at most a countable number of points in the neighbourhood of which $s_n(x)$, regarded as a function of n and x, has not both its upper and its lower bound finite, and if, further, $U(x)$ or $L(x)$, or both, is the same type of function as it would be if there were no exceptional points, viz., continuous everywhere, then*

$$\int u(x)\,dx - U(x) \quad \text{and} \quad U(x) - \int l(x)\,dx,$$

or

$$\int u(x)\,dx - L(x) \quad \text{and} \quad L(x) - \int l(x)\,dx,$$

or all these, are positive non-decreasing functions of x. Hence, in particular, $U(x)$ or $L(x)$ or both, lie between the integrals of the upper and lower functions $u(x)$ and $l(x)$.

15. Denoting $\int l(x)\,dx$ by G and $\int u(x)\,dx$ by H, we easily see that any one of the following scheme of inequalities may hold :—

$$G < L < U < H,$$
$$G < L < H < U,$$
$$G < H < L < U,$$
$$L < G < U < H,$$
$$L < G < H < U,$$
$$L < U < G < H,$$

where the sign of inequality may anywhere become the sign of equality.

16. From the preceding theorems we at once deduce the following, which, on account of its importance, is here stated as a separate theorem.

THEOREM 9.—*If the points in the neighbourhood of which $s_n(x)$, regarded as a function of x and n, is unbounded, are at most countably infinite, and if $u(x)$ and $l(x)$ differ only at a set of content zero, while $U(x)$ and $L(x)$ are continuous, the integral series converges to the continuous function constituted by the common integral of $l(x)$ and $u(x)$.*

* Supposing $l(x)$ and $u(x)$ to have finite integrals, which will of itself be fulfilled if both bounds are finite.

This result is, of course, a very special case of those we have just given ; it includes, however, all previous results of the kind given by other writers.

We notice, moreover, that in this case, when $u(x)$ and $l(x)$ differ only at a set of content zero, if the points in the neighbourhood of which $s_n(x)$, regarded as a function of x and n, is unbounded, have the potency of the continuum, but those in the neighbourhood of which the upper bound is $+\infty$ are not of this potency, then the result of term-by-term integration, if not what we should wish it to be, is too small. On the other hand, if those points where the lower bound is $-\infty$ are not of the potency of the continuum, the result, if not the desired one, is too great.

It is only when both these types of points have the potency of the continuum that we can assert nothing.

ON THE DISCONTINUITIES OF A FUNCTION OF ONE OR MORE REAL VARIABLES

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1. The modern definition of a function of one or more real variables is of such a general nature that one is inclined at first to imagine that there can be, in the nature of things, little or no connection between the character of the discontinuity of the function at a point and the mode of distribution of the discontinuities. It was with some surprise therefore that I remarked that *the points at which the upper and lower limits of a function of single real variable can differ on the right and on the left are in no case more than countable*, so that, in particular, discontinuities of the first kind, sometimes called "ordinary discontinuities," are relatively speaking exceptional, while a symmetrical discontinuity of the second kind constitutes the normal type.*

Subsequently I perceived this result to be only a particular case of the more general one :—*the limits of a function of a single variable at a point, obtained by all modes of approach to the point on the right, are identical with those obtained by all modes of approach on the left, except at most at a countable set of points.*†

One is always glad to obtain a second proof of a new theorem, especially in the Theory of Sets of Points, where the reasoning is so subtle that almost all who have written on it have at one time or another stumbled. Such a second proof is furnished by the reasoning adopted in the present Note. My main object is, however, the proof of the following theorem, which, it will be seen, constitutes the analogous result for functions of two or more variables.

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* W. H. Young, "Some Results in the Theory of Functions of a Real Variable" (1907) *Report of the British Association*, p. 445 (abstract) : "On the Distinction of Right and Left at Points of Discontinuity," *Quart. Jour. of Math.*, Vol. XXXIX, pp. 67–83.

† W. H. Young, "Sulle due funzioni a più valori costituite dai limiti d'una funzione d'una variabile a destra e a sinistra di ciascun punto" (1908), *Rend. della R. Acc. dei Lincei*, T. XVII, pp. 582-587.

The whole set of limits of a function $f(x)$ of any number of variables, denoted in their ensemble by x, at a point P, may be obtained by different modes of approach to the point P confined to a completely open neighbourhood of pre-assigned form with P as boundary point, except possibly at a certain set of the first category, depending on the particular function and on the pre-assigned form of neighbourhood.*

That this theorem is not a mere generalisation of the corresponding theorem for functions of a single variable will not be surprising if we reflect on one of the fundamental distinctions between the division of a straight line by means of segments and of the plane or higher space by means of compartments. In the former case the end-points of the intervals have the potency of the natural numbers; in the second case, however, the analogous points fill up lines, and have therefore, as also in higher space, the potency of the continuum. It is precisely this fact which is responsible for the modification in the theorem as we pass from the straight line to higher space. That the exceptional points considered in the theorem are in the one-dimensional case countable is, so to speak, an accidental fact; the general fact is that they form a set of the first category.

The following curious result, which seems to me not without importance in the general theory of limits, is an immediate consequence :—*Excepting only at the points of a certain set of the first category, all the limits of $f(x)$, where there are any number of variables x, may be obtained by means of sequences of points tangential at the point to a fixed direction, previously chosen.*

In particular, *if there is a fixed direction tangential to which $f(x)$ has an unique limit at every point, the function is necessarily only pointwise discontinuous.*

It should be noticed that in all these theorems the modes of approach are by means of regions. The main result is capable of the following mode of expression :—*There is symmetry for approach by means of regions at every point except at a set of the first category.* Here, however, it must be remembered that the set of the first category depends not only on the function, but also on the choice of the regions grouped round a point.

Another immediate result is that, *except possibly at a set of the first category, $f(P)$ lies between its upper and lower limits for all modes of approach to the point P in a region of prescribed form with P as boundary*

* For the definition of such a set see, for example, Young's *Theory of Sets of Points*, Camb. Univ. Press (1906), pp. 70, 71.

point. To obtain this from the main result, we require to assume that $f(x)$ *lies between the upper and lower limits of* $f(x)$ *at each point except at a countable set of points.* A proof of this theorem valid for any number of variables x,* is accordingly given in § 3.

2. The reasoning of the present paper depends on the systematic use of the two facts:

(A) *If a number* a *is greater than a number* b, *there is a rational number greater than* b *and less than* a.

(B) *The rational numbers are countable.*

The countable order of the rational numbers may be any conveniently chosen one; for instance, they may be read in columns from the following wedge:—

$$\left.\begin{array}{rrrrrr} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4}, & \frac{1}{5}, \ \ldots \\ 1+\frac{1}{2}, & 1+\frac{1}{3}, & 1+\frac{2}{3}, & 1+\frac{1}{4}, & 1+\frac{3}{4}, & \ldots \\ -1+\frac{1}{2}, & -1+\frac{1}{3}, & -1+\frac{2}{3}, & -1+\frac{1}{4}, & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{array}\right\}. \qquad (1)$$

Here the first row is the usual order of the rational numbers between 0 and 1, in which the denominators form a monotone series, and so do the numerators corresponding to any the same denominator: the succeeding rows are got by adding in succession the various integers, alternately positive and negative. When arranged in this order we shall denote the rational numbers by $N_1, N_2, \ldots, N_r, \ldots$.

3. The principles employed are exhibited in their simplest form in the proof of the following theorem:—

Theorem 1.—*If* $f(x)$ *is a function of any number of variables, denoted in their ensemble by* x, *and* $\phi(x)$ *is its associated upper limiting function,*† *the points at which*

$$\phi(x) < f(x), \qquad (2)$$

if any, form a countable set.

Let S_r denote the set of those points, if any, such that

$$\phi(x) < N_r < f(x),$$

* The original statement and proof of this theorem, given in the paper first quoted, was for one dimension.

† That is, the function whose value at each point is the upper limit of $f(x)$—that is, the lower bound of the upper bound of $f(x)$ at all points except P in an interval or region with P as internal point.

while no rational number of the series (1) with index less than r is greater than $\phi(x)$ and less than $f(x)$. This set S_r is then perfectly determined, and any point P at which

$$\phi(P) < f(P)$$

belongs to one, and one only, of the sets S_r. Thus the whole set of points, if any, at which

$$\phi(x) < f(x) \tag{2}$$

is the sum $S_1+S_2+\ldots+S_r+\ldots$, of these sets, and is therefore countable if S_r is countable for all values of the integer r; this we shall now prove to be the case.

Let P be any point of the set S_r. Then, by the definition of $\phi(P)$, we can find a whole interval or region with P as internal point, such that at every point x of it, except P,

$$f(x) < N_r.$$

so that no point of it, except P, belongs to the set S_r. Thus P is an isolated point of the set S_r. Hence S_r consists entirely of isolated points, and is therefore countable,* which proves the theorem.

4. THEOREM 2.—*If $\theta(P)$ denote the upper limit at a point P of a function $f(x)$ of two or more variables denoted in their ensemble by x, that upper limit being determined with respect to a certain neighbourhood having P as boundary point, e.g., a $(+, +)$-axial neighbourhood,† and $\phi(x)$ is the associated upper limiting function of f, then*

$$\theta(x) = \phi(x), \tag{3}$$

except at a set of points of the first category.

Let S_r denote the set of all those points x at which (3) does not hold, and such that N_r is the first of the succession (1) of rational numbers which is greater than $\theta(x)$ and less than $\phi(x)$. Then any point at which

* By a theorem of Cantor's. Young's *Theory of Sets of Points*, p. 42.

† In the case of two independent variables we suppose the axes of coordinates to be horizontal and vertical, with positive values of x_1 on the right, and positive values of x_2 above. The parallels to the axes through any point P are then said to form *the axial cross* through P, and divide the neighbourhood of P into four quadrants, of which the $(+, +)$-quadrant has P as left-hand bottom corner. By *a $(+, +)$-axial neighbourhood* of P is meant any completely open rectangle in this $(+, +)$-quadrant, bounded on the left and at the bottom by the axial cross through P.

A similar notation enables us to distinguish axial neighbourhoods in the other quadrants. When there are more variables, we can in like manner distinguish the various axial neighbourhoods at a point.

(3) does not hold belongs to one, and only one, of the sets S_r, so that the set of points at which (3) does not hold is the sum $S_1+S_2+\ldots+S_r+\ldots$ of the sets S_r, and is therefore a set of the first category if S_r is dense nowhere for every integer r. This we shall now prove to be the case.

Let d denote any completely open region. Then either in d there is no point of S_r, or there is such a point P. In the latter case

$$\theta(P) < N_r < \phi(P).$$

Hence, by the definition of θ, there is a completely open region d', with P as one of its boundary points, sufficiently small to lie inside the region d, and such that at every internal point of d',

$$f(x) < N_r;$$

and therefore

$$\phi(x) \leqslant N_r,$$

so that no point internal to d' is a point of S_r.

Thus, whatever region d be taken, there is a region internal to it containing no points of the set S_r; that is, the set S_r is dense nowhere, which proves the theorem.

COR.—*If there are two independent variables, and* ϕ_{++}, ϕ_{+-}, ϕ_{-+}, ϕ_{--}, *denote the associated upper limiting functions taken with respect to the axial neighbourhoods at* P, *then*

$$\phi_{++} = \phi_{+-} = \phi_{-+} = \phi_{--} = \phi,$$

except at a set of the first category.

5. In the case when there is only one independent variable, we can go a step further and prove the theorem already referred to, that

$$\phi_L = \phi_R,$$

except at a countable set of points.

For the small region d' of the preceding proof is then an interval with P as end-point, so that P is an isolated point or only a one-sided limiting point of the set S_r. The set S_r contains therefore no points which are limiting points of the set on both sides, and is therefore a countable set.* Hence the set $S_1+S_2+\ldots+S_r+\ldots$, at which (3) does not hold, is also a countable set.

6. THEOREM 3.—*If* $f(x)$ *is a function of two or more real variables* x, *its limits at any point* P *obtained from any one of the axial neighbour-*

* Young's *Theory of Sets of Points*, p. 42, § 20, Cor.; the intervals of that corollary being taken to be the black intervals of the set got by closing S_r.

hoods are the same as those obtained from any other axial neighbourhood, unless P belongs to a certain set of the first category, depending on the particular function f(x).

Let P be any point at which there is a limit obtained from one of the axial neighbourhoods, say a $(+, +)$-limit, which is not a limit obtained from some other, say not a $(+, -)$-limit. Then there is a gap in the closed set of the $(+, -)$-limits at P, that is, there are two $(+, -)$-limits $G(P)$ and $H(P)$, between which there do not lie any $(+, -)$-limits, while there does lie at least one $(+, +)$-limit.

This gap determines therefore a pair of integers (i, j), such that N_i is the first rational number of the succession (1), which is greater than $G(P)$ and less than $H(P)$, and such that between it and $H(P)$ there is at least one $(+, +)$-limit; while N_j is the first rational number greater than N_i and less than $H(P)$, and such that between it and N_i there is at least one $(+, +)$-limit.

Hence, if $S_{i,j}$ denote the set of all those points P at which the particular pair of integers (i, j) is so determined, every point of every one of the sets $S_{i,j}$ for all integers i and j, is an unsymmetrical discontinuity of $f(x)$ at which there is a $(+, +)$-limit which is not at the same time a $(+, -)$-limit. Conversely, every such unsymmetrical discontinuity belongs to *at least one* of the sets $S_{i,j}$. Since there is precisely a countably infinite number of the sets $S_{i,j}$, it follows that these unsymmetrical discontinuities form a set of the first category, provided each set $S_{i,j}$ is dense nowhere. This we shall now prove to be the case.

For this purpose it is sufficient to shew that if a completely open region contains any point P of $S_{i,j}$, it contains a $(+, -)$-neighbourhood of P free from points of $S_{i,j}$. This we shall prove by a *reductio ad absurdum.*

Suppose there were a sequence of points $P_1, P_2, \ldots$ lying in the $(+, -)$-neighbourhood of P, with P as limiting point, and each, as well as P, belonging to the set $S_{i,j}$. Then there is a $(+, +)$-limit at P_n lying in the closed interval (N_i, N_j), and therefore a determinate one $g(P_n)$ which is nearest to N_i since such limits form a closed set.

Since $g(P_n)$ is a limit of $f(x)$ at P_n, there is a sequence of points along which $f(x)$ has at P_n the unique limit $g(P_n)$, and since

$$N_i \leqslant g(P_n) \leqslant N_j,$$

we may commence this sequence at such a point that at every point of it

$$N_i - 2^{-n} < f(x) < N_j + 2^{-n}.$$

Moreover, since P_n is a point of a $(+, -)$-neighbourhood of P, we may

take it that every point of this sequence belongs also to a $(+, -)$-neighbourhood of P. Let this sequence be

$$Q_{n,1}, \quad Q_{n,2}, \quad \ldots, \quad Q_{n,r}, \quad \ldots .$$

Since P is the limiting point of the sequence $P_1, P_2, \ldots, P_n, \ldots$, and each point P_n is the limiting point of the sequence

$$Q_{n,1}, \quad Q_{n,2}, \quad \ldots, \quad Q_{n,r}, \quad \ldots ,$$

we can find a point from each of these sequences, so as to form a sequence,

$$Q_{1,r_1}, \quad Q_{2,r_2}, \quad \ldots, \quad Q_{n,r_n}, \quad \ldots ,$$

having P as limiting point. As we pass along this sequence the limits of $f(x)$ which we obtain, being $(+, -)$-limits, cannot lie in the closed interval (N_i, N_j). But, by the preceding inequality,

$$N_i - 2^{-n} < f(Q_{n,r_n}) < N_j + 2^{-n},$$

so that the limits in question cannot lie outside the interval (N_i, N_j). Thus the assumption that there is such a sequence $P_1, P_2, \ldots$ leads to an *impasse* and must be abandoned.

Hence there must be a $(+, -)$-neighbourhood of P entirely free of points of the set $S_{i,j}$, which proves that $S_{i,j}$ is dense nowhere, and therefore that the set of points at which there is a $(+, +)$-limit which is not a $(+, -)$-limit is a set of the first category.

Taking the axial quadrants in pairs, this shews that the points at which the limits obtained from any one of the axial neighbourhoods are not precisely the same as those obtained from any other, form a finite number of sets of the first category, and therefore form a set of the first category. Q.E.D.

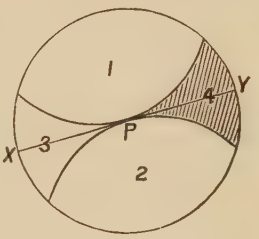
7. The preceding theorem has been stated for functions of two or more variables, and the proof has been given as if there were two variables. It is clear, however, that the proof is valid for any number of dimensions, only requiring a proper interpretation of the symbols $(+, +)$ and $(+, -)$, or the substitution for them of other symbols.

In particular, if there is only a single independent variable we get, as in § 5, an alternative proof of the theorem of the Lincei paper quoted in the introduction.

Again, it was only for the sake of clearness that the neighbourhoods used in defining the limits were taken to be axial neighbourhoods. The argument would hold as it stands if we assumed any other law of neighbourhoods by which the forms of neighbourhood at any point are non-overlapping.

In this case, in fact, there will be a finite or countable infinite set of forms of neighbourhood at a point, just as there were four axial neighbourhoods at a point in a plane. Hence, by the preceding proof, taking the neighbourhoods two by two, we get a countably infinite set of sets of the first category, that is, a set of the first category, of exceptional points.

In particular, let us choose the following law of neighbourhoods :—



Let a circle be drawn with P as centre (see figure), whose diameter in a fixed direction is XY. Draw two other circles with the same radius, touching XY at P. We thus determine four neighbourhoods at P, numbered 1, 2, 3, and 4 in the figure. Since the limits obtained from the neighbourhood 4 are the same as in the neighbourhoods 1, 2, and 3, except at a certain set of the first category, it follows that *excepting only at the points of a certain set of the first category, all the limits of a function of two variables at any point may be obtained by means of sequences of points tangential at the point to a fixed direction, previously chosen.*

This theorem is evidently true for functions of any number of variables, in the case of three variables we need only rotate the accompanying figure about the line XY to get suitably forms of neighbourhood, and similarly we can define suitable forms of neighbourhood when there are more variables.

THE SOLUTION OF THE HOMOGENEOUS LINEAR DIFFERENCE EQUATION OF THE SECOND ORDER

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1. It is proposed, in this paper, to examine the character of the solution of the linear difference equation

$$A(x)f(x+1)-B(x)f(x)+C(x)f(x-1) = 0, \tag{1}$$

where $A(x)$, $B(x)$, $C(x)$ are known uniform functions of the complex variable x. The ultimate object of the present investigation is to obtain a solution of the equation (1) which is of practical use in applications to the theory of certain linear differential equations.

It has been usual to consider linear difference equations in which the variable x is restricted to be a positive integer. References to investigations of this kind are given by Boole*; the equations considered are usually of the first order, but some very particular equations of the second order have been the subjects of investigation.

The first investigations in which the variable was supposed to assume all complex values appear to be due to Poincaré,† who has given a number

* Boole, *Finite Differences*, 3rd edition (1880), pp. 258 *et seq.*

† Poincaré, *American Journal of Mathematics* (1885) Vol. VII, pp. 203–258.

of general theorems, true for equations of any order, and to Guichard,* who has investigated the equation of the first order. A paper by Ford† has recently appeared in which asymptotic expansions of particular solutions of difference equations of the second order are obtained (the variable being supposed to be a positive integer). Some of his results can be obtained immediately from the results of the present paper.

The only complete investigations of linear difference equations of the second order, when the variable can assume all complex values, are connected with the particular case when $A(x)$, $B(x)$, $C(x)$ are linear functions of x. This case has been considered in detail by various writers.‡

The restrictions which will be made on the nature of the functions $A(x)$, $B(x)$, $C(x)$ are the following :—

(i) That they are such that the function

$$\frac{A(x)\,C(x+1)}{B(x)\,B(x+1)},$$

whenever $|x|$ is greater than, or equal to, a certain finite quantity μ, is a regular analytic function of x^{-1}, and is, consequently, capable of being expanded in the form

$$\frac{A(x)\,C(x+1)}{B(x)\,B(x+1)} = \lambda_0 + \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \ldots.$$

(ii) That the subsidiary difference equations of the first order

$$B(x)\,u(x) = C(x)\,u(x-1),$$
$$B(x)\,u_1(x) = A(x)\,u_1(x+1)$$

are such that their solutions may be regarded as known. Now, the difference equation of the first order

$$\phi(x)\,f(x+1) - \chi(x)\,f(x) = 0,$$

where $\phi(x)$, $\chi(x)$ are meromorphic functions of x, is the subject of a paper by Barnes§ ; consequently the solution of the subsidiary equations may be regarded as known when $B(x)/C(x)$, $A(x)/B(x)$ are meromorphic functions of x. Also, the difference equation of the first order

$$\phi(x)\,f(x+1) - \chi(x)\,f(x) = 0,$$

in which the functions $\phi(x)$, $\chi(x)$ are such that a finite integer n can be

* Guichard, *Annales de l'Ecole Normale Supérieure* (1887), Sér. 3, T. IV, pp. 361–380.

† Ford, *Transactions of the American Mathematical Society*, Vol. 10 (1909), pp. 319–336.

‡ Heine, *Kugelfunctionen* (1878), pp. 115–120 ; Heymann, *Crelle*, Bd. CXXII, pp. 166 *et seq.* ; Barnes, *Messenger of Mathematics*, Vol. XXXIV, pp. 52–71.

§ Barnes, *Proceedings of the London Mathematical Society*, Ser. 2., Vol. II, pp. 438–469.

found such that $x^n \phi(x)/\chi(x)$ is a regular analytic function of x when $|x|$ is large, is the subject of a paper by the writer*; consequently the solution of the subsidiary equations may be regarded as known when $x^m B(x)/C(x)$, $x^n A(x)/B(x)$ are analytic functions of x when $|x|$ is large.

As has been already mentioned, linear difference equations of the form (1) occur largely in the theory of linear differential equations; in such cases $A(x)$, $B(x)$, $C(x)$ are polynomials in x, so that the condition (ii.) is satisfied.

The solution of the equation (1) when x is restricted to be a positive integer has been obtained already by the writer.† We now assume that x is a general complex quantity. It is found that four different cases arise for consideration, each of which requires a separate investigation; these are referred to as (I), (II), (III), and (IIIa).

The distinctions between these cases depend on the different forms taken by the asymptotic expansion, when $|x|$ is large, of a certain infinite continued fraction, the n-th element of which depends on $A(x+n)$, $B(x+n)$, $C(x+n)$.

Infinite continued fractions of this form have not hitherto received much attention. A comparatively simple type of such a fraction has been considered by Stieltjes.‡ De Montessus§ has investigated the convergence of continued fractions (similar to those of §§ 4–6 of this paper) when $A(x)$, $B(x)$, $C(x)$ are polynomials in x. Some results equivalent to those of De Montessus have been stated by Nörlund.‖ Pringsheim¶ has proved the important theorem that the continued fraction

$$\frac{a_1}{1+}\,\frac{a_2}{1+}\,\frac{a_3}{1+}\ldots$$

is unconditionally convergent, if

$$|a_{2n-1}| + |a_{2n}| \leqslant \tfrac{1}{2}.$$

This theorem will be employed in §§ 4, 5, but it is not sufficient for the purposes of §§ 6, 7.

It is to be noticed that, if $f_1(x)$, $f_2(x)$ are two solutions of (1), $f_1(x)$ and $f_2(x)$ being functions of x whose ratio is not a periodic function with period

* *Quarterly Journal of Mathematics*, Vol. XLI (1909), pp. 10–20.

† *Ibid.*, pp. 50–55.

‡ Stieltjes, *Annales de la Faculté des Sciences de Toulouse*, T. VII (J), (1893), and T. VIII (A), (1894).

§ De Montessus, *Rendiconti del Circolo Matematico di Palermo*, T. XIX (1905), pp. 229 *et seq.* *Acta Mathematica*, Vol. 32 (1909), pp. 257–281.

‖ Nörlund, *Comptes Rendus*, T. CXLVII (1908), pp. 585–587.

¶ Pringsheim, *Sitzungsberichte der K. B. Akademie zu München*, Bd. XXVIII (1898), pp. 295–324.

unity, the general solution of (1) is

$$f(x) = \varpi_1(x) f_1(x) + \varpi_2(x) f_2(x),$$

where ϖ_1, ϖ_2 are periodic functions of x with period unity.

We shall, accordingly, endeavour to obtain only two particular solutions of equation (1). It will be shewn that, in the cases considered, two solutions are obtainable; but it does not appear possible to prove that their ratio cannot be a periodic function of x with period unity.

2. We now proceed to the solution of the difference equation (1).

Let $u(x)$ be any solution of the difference equation

$$u(x)\, B(x) = u(x-1)\, C(x). \tag{2}$$

Put $$f(x) = u(x)\, \phi(x),$$

and the equation (1) becomes

$$\phi(x-1) = \phi(x) - \frac{A(x)\, C(x+1)}{B(x)\, B(x+1)}\, \phi(x+1),$$

which may be written in the form

$$\phi(x-1) = \phi(x) - v(x)\, \phi(x+1)$$

on putting $$\frac{A(x)\, C(x+1)}{B(x)\, B(x+1)} = v(x)$$

for brevity. This last equation may be written

$$\frac{\phi(x)}{\phi(x-1)} = \frac{1}{1 - v(x)\, \dfrac{\phi(x+1)}{\phi(x)}}. \tag{2a}$$

We are thus led to consider the convergence and other properties of the infinite continued fraction

$$\psi(x) = \frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \frac{v(x+2)}{1} - \ldots. \tag{3}$$

In some cases it is possible to obtain the properties of $\psi(x)$ from those of the finite continued fraction

$$\psi_n(x) = \frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots - \frac{v(x+n-1)}{1};$$

it is possible to express $\psi_n(x)$ as the quotient of two rational functions of $v(x)$, $v(x+1)$, ...; but, except in the special case when $\operatorname{Lt}_{x=\infty} x\, v(x) = 0$, these rational functions have no finite limit when n is increased indefinitely. Hence it seems desirable in all cases to obtain the properties of $\psi(x)$ by the same general methods, which are given in the sequel.

We shall now suppose that, when $|x|$ is greater than or equal to a certain finite quantity μ, $v(x)$ can be expanded into the absolutely convergent series

$$v(x) = \lambda_0 + \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \dots. \tag{4}$$

We shall also suppose that the point x lies outside the circle $|x| = \mu$, and, further, that $R(x) > -\frac{1}{2}$; this last condition ensures that the quantities $|x|$, $|x+1|$, $|x+2|$, ... form a steadily increasing sequence.

A slight generalisation of Weierstrass' factor theorem* will be found to be of great use, viz., that, if the series $\sum_{n=0}^{\infty} u_n$ be not absolutely convergent, while the two series $\sum_{n=0}^{\infty} (u_n - \alpha_n)$ and $\sum_{n=0}^{\infty} u_n^2$ are absolutely convergent, the infinite product

$$\prod_{n=0}^{\infty} [(1+u_n) \exp(-\alpha_n)]$$

converges absolutely.

The proof of this theorem is not at all difficult, and it can be set forth in the same manner as the proof of Weierstrass' factor theorem already quoted; accordingly, it seems unnecessary to give a formal proof. The theorem will, in the sequel, be referred to as Weierstrass' theorem.

We shall also need the theorem that, if, when $|x| \geqslant \mu$,

$$S = \frac{a_1}{x} + \frac{a_2}{x^2} + \dots$$

is a convergent series, then a finite quantity μ' can be found such that, when $|x| > \mu'$, $|S|$ is less than an arbitrary finite non-vanishing quantity, assigned beforehand. This theorem will be referred to as the fundamental continuity theorem.

A matter of notation has to be mentioned: if $\xi = \xi + i\eta$, where ξ, η are real quantities, we write

$$\xi = R(\xi), \qquad \eta = I(\xi).$$

3. We are now in a position to proceed to the consideration of the asymptotic† expansion of $\psi(x)$. The four distinct cases to be investigated

---

* Forsyth, *Theory of Functions*, § 51 (2nd edition).

† The term "asymptotic" is here, as elsewhere, employed only in that sense in which it is used by Poincaré.

are as follows:—

(I). $\lambda_0 = \lambda_1 = 0$.

(II). $\lambda_0 = 0$, $\lambda_1 \neq 0$.

(III). $\lambda_0 \neq 0$, $\lambda_0 \neq$ any real quantity greater than or equal to $\frac{1}{4}$.

(IIIa). $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, $\lambda_2 \neq$ any real quantity greater than or equal to $\frac{1}{16}$.

In cases other than these, the solution given is not valid, as will appear later.

In each investigation it has been deemed expedient to use the same symbols in different, but kindred, senses; thus ϵ_x as used in § 4 has not the same meaning as when used in § 6. This, however, will not lead to any confusion.

4. CASE (I).—$\lambda_0 = \lambda_1 = 0$.

We shall suppose, for the sake of generality, that the first h coefficients λ all vanish, h being an integer greater than unity.

We then have

$$v(x) = \frac{\lambda_h + \epsilon_x}{x^h},$$

where $|\epsilon_x|$ has a certain finite upper limit, and $|x\epsilon_x|$ tends to a finite limit as $|x|$ tends to infinity. This formula is true for all values of x satisfying the inequalities $|x| > \mu$, $R(x) > -\frac{1}{2}$. Let the upper limit of $|x\epsilon_x|$ for this range of values of x be E.

We proceed to prove that in this case $\psi(x)$, as defined by (3), asymptotically approaches the value given by

$$\psi(x) \sim 1 + \frac{\lambda_h}{x^h}.$$

Consider quantities η, η' connected by the relation

$$1 + \frac{\lambda_h + \eta}{x^h} = \frac{1}{1 - \dfrac{\lambda_h + \epsilon_x}{x^h}\left\{1 + \dfrac{\lambda_h + \eta'}{(x+1)^h}\right\}}.$$

On reduction this becomes

$$\lambda_h + \eta = x^h\left[1 - \frac{\lambda_h + \epsilon_x}{x^h} - \frac{(\lambda_h + \epsilon_x)(\lambda_h + \eta')}{x^h(x+1)^h}\right]^{-1} - x^h.$$

This may be written in the form

$$\eta = \epsilon_x + \frac{(\lambda_h + \epsilon_x)(\lambda_h + \eta')}{(x+1)^h} + \frac{(\lambda_h + \epsilon_x)^2}{x^h}\left[1 + \frac{\lambda_h + \eta'}{(x+1)^h}\right]^2\left[1 - \frac{\lambda_h + \epsilon_x}{x^h}\left\{1 + \frac{\lambda_h + \eta'}{(x+1)^h}\right\}\right]^{-1}.$$

And hence

$$\eta-\epsilon_x-\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}=\frac{\lambda_h+\epsilon_x}{(x+1)^h}\left\{\eta'-\epsilon_{x+1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right\}$$

$$+\frac{(\lambda_h+\epsilon_x)^2}{x^h}\left[1+\frac{\lambda_h+\eta'}{(x+1)^h}\right]^2\left[1-\frac{\lambda_h+\epsilon_x}{x^h}\left\{1+\frac{\lambda_h+\eta'}{(x+1)^h}\right\}\right]^{-1}$$

$$+\frac{\lambda_h+\epsilon_x}{(x+1)^h}\left[\epsilon_{x+1}+\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right]$$

and $$|x+1|>|x|.$$

We can find a finite quantity β greater than unity, μ, and $|\lambda_h|+E$, such that, when $|x|>\beta$ and $R(x)>-\frac{1}{2}$,

$$\frac{|\lambda_h|+|\epsilon_x|}{|x|^h}<\tfrac{1}{3}.$$

This follows from the fundamental continuity theorem.

Further, let the upper limit of $|\lambda_h+\epsilon_x|$ for this range of values of x be Θ_1, and choose η' so that $|\lambda_h+\eta'|<$ a certain finite quantity K such that

$$|\lambda_h|+E<K<\beta. \tag{5}$$

We then get

$$\left|\eta-\epsilon_x-\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right|$$

$$<\tfrac{1}{3}\left|\eta'-\epsilon_{x+1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right|+\frac{\Theta_1^2}{|x|^h}2^2(1-\tfrac{2}{3})^{-1}+\frac{\Theta_1}{|x+1|^h}\left|\epsilon_{x+1}+\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right|.$$

We can find a finite quantity Θ, independent of x, such that, when $|x|>\beta$ and $R(x)>-\frac{1}{2}$,

$$\frac{12\Theta_1^2}{|x|^h}+\frac{\Theta_1}{|x+1|^h}\left|\epsilon_{x+1}+\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right|<\frac{\Theta}{|x|^h}.$$

And we then have

$$\left|\eta-\epsilon_x-\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right|<\tfrac{1}{3}\left|\eta'-\epsilon_{x+1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+1})}{(x+2)^h}\right|+\frac{\Theta}{|x|^h}. \tag{6}$$

Before proceeding further, it is expedient to impose an additional limitation on $|x|$.

We can find a finite quantity Θ_2, independent of x, such that, when $|x|>\beta$ and $R(x)>-\frac{1}{2}$,

$$\left|\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right|<\frac{\Theta_2}{|x|^h}.$$

Also, we can find a finite quantity $\gamma \geqslant \beta$ such that, when $|x| > \gamma$ and $R(x) > -\frac{1}{2}$,

$$\left|\lambda_h+\epsilon_x+\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right|+\tfrac{1}{3}\Theta_2|x|^{-h}+\tfrac{3}{2}\Theta|x|^{-h} < K, \qquad (7)^*$$

and

$$|v(x)| < \tfrac{1}{4}. \qquad (7a)$$

We shall now suppose that $|x| > \gamma$ *and* $R(x) > -\frac{1}{2}$.

Now, if we have the series of inequalities

$$\begin{aligned} U_{x+n} &< \tfrac{1}{3}U_{x+n+1}+\frac{\Theta}{|x+n|^h}, \\ U_{x+n+1} &< \tfrac{1}{3}U_{x+n+2}+\frac{\Theta}{|x+n+1|^h}, \\ \cdots \quad &\cdots \quad \cdots \quad \cdots \quad \cdots \\ U_{x+m} &< \tfrac{1}{3}U_{x+m+1}+\frac{\Theta}{|x+m|^h}, \end{aligned}$$

where n is a positive integer or zero, and m is an integer greater than n, the following series of inequalities is, *a priori*, satisfied:—

$$\begin{aligned} U_{x+n} &< \tfrac{1}{3}U_{x+n+1}+\frac{\Theta}{|x+n|^h}, \\ U_{x+n+1} &< \tfrac{1}{3}U_{x+n+2}+\frac{\Theta}{|x+n|^h}, \\ \cdots \quad &\cdots \quad \cdots \quad \cdots \quad \cdots \\ U_{x+m} &< \tfrac{1}{3}U_{x+m+1}+\frac{\Theta}{|x+n|^h}, \end{aligned}$$

since $$|x+m| > |x+m-1| > \ldots > |x+n+1| > |x+n|.$$

Hence we deduce without any difficulty that

$$U_{x+n} < \tfrac{3}{2}\left\{1-(\tfrac{1}{3})^{m+1-n}\right\}\frac{\Theta}{|x+n|^h}+(\tfrac{1}{3})^{m+1-n}U_{x+m+1}. \qquad (8)$$

Now, if we have the following series of equations defining

$$\eta_{x+m},\ \eta_{x+m-1},\ \ldots,\ \eta_{x+n};$$

η_{x+m+1} being supposed known,

$$1+\frac{\lambda_h+\eta_{x+n+r}}{(x+n+r)^h} = \frac{1}{1-\dfrac{\lambda_h+\epsilon_{x+n+r}}{(x+n+r)^h}\left\{1+\dfrac{\lambda_h+\eta_{x+n+r+1}}{(x+n+r+1)^h}\right\}}, \qquad (9)$$

where $$r = 0, 1, 2, \ldots, m-n,$$

* The reason for choosing this inequality will be apparent later [see equation (11)].

and if $\eta_{x+n+1}, \eta_{x+n+2}, \ldots, \eta_{x+m+1}$ be such that

$$|\lambda_h+\eta_{x+n+1}| < K,$$
$$|\lambda_h+\eta_{x+n+2}| < K,$$
$$\cdots \quad \cdots \quad \cdots$$
$$|\lambda_h+\eta_{x+m+1}| < K,$$

(η_{x+m+1} will be taken equal to ϵ_{x+m+1} for convenience), then the series of inequalities

$$\left|\eta_{x+n+r}-\epsilon_{x+n+r}-\frac{\lambda_h(\lambda_h+\epsilon_{x+n+r})}{(x+n+r+1)^h}\right|$$
$$< \tfrac{1}{3}\left|\eta_{x+n+r+1}-\epsilon_{x+n+r+1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+n+r+1})}{(x+n+r+2)^h}\right|+\frac{\Theta}{|x+n|^h}$$

will hold by reason of (6); where r, as before, takes the values 0, 1, 2, ... $m-n$.

Applying the result (8) to this inequality, we get

$$\left|\eta_{x+n}-\epsilon_{x+n}-\frac{\lambda_h(\lambda_h+\epsilon_{x+n})}{(x+n+1)^h}\right|$$
$$< \tfrac{3}{2}\left\{1-(\tfrac{1}{3})^{m+1-n}\right\}\frac{\Theta}{|x+n|^h}+(\tfrac{1}{3})^{m+1-n}\left|\eta_{x+m+1}-\epsilon_{x+m+1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+m+1})}{(x+m+2)^h}\right|$$
$$< \tfrac{3}{2}\frac{\Theta}{|x+n|^h}+\tfrac{1}{3}\left|\frac{\lambda_h(\lambda_h+\epsilon_{x+m+1})}{(x+m+2)^h}\right|$$
$$< \left\{\tfrac{3}{2}\Theta+\tfrac{1}{3}\Theta_2\right\}|x+n|^{-h}. \qquad (10)$$

By this result

$$|\lambda_h+\eta_{x+n}| < \left|\lambda_h+\epsilon_{x+n}+\frac{\lambda_h(\lambda_h+\epsilon_{x+n})}{(x+n+1)^h}\right|+\left\{\tfrac{3}{2}\Theta+\tfrac{1}{3}\Theta_2\right\}|x+n|^{-h},$$

i.e.,
$$|\lambda_h+\eta_{x+n}| < K < |x+n| \qquad (11)$$

by virtue of (5) and (7).

And hence [when $|x+n-1| > \gamma$ and $R(x+n-1) > -\frac{1}{2}$], if η_{x+n-1} be defined by putting $r=-1$ in (9), we can deduce the inequality

$$\left|\eta_{x+n-1}-\epsilon_{x+n-1}-\frac{\lambda_h(\lambda_h+\epsilon_{x+n-1})}{(x+n)^h}\right|$$
$$< \tfrac{1}{3}\left|\eta_{x+n}-\epsilon_{x+n}-\frac{\lambda_h(\lambda_h+\epsilon_{x+n})}{(x+n+1)^h}\right|+\frac{\Theta}{|x+n-1|^h}$$

in the same way as we deduced (6).

And hence, in the same manner as we deduced (11), we can deduce

$$|\lambda_h+\eta_{x+n-1}| < K < |x+n-1|.$$

Now we see that $|\lambda_h+\eta_{x+m+1}| < K,$

by using (5). And hence $|\lambda_h+\eta_{x+m}| < K$ from (11) and (10); and so, by induction, we get

$$|\lambda_h+\eta_{x+n}| < K < |x+n|,$$

where $n = 0, 1, 2, \ldots, m.$

And, further, the inequality (10) now holds for all the values $0, 1, \ldots, m$ of n.

Putting $n = 0$ in the inequality, we get

$$\left|\eta_x-\epsilon_x-\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right| < \{\tfrac{3}{2}\Theta+\tfrac{1}{3}\Theta_2\}\,|x|^{-h},$$

where $$1+\frac{\lambda_h+\eta_x}{x^h} = \frac{1}{1-}\,\frac{v(x)}{1-}\ldots-\frac{v(x+m)}{1}\,\{1+v(x+m+1)\}$$

and Θ, Θ_2 are finite quantities independent of m and x.

Making m indefinitely great, we see that, if $\psi(x)$ be defined as

$$\operatorname*{Lt}_{m=\infty}\left[\frac{1}{1-}\,\frac{v(x)}{1-}\ldots-\frac{v(x+m)}{1}\,\{1+v(x+m+1)\}\right], \tag{12}$$

where $$v(x) = \frac{\lambda_h}{x^h}+\frac{\lambda_{h+1}}{x^{h+1}}+\ldots, \tag{12a}$$

then, when $|x| >$ a certain finite quantity γ, and $R(x) > -\frac{1}{2}$, $\psi(x)$ converges unconditionally by Pringsheim's theorem (see page 127), since $|v(x+n)| < \frac{1}{4}$ by (7a); and the value of $\psi(x)$ is given by

$$\psi(x) = 1+\frac{\lambda_h+\eta_x}{x^h},$$

where $$\left|\eta_x-\epsilon_x-\frac{\lambda_h(\lambda_h+\epsilon_x)}{(x+1)^h}\right| < \frac{\Theta_3}{|x|^h},$$

where Θ_3 is a finite quantity. That is to say, if $h \geqslant 2$, an asymptotic expression for $\psi(x)$, is

$$\psi(x) = 1+\frac{\lambda_h}{x^h}+J, \tag{13}$$

where $x^{h+1}J$ is finite and tends uniformly to a finite limit as $|x|$ tends to infinity.

Now consider the infinite product

$$\chi(x) = \prod_{n=0}^{\infty}\{\psi(x+n)\},$$

it being assumed that $-R(x)$ is not an indefinitely great positive quantity.

We write this equation in the form

$$\chi(x) = \prod_{n=0}^{m-1} \{\psi(x+n)\} \cdot \prod_{n=m}^{\infty} \{\psi(x+n)\},$$

where m is now the smallest integer which makes $|x+m| > \gamma$ and $R(x+m) > -\frac{1}{2}$.

Then, using (13), it follows from Weierstrass' theorem that the infinite product converges absolutely.

Now consider the quantity $1/\psi(x+n)$ when $0 \leqslant n < m$. Since

$$\frac{1}{\psi(x+n-1)} = 1 - v(x+n-1)\,\psi(x+n),$$

if the function $v(x)$ have all its singularities (inside the circle $|x| = \gamma$) isolated, $\psi(x+n)$ will become infinite or zero for certain isolated values of x, but will in general be finite. With the exception of these points, the function $\chi(x)$ is regular in the finite part of the x-plane.

Also

$$\frac{\chi(x)}{\chi(x+1)} = \frac{1}{1} - \frac{v(x)\,\chi(x+1)}{\chi(x+2)}.$$

Consequently, by (2a), a solution of the difference equation (1) is given by

$$f(x) = u(x)/\chi(x+1),$$

i.e.,
$$f(x) = u(x) \prod_{n=1}^{\infty} [\{\psi(x+n)\}^{-1}], \tag{I}$$

where $u(x)$ is any solution of

$$u(x)\,B(x) = u(x-1)\,C(x),$$

and where $\psi(x)$ and $v(x)$ are defined by (12) and (12a).

5. CASE (II).—$\lambda_0 = 0$, $\lambda_1 \neq 0$.

In the preceding analysis of § 4, we have not made use of the fact that h is at least equal to 2 in proving the equation (12), but in various places it has been assumed that $h > 0$ [*e.g.*, formula (7)].

Accordingly all the analysis of § 4 up to equation (12) is at our disposal when h is equal to unity, and, to summarise the results obtained, we have the theorem that, if

$$\psi(x) = \operatorname{Lt}_{m=\infty} \left[\frac{1}{1} - \frac{v(x)}{1} - \ldots \frac{v(x+m)}{1}\{1+v(x+m+1)\}\right], \tag{14}$$

where
$$v(x) = \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \ldots, \tag{14a}$$

then, when $|x| >$ a certain finite quantity γ and $R(x) > -\frac{1}{2}$,

$$\psi(x) = 1 + \frac{\lambda_1 + \eta_x}{x},$$

where $$\left| \eta_x - \epsilon_x - \frac{\lambda_1(\lambda_1 + \epsilon_x)}{x+1} \right| < \frac{\Theta_3}{|x|},$$

i.e., a finite quantity Θ_4, independent of x, can be found such that

$$|\eta_x| < \frac{\Theta_4}{|x|}.$$

That is to say, in this case, an asymptotic expression for $\psi(x)$ is

$$\psi(x) = 1 + \frac{\lambda_1}{x} + J, \tag{15}$$

where $|x^2 J|$ is finite and has a finite upper limit for the values of x under consideration.

Now consider the infinite product

$$\chi(x) = \prod_{n=0}^{\infty} \left[\psi(x+n) \exp\left\{ \frac{-\lambda_1}{(x+n)} \right\} \right],$$

it being assumed that $-R(x)$ is not an indefinitely great positive quantity.

We write this equation in the form

$$\chi(x) = \prod_{n=0}^{m-1} \left[\psi(x+n) \exp\left\{ \frac{-\lambda_1}{x+n} \right\} \right] \prod_{n=m}^{\infty} \left[\psi(x+n) \exp\left\{ \frac{-\lambda_1}{(x+n)} \right\} \right],$$

where m is the smallest integer which makes

$$|x+m| > \gamma \quad \text{and} \quad R(x+m) > -\tfrac{1}{2}.$$

Then, using (15), it follows from Weierstrass' theorem that the infinite product converges absolutely.

Now consider the quantity $1/\psi(x+n)$ when $0 \leqslant n < m$. Since

$$\frac{1}{\psi(x+n-1)} = 1 - v(x+n-1)\,\psi(x+n),$$

if the function $v(x)$ have all its singularities (inside the circle $|x| = \gamma$) isolated, $\psi(x+n)$ will become infinite or zero for certain isolated values of x, but will in general be finite. With the exception of these points and the points $x = 0, -1, -2, \ldots$, the function $\chi(x)$ is analytic in the finite part of the x-plane.

If now we write $\chi(x) = w(x)/\phi(x-1)$,

since $\psi(x) = \phi(x)/\phi(x-1)$,

we get the difference equation of the first order

$$w(x+1) = w(x) \exp\left(\frac{\lambda_1}{x}\right),$$

to determine the function $w(x)$.

A solution of this equation is

$$w(x) = \exp\{\lambda_1 \Gamma'(x)/\Gamma(x)\}.$$

Hence, in this case, a solution of the difference equation (1) is

$$f(x) = u(x) \exp\{\lambda_1 \Gamma'(x+1)/\Gamma(x+1)\} \prod_{n=1}^{\infty} [\{\psi(x+n)\}^{-1} \exp\{\lambda_1/(x+n)\}], \tag{II},$$

where $u(x)$ is any solution of the first order difference equation

$$u(x)\, B(x) = u(x-1)\, C(x),$$

and where $\psi(x)$, $v(x)$ are defined by (14) and (14a).

6. CASE (III).—$\lambda_0 \neq 0$, $\lambda_0 \neq$ a real quantity greater than or equal to $\frac{1}{4}$.

A modification of the method employed in § 4 is effective in determining the asymptotic expansion of $\psi(x)$ when $\lambda_0 \neq 0$; the analysis, however, breaks down when λ_0 is a real quantity greater than $\frac{1}{4}$; and if $\lambda_0 = \frac{1}{4}$ and $\lambda_1 \neq 0$, the analysis also breaks down; in this second case (viz., $\lambda_0 = \frac{1}{4}$, $\lambda_1 \neq 0$) it is not improbable that the asymptotic expansion of $\psi(x)$ involves terms which are not uniform functions of x; the multiform functions being, in fact, $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$, ...; but whether this is actually the case or not, I cannot, at present, say.

Let quantities α_1, α_2 be defined as the roots of the equation

$$z^2 - z + \lambda_0 = 0.$$

With the limitation laid on λ_0, $|\alpha_1| \neq |\alpha_2|$; accordingly, let α_1, α_2 be so chosen that $|\alpha_1| > |\alpha_2|$.

We are now in a position to prove that, when $R(x) > -\frac{1}{2}$, if a certain interpretation be put on the meaning of the infinite continued fraction

$$\psi(x) = \frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots,$$

then $\psi(x)$ asymptotically approaches the value

$$\frac{1}{\alpha_1}\left\{1 + \frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)x}\right\}.$$

When $|x|>\mu$ and $R(x)>-\frac{1}{2}$, we have

$$v(x)=\lambda_0+\frac{\lambda_1+\epsilon_x}{x},$$

where $|\epsilon_x|$ has a certain finite upper limit and $|x\epsilon_x|$ tends to a finite limit as $|x|$ tends to infinity. Let the upper limit of $|x\epsilon_x|$ for the possible range of values of x be E, and let the upper limit of $|\epsilon_x|$ be E'.

Now consider quantities η, η' satisfying the relation

$$\frac{1}{\alpha_1}\left\{1+\frac{\lambda_1+\eta}{\alpha_1(\alpha_1-\alpha_2)x}\right\}=\frac{1}{1}-\frac{1}{\alpha_1}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{1+\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}.$$

On reduction this becomes

$$\lambda_1+\eta$$
$$=\alpha_1^2(\alpha_1-\alpha_2)x\left[1-\frac{1}{\alpha_1}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{1+\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^{-1}-\alpha_1(\alpha_1-\alpha_2)x.$$

If we use the identity

$$(1-U)^{-1}\equiv 1+U+U^2+U^3(1-U)^{-1},$$

we may write this equation in the form

$$\eta=\frac{\alpha_1-\alpha_2}{\alpha_1}\epsilon_x-\frac{\lambda_1\alpha_2}{\alpha_1(x+1)}+\frac{\lambda_1(\lambda_1+\epsilon_x)}{\alpha_1^2(x+1)}+\frac{(\alpha_1-\alpha_2)(\lambda_1+\epsilon_x)^2}{\alpha_1^3x}+\frac{\lambda_0^2\lambda_1^2}{\alpha_1^5(\alpha_1-\alpha_2)(x+1)}$$

$$+\frac{2\lambda_1}{\alpha_1^4(x+1)}(\lambda_1+\epsilon_x)\left(\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right)$$

$$+\left[\frac{\alpha_2}{\alpha_1}-\left(\frac{\alpha_2}{\alpha_1}-\frac{\lambda_1+\epsilon_x}{\alpha_1^2}\right)\frac{1}{x+1}+\frac{2(\lambda_1+\epsilon_x)}{\alpha_1^4(x+1)}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\right.$$

$$\left.+\frac{2\lambda_1 x}{\alpha_1^5(\alpha_1-\alpha_2)(x+1)^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}^2\right]\eta'$$

$$+\frac{\eta'^2}{\alpha_1^5(\alpha_1-\alpha_2)}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}^2\frac{x}{(x+1)^2}$$

$$+\alpha_1(\alpha_1-\alpha_2)x\left[\frac{\lambda_1+\epsilon_x}{\alpha_1^2x}+\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^3$$

$$\times\left[1-\frac{\lambda_1+\epsilon_x}{\alpha_1^2x}-\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^{-1}$$

+ some terms whose numerators involve λ_0, λ_1 and ϵ_x, and whose denominators involve terms of order x^2 at least, multiplied by powers of α_1 and $\alpha_1-\alpha_2$.

This may be rewritten in the form

$$\eta-\epsilon_x+\frac{\lambda_1\alpha_2}{(\alpha_1-\alpha_2)(x+1)}-\frac{\lambda_1(\lambda_1+\epsilon_x)}{\alpha_1(\alpha_1-\alpha_2)(x+1)}-\frac{(\lambda_1+\epsilon_x)^2}{\alpha_1^2 x}$$

$$-\frac{2\lambda_1}{\alpha_1^3(\alpha_1-\alpha_2)(x+1)}\{\lambda_1+\epsilon_x\}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}-\frac{\lambda_0^2\lambda_1^2}{\alpha_1^4(\alpha_1-\alpha_2)^2(x+1)}$$

$$=\left[\frac{\alpha_2}{\alpha_1}-\left(\frac{\alpha_2}{\alpha_1}-\frac{\lambda_1+\epsilon_x}{\alpha_1^2}\right)\frac{1}{x+1}+\frac{2(\lambda_1+\epsilon_x)}{\alpha_1^4(x+1)}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\right.$$

$$\left.+\frac{2\lambda_1 x}{\alpha_1^5(\alpha_1-\alpha_2)(x+1)^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}^2\right]$$

$$\times\left[\eta'-\epsilon_{x+1}+\frac{\lambda_1\alpha_2}{(\alpha_1-\alpha_2)(x+2)}-\frac{\lambda_1(\lambda_1+\epsilon_{x+1})}{\alpha_1(\alpha_1-\alpha_2)(x+2)}-\frac{(\lambda_1+\epsilon_{x+1})^2}{\alpha_1^2(x+1)}\right.$$

$$\left.-\frac{2\lambda_1}{\alpha_1^3(\alpha_1-\alpha_2)(x+2)}\{\lambda_1+\epsilon_{x+1}\}\left\{\lambda_0+\frac{\lambda_1+\epsilon_{x+1}}{x+1}\right\}-\frac{\lambda_0^2\lambda_1^2}{\alpha_1^4(\alpha_1-\alpha_2)^2(x+2)}\right]$$

$$+\frac{\alpha_2}{\alpha_1}\left[\epsilon_{x+1}-\epsilon_x+\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)}\left\{\frac{\epsilon_{x+1}}{x+2}-\frac{\epsilon_x}{x+1}\right\}\right.$$

$$\left.+\frac{1}{\alpha_1^2}\left\{\frac{2\lambda_1\epsilon_{x+1}+\epsilon_{x+1}^2}{x+1}-\frac{2\lambda_1\epsilon_x+\epsilon_x^2}{x}\right\}+\frac{2\lambda_0\lambda_1}{\alpha_1^3(\alpha_1-\alpha_2)}\left\{\frac{\epsilon_{x+1}}{x+1}-\frac{\epsilon_x}{x}\right\}\right]$$

$$+\alpha_1(\alpha_1-\alpha_2)x\left[\frac{\lambda_1+\epsilon_x}{\alpha_1^2 x}+\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^3$$

$$\times\left[1-\frac{\lambda_1+\epsilon_x}{\alpha_1^2 x}-\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^{-1}$$

$$+\frac{\eta'^2}{\alpha_1^5(\alpha_1-\alpha_2)}\frac{x}{(x+1)^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}^2$$

+ some terms the moduli of whose numerators are less than certain finite quantities independent of x, when $|x|>\mu$ and $R(x)>-\frac{1}{2}$, and whose denominators involve terms of order x^2 at least, multiplied by powers of α_1 and $\alpha_1-\alpha_2$. (16)

This may be rewritten, for brevity, in the form

$$\eta-\xi_x=\left(\frac{\alpha_2}{\alpha_1}-\xi_x\right)(\eta'-\xi_{x+1})+\frac{\eta'^2}{\alpha_1^5(\alpha_1-\alpha_2)}\frac{x}{(x+1)^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}^2$$

$$+\alpha_1(\alpha_1-\alpha_2)x\left[\frac{\lambda_1+\epsilon_x}{\alpha_1^2 x}+\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^3$$

$$\times\left[1-\frac{\lambda_1+\epsilon_x}{\alpha_1^2 x}-\frac{1}{\alpha_1^2}\left\{\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right\}\left\{\frac{\lambda_1+\eta'}{\alpha_1(\alpha_1-\alpha_2)(x+1)}\right\}\right]^{-1}$$

+ some terms of the type described in equation (16). (17)

The values of ξ_x, ζ_x may be seen by comparing this formula with (16); it is not necessary to rewrite their values in full.

It may now conveniently be noted that $a_1 \neq 0$, for $a_1 = 0$ implies that $\lambda_0 = 0$; $a_1 - a_2 \neq 0$, for $a_1 = a_2$ implies that $\lambda_0 = \frac{1}{4}$; $|(a_2/a_1)| \neq 1$, for $|(a_2/a_1)| = 1$ implies that λ_0 is a real quantity greater than $\frac{1}{4}$; we may accordingly take that determination of a_1, a_2 which makes $|(a_2/a_1)| < 1$.

Let $|(a_2/a_1)| = \sigma'$, so that $\sigma' < 1$.

We can find a finite quantity β greater than 1, μ and

$$\{|\lambda_1| + E'\}\{3|(\lambda_0/a_1^2)| + 1\}\{|a_1(a_1 - a_2)|\}^{-1},$$

such that, when $|x| > \beta$ and $R(x) > -\frac{1}{2}$,

$$\left|\frac{\lambda_1 + \epsilon_x}{a_1^2 x}\right| < \tfrac{1}{3}, \qquad \sigma' + |\xi_x| < \sigma, \tag{18}$$

where σ is an arbitrary quantity subject to the inequalities $\sigma' < \sigma < 1$; these results follow from the fundamental continuity theorem; also, we shall choose η' so that $|\lambda_1 + \eta'| < |\lambda_1| + E'$.

Further, we shall choose an arbitrary quantity K subject only to the limitations

$$|\lambda_1| + E' < K < \frac{|a_1^3(a_1 - a_2)|}{3|\lambda_0| + |a_1|^2}\beta. \tag{19}$$

We then get

$$|\eta - \zeta_x| < \sigma|\eta' - \zeta_{x+1}| + \frac{(K + |\lambda_1|)^2}{|a_1^5(a_1 - a_2)|}\frac{v^2}{|x|} + \frac{\theta}{|x|^2}$$
$$+ \frac{|a_1(a_1 - a_2)|}{|x|^2}\left[\left|\frac{\lambda_1 + \epsilon_x}{a_1^2}\right| + \frac{vK}{|a_1^3(a_1 - a_2)|}\right]^3 [1 - \tfrac{1}{3} - \tfrac{1}{3}]^{-1}$$

on using (18), (19), and the fundamental continuity theorem; where θ is some finite quantity depending on $|\lambda_0|$, $|\lambda_1|$, B, E', and v is the upper limit of $|v(x)|$. That is to say,

$$|\eta - \zeta_x| < \sigma|\eta' - \xi_{x+1}| + \frac{\Theta_1}{|x|} + \frac{\Theta}{|x|^2}, \tag{20}$$

where Θ, Θ_1 are finite and independent of x.

Before proceeding further, it will be expedient to impose an additional limitation on $|x|$.

When $|x| > \beta$ and $R(x) > -\frac{1}{2}$, we can find a positive quantity Θ_2, independent of x, such that

$$|\xi_{x+1}| < \frac{\Theta_2}{\sigma|x|}. \tag{21}$$

Also we can find a finite quantity γ, greater than or equal to β, such that when $|x| > \gamma$ and $R(x) > -\frac{1}{2}$,

$$|\lambda_1+\xi_x| + \frac{1}{1-\sigma}\left[\frac{\Theta}{|x|^2}+\frac{\Theta_1}{|x|}\right]+\frac{\Theta_2}{\sigma|x|} < K, \qquad (22)^*$$

(remembering that $|\lambda_1| < K$).

We shall now suppose that $|x| > \gamma$ *and* $R(x) > -\frac{1}{2}$.

Now, if we have the series of inequalities

$$U_{x+n} < \sigma U_{x+n+1} + \frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|},$$

$$U_{x+n+1} < \sigma U_{x+n+2} + \frac{\Theta}{|x+n+1|^2}+\frac{\Theta_1}{|x+n+1|},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$U_{x+m} < \sigma U_{x+m+1} + \frac{\Theta}{|x+m|^2}+\frac{\Theta_1}{|x+m+1|},$$

where n is a positive integer or zero, and m is an integer greater than n, the following series of inequalities is *a priori* satisfied

$$U_{x+n} < \sigma U_{x+n+1} + \frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|},$$

$$U_{x+n+1} < \sigma U_{x+n+2} + \frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$U_{x+m} < \sigma U_{x+m+1} + \frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|},$$

since $\quad |x+m| > |x+m-1| > \ldots > |x+n+1| > |x+n|.$

Hence we deduce without any difficulty that

$$U_{x+n} < \frac{1-\sigma^{m+1-n}}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\sigma^{m+1-n}U_{x+m+1}. \qquad (23)$$

Now, if we have the following series of equations defining

$$\eta_{x+m},\ \eta_{x+m-1},\ \ldots,\ \eta_{x+n};$$

The reason for choosing this inequality will be apparent later [see equation (27)].

η_{x+m+1} being supposed known,

$$\frac{1}{a_1}\left\{1+\frac{\lambda_1+\eta_{x+n+r}}{a_1(a_1-a_2)(x+n+r)}\right\}$$

$$=\frac{1}{1-}\left\{\lambda_0+\frac{\lambda_1+\epsilon_{x+n+r}}{x+n+r}\right\}\frac{1}{a_1}\left\{1+\frac{\lambda_1+\eta_{x+n+r+1}}{a_1(a_1-a_2)(x+n+r+1)}\right\}, \quad (24)$$

where $$r=0, 1, 2, \ldots, m-n,$$

and if η_{x+n+1}, η_{x+n+2}, ..., η_{x+m+1} be such that

$$|\lambda_1+\eta_{x+n+1}|<K,$$
$$|\lambda_1+\eta_{x+n+2}|<K,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$|\lambda_1+\eta_{x+m+1}|<K,$$

(η_{x+m+1} will be taken equal to zero for convenience), then the series of inequalities

$$|\eta_{x+n+r}-\xi_{x+n+r}|<\sigma|\eta_{x+n+r+1}-\xi_{x+n+r+1}|+\frac{\Theta}{|x+n+r|^2}+\frac{\Theta_1}{|x+n+r|}, \quad (25)$$

will hold by reason of (20); where r, as before, takes the values 0, 1, 2, ..., $m-n$.

Applying the result (23) to (25), we get

$$|\eta_{x+n}-\xi_{x+n}|<\frac{1-\sigma^{m+1-n}}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\sigma^{m+1-n}|\eta_{x+m+1}-\xi_{x+m+1}|$$

$$<\frac{1}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\sigma|\xi_{x+m+1}|$$

$$<\frac{1}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\frac{\Theta_2}{|x+m|} \quad \text{[by reason of (21)]}$$

$$<\frac{1}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\frac{\Theta_2}{|x+n|}. \quad (26)$$

By this result

$$|\lambda_1+\eta_{x+n}|<|\lambda_1+\xi_{x+n}|+|\eta_{x+n}-\xi_{x+n}|$$

$$<|\lambda_1+\xi_{x+n}|+\frac{1}{1-\sigma}\left[\frac{\Theta}{|x+n|^2}+\frac{\Theta_1}{|x+n|}\right]+\frac{\Theta_2}{|x+n|},$$

i.e., $$|\lambda_1+\eta_{x+n}|<K<\frac{|a_1^3(a_1-a_2)|}{3|\lambda_0|+|a_1^2|}|x+n|, \quad (27)$$

by reason of (19) and (22).

Accordingly, if η_{x+n-1} be defined by putting $r=-1$ in (24), then provided $|x+n-1|>\gamma$ and $R(x+n-1)>-\frac{1}{2}$, we can deduce the inequality

$$|\eta_{x+n-1}-\xi_{x+n-1}|<\sigma|\eta_{x+n}-\xi_{x+n}|+\frac{\Theta}{|x+n-1|^2}+\frac{\Theta_1}{|x+n-1|},$$

in the same way as we deduced (20).

And hence, in the same manner as we deduced (27), we can deduce

$$|\lambda_1+\eta_{x+n-1}|<K<\frac{|a_1^3(a_1-a_2)|}{3|\lambda_0|+|a_1^2|}|x+n-1|. \tag{28}$$

Now it is known that $|\lambda_1+\eta_{x+m+1}|<K$, remembering that $\eta_{x+m+1}=0$.

And hence $|\lambda_1+\eta_{x+n}|<K$, from (20) and (28), and so, by induction from the results (27) and (28), we get

$$|\lambda_1+\eta_{x+n}|<K<\frac{|a_1^3(a_1-a_2)|}{3|\lambda_0|+|a_1^2|}|x+n|,$$

where $n=0, 1, 2, \ldots, m$.

And, further, the inequality (26) now holds for all the values $0, 1, 2, \ldots, m$ of n.

Putting $n=0$ in the inequality, we get

$$|\eta_x-\xi_x|<\frac{1}{1-\sigma}\left[\frac{\Theta}{|x|^2}+\frac{\Theta_1}{|x|}\right]+\frac{\Theta_2}{|x|},$$

and σ, Θ, Θ_1, Θ_2 are finite quantities independent of m and x.

We now see that, if $\psi_m(x)$ be defined as the value of the continued fraction

$$\frac{1}{1}-\frac{v(x)}{1}-\frac{v(x+1)}{1}-\ldots-\frac{v(x+m)}{a_1}\left\{1+\frac{\lambda_1}{a_1(a_1-a_2)(x+m+1)}\right\}, \tag{29}$$

where
$$v(x)=\lambda_0+\frac{\lambda_1}{x}+\frac{\lambda_2}{x^2}+\ldots, \tag{29a}$$

then, when $|x|>$ a certain finite quantity γ and $R(x)>-\frac{1}{2}$,

$$\psi_m(x)=\frac{1}{a_1}\left\{1+\frac{\lambda_1+\eta_x}{a_1(a_1-a_2)x}\right\},$$

where
$$\left|\eta_x-\epsilon_x+\frac{\lambda_1 a_2}{(a_1-a_2)(x+1)}-\frac{\lambda_1(\lambda_1+\epsilon_x)}{a_1(a_1-a_2)(x+1)}-\frac{(\lambda_1+\epsilon_x)^2}{a_1^2 x}\right.$$
$$\left.-\frac{2\lambda_1}{a_1^3(a_1-a_2)(x+1)}(\lambda_1+\epsilon_x)\left(\lambda_0+\frac{\lambda_1+\epsilon_x}{x}\right)-\frac{\lambda_0^2\lambda_1^2}{a_1^4(a_1-a_2)^2(x+1)}\right|$$
$$<\frac{1}{1-\sigma}\left[\frac{\Theta}{|x|^2}+\frac{\Theta_1}{|x|}\right]+\frac{\Theta_2}{|x|}.$$

I.e.,
$$\psi_m(x) = \frac{1}{a_1} + \frac{\lambda_1}{a_1^2(a_1-a_2)x} + J, \tag{30}$$

where $|x^2 J|$ is finite and has a finite upper limit for the values of x and m under consideration.

We now shew that $\psi_m(x) \to$ a *definite* limit, $\psi(x)$, as $m \to \infty$.

We shall first shew that the quantities

$$\left| \prod_{t=0}^{m-2} \left[\lambda_0 v(x+t) \exp\left\{ \frac{-\lambda_1}{\lambda_0(x+t)} \right\} \right] \right|,$$

$$\left| \prod_{t=0}^{m} \{\psi_{m-t}(x+t)\}\, a_1 \exp\left\{ \frac{-\lambda_1}{a_1(a_1-a_2)(x+t)} \right\} \right|,$$

have finite upper limits depending on γ, but independent of m and x.

Consider the second of the two quantities: by (30),

$$\left| \psi_{m-t}(x+t)\, a_1 \exp\left\{ \frac{-\lambda_1}{a_1(a_1-a_2)(x+t)} \right\} \right| < 1 + \frac{K}{|x+t|^2},$$

where K depends on γ, but is independent of x and m; it being supposed that $t \leqslant m$.

Hence

$$\begin{aligned} \left| \prod_{t=0}^{m} \{\psi_{m-t}(x+t)\}\, a_1 \exp\left\{ \frac{-\lambda_1}{a_1(a_1-a_2)(x+t)} \right\} \right| &< \prod_{t=0}^{m} \left\{ 1 + \frac{K}{|x+t|^2} \right\} \\ &< \prod_{t=0}^{\infty} \left\{ 1 + \frac{K}{|x+t|^2} \right\} \\ &< J_2 \end{aligned}$$

(where J_2 is a finite quantity depending on K and γ), since $\sum_{t=0}^{\infty} |x+t|^{-2}$ is absolutely and uniformly convergent.

In a similar manner, using (29a), we may prove that

$$\left| \prod_{t=0}^{m-2} \left[\lambda_0 v(x+t) \exp\left\{ \frac{-\lambda_1}{\lambda_0(x+t)} \right\} \right] \right| < J_1,$$

where J_1 is finite and depends on γ, but not on x or m.

Now put, for brevity,

$$w(x) = \frac{v(x)}{a_1} \left\{ 1 + \frac{\lambda_1}{a_1(a_1-a_2)(x+1)} \right\},$$

and let the r-th convergent of $\psi_m(x)$ be ${}_mP_r/{}_mQ_r$, taking $w(x+m)$ to be the last element of $\psi_m(x)$.

Then, by elementary algebra,

$$\psi_m(x)-\psi_{m-1}(x)$$

$$=\frac{(-)^m\left\{\prod_{t=0}^{m-2}v(x+t)\right\}[v(x+m-1)-w(x+m-1)+w(x+m-1)w(x+m)]}{{}_mQ_{m+2}\cdot{}_{m-1}Q_{m+1}}$$

and

$${}_mQ_{m+2}=\prod_{t=0}^{m}\{\psi_{m-t}(x+t)\}^{-1}.$$

Hence, using the theorems just proved,

$$|\psi_m(x)-\psi_{m-1}(x)|<J_1J_2^2\left|\frac{\lambda_0^{m-1}}{\alpha_1^{2m}}\exp\left\{\sum_{t=0}^{m}\left(\frac{\lambda_1}{\lambda_0}+\frac{2\lambda_1}{\alpha_1(\alpha_1-\alpha_2)}\right)\frac{1}{x+t}\right\}\right|J_3,$$

where J_3 is a finite quantity depending on γ.

Hence, using Mellin's* asymptotic expansion of the logarithmic derivate of the gamma function, and taking $|\arg(x+m+1)|$ and $|\arg(x)|$ to be less than π, we get

$$|\psi_m(x)-\psi_{m-1}(x)|<J_4\left|\frac{\alpha_2}{\alpha_1}\right|^m\left|\exp\left\{\frac{\lambda_1}{\lambda_0(\alpha_1-\alpha_2)}\left(\frac{\Gamma'(x+m+1)}{\Gamma(x+m+1)}-\frac{\Gamma'(x)}{\Gamma(x)}\right)\right\}\right|$$

$$<J_5\left|\frac{\alpha_2}{\alpha_1}\right|^m\left|\exp\left\{\frac{\lambda_1}{\lambda_0(\alpha_1-\alpha_2)}\log\frac{x+m+1}{x}\right\}\right|,$$

where J_4 and J_5 are finite quantities.

Also (noting the limitations imposed on x) when m is a positive integer

$$\left|\exp\left\{\frac{\lambda_1}{\lambda_0(\alpha_1-\alpha_2)}\log\frac{x+m+1}{x}\right\}\right|<Am^p,$$

where p, A are real finite quantities depending on γ, α_1, α_2, λ_1 only.

Consequently,

$$|\psi_m(x)-\psi_n(x)|<|\psi_m(x)-\psi_{m-1}(x)|+\ldots+|\psi_{n+1}(x)-\psi_n(x)|$$

$$<AJ_5\sum_{t=n+1}^{m}t^p|\alpha_2\alpha_1^{-1}|^t,$$

and, since $|\alpha_2\alpha_1^{-1}|<1$, this tends uniformly to zero as m, n both tend to infinity. That is to say, $\operatorname{Lt}_{m=\infty}\psi_m(x)$ is a definite quantity $\psi(x)$, and, by (30), the asymptotic expansion of $\psi(x)$ is

$$\psi(x)=\frac{1}{\alpha_1}+\frac{\lambda_1}{\alpha_1^2(\alpha_1-\alpha_2)x}+J, \tag{30a}$$

where $|x^2J|$ is finite and has a finite upper limit for the values of x under consideration.

* *Acta Societatis Fennicae*, Vol. XXIX.

Now consider the infinite product

$$\chi(x) = \prod_{n=0}^{\infty}\left[a_1\psi(x+n)\exp\left\{\frac{-\lambda_1}{a_1(a_1-a_2)(x+n)}\right\}\right],$$

it being assumed that $-R(x)$ is not an indefinitely great positive quantity.

We write this equation in the form

$$\chi(x) = \prod_{n=0}^{m-1}\left[a_1\psi(x+n)\exp\left\{\frac{-\lambda_1}{a_1(a_1-a_2)(x+n)}\right\}\right] \times \prod_{n=m}^{\infty}\left[a_1\psi(x+n)\exp\left\{\frac{-\lambda_1}{a_1(a_1-a_2)(x+n)}\right\}\right],$$

where m is now the smallest integer which makes $|x+m|>\gamma$ and $R(x+m)>-\frac{1}{2}$.

Then, using (30a), it follows from Weierstrass' theorem that the infinite product converges absolutely.

Now, consider the quantity

$$a_1\psi(x+n)\exp\left\{-\frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right\},$$

when $0 \leqslant n < m$.

Since $$\frac{1}{\psi(x+n-1)} = 1-v(x+n-1)\,\psi(x+n),$$

if the function $v(x)$ have all its singularities (within the circle $|x|=\gamma$) isolated, $\psi(x+n)$ will become infinite or zero for certain isolated values of x, but will, in general, be finite. Accordingly, with the exception of these points, and the points $x=0, -1, -2, \ldots$, the function $\chi(x)$ is analytic in the finite part of the x plane.

If now we write $$\chi(x) = w(x)/\phi(x-1),$$

since $$\psi(x) = \phi(x)/\phi(x-1),$$

we get the difference equation of the first order

$$w(x+1) = a_1^{-1}w(x)\exp\{\lambda_1/a_1(a_1-a_2)\,x\},$$

to determine the function $w(x)$.

A solution of this equation is

$$w(x) = \exp\left\{-(x-1)\log a_1 + \frac{\lambda_1}{a_1(a_1-a_2)}\,\frac{\Gamma'(x)}{\Gamma(x)}\right\}.$$

Hence, in this case, a solution of the difference equation (1) is

$$f(x) = u(x) \exp\left[-x \log a_1 + \frac{\lambda_1}{a_1(a_1-a_2)} \frac{\Gamma'(x+1)}{\Gamma(x+1)}\right] \times \prod_{n=1}^{\infty}\left[\frac{1}{a_1 \psi(x+n)} \exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right\}\right], \qquad \text{(III)}$$

where $u(x)$ is any solution of the first order difference equation

$$u(x)\, B(x) = u(x-1)\, C(x),$$

and where $\psi(x)$, $v(x)$ are defined by (29) and (29a), and a_1, a_2 are the roots of $a^2-a+\lambda_0 = 0$, taken in such a way that $|a_1| > |a_2|$.

7. CASE (IIIa).—$\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, λ_2 not a real quantity greater than or equal to $\frac{1}{16}$.

The asymptotic expansion of $\psi(x)$ with these values of λ_0, λ_1, λ_2 can be obtained by methods of a nature similar to those employed in §§ 4 and 6. The necessary analysis is more elaborate on account of the fact that the modulus of a certain quantity (which corresponds to the quantity called σ in § 6) is not appreciably different from unity.

We shall now prove that, subject to a certain interpretation of the meaning of the infinite continued fraction

$$\psi(x) = \frac{1}{1-}\ \frac{v(x)}{1-}\ \frac{v(x+1)}{1-}\ \cdots,$$

when $R(x) > -\frac{1}{2}$, $\psi(x)$ asymptotically approaches the value $2+\frac{4a}{x}$, where a, a' are the roots of the quadratic equation

$$a^2-\tfrac{1}{2}a+\lambda_2 = 0,$$

so chosen that $R(a) < R(a')$; this choice can always be made, on account of the limitation placed on λ_2.

When $|x| > \mu$ and $R(x) > -\frac{1}{2}$, we have

$$v(x) = \tfrac{1}{4} + \frac{\lambda_2+\epsilon_x}{x^2},$$

where $|\epsilon_x|$ has a finite upper limit E', and $|x\epsilon_x|$ has a finite upper limit E.

Now consider quantities η, η' satisfying the relation

$$2+\frac{4(a+\eta)}{x} = \frac{1}{1-}\left\{\tfrac{1}{4}+\frac{\lambda_2+\epsilon_x}{x^2}\right\}\left\{2+\frac{4(a+\eta')}{x+1}\right\}.$$

On reduction this becomes

$$2(\alpha+\eta) = x\left[1-\frac{2(\alpha+\eta')}{x+1}-\frac{4(\lambda_2+\epsilon_x)}{x^2}-\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^{-1}-x.$$

Using the identity

$$(1-U)^{-1} \equiv 1+U+U^2+U^3(1-U)^{-1},$$

this may be written in the form

$$\begin{aligned}2(\alpha+\eta) = \quad & x\left[\frac{2(\alpha+\eta')}{x+1}+\frac{4(\lambda_2+\epsilon_x)}{x^2}+\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right] \\ & +x\left[\frac{2(\alpha+\eta')}{x+1}+\frac{4(\lambda_2+\epsilon_x)}{x^2}+\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^2 \\ & +x\left[\frac{2(\alpha+\eta')}{x+1}+\frac{4(\lambda_2+\epsilon_x)}{x^2}+\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^3 \\ & \times\left[1-\frac{2(\alpha+\eta')}{x+1}-\frac{4(\lambda_2+\epsilon_x)}{x^2}-\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^{-1}.\end{aligned}$$

We may rewrite this equation in the form

$$\begin{aligned}\eta = & \frac{2\lambda_2(2x+1)}{x(x+1)^2}-\frac{\alpha}{(x+1)^2}+\frac{2\epsilon_x}{x} \\ & +\eta'\left[1+\frac{1}{x+1}\left\{4\alpha-1-\frac{4\alpha}{x+1}+\frac{12(\lambda_2+\epsilon_x)}{x}+\frac{32\alpha(\lambda_2+\epsilon_x)}{x(x+1)}+\frac{32(\lambda_2+\epsilon_x)^2}{x^3}\right.\right. \\ & \qquad \left.\left.+\frac{64\alpha(\lambda_2+\epsilon_x)^2}{x^3(x+1)}\right\}+\frac{2x\eta'}{(x+1)^2}\left\{1+\frac{4(\lambda_2+\epsilon_x)}{x^2}\right\}^2\right] \\ & +\frac{x}{2}\left[\frac{2(\alpha+\eta')}{x+1}+\frac{4(\lambda_2+\epsilon_x)}{x^2}+\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^3 \\ & \times\left[1-\frac{2(\alpha+\eta')}{x+1}-\frac{4(\lambda_2+\epsilon_x)}{x^2}-\frac{8(\lambda_2+\epsilon_x)(\alpha+\eta')}{x^2(x+1)}\right]^{-1} \\ & +\text{some terms whose numerators involve } \alpha,\ \lambda_2,\ \epsilon_x \text{ and whose denominators involve terms of order } x^2 \text{ at least.} \qquad (31)\end{aligned}$$

Now we can find a finite quantity β, greater than μ and $16|\alpha|$, such

that when $|x| > \beta$ and $R(x) > -\frac{1}{2}$,

$$\left|\frac{4(\lambda_2+\epsilon_x)}{x^2}\right| < \tfrac{1}{3}, \qquad \left|\frac{4(\lambda_2+\epsilon_x)}{x}\right| < 1, \tag{32}$$

$$\left| -\frac{4a}{x+1} + \frac{12(\lambda_2+\epsilon_x)}{x} + \frac{32a(\lambda_2+\epsilon_x)}{x(x+1)} + \frac{32(\lambda_2+\epsilon_x)^2}{x^3} + \frac{64a(\lambda_2+\epsilon_x)^2}{x^3(x+1)} \right|$$
$$< \tfrac{2}{9}\,|R(4a-1)|. \tag{33}$$

These possibilities follow from the fundamental continuity theorem; further, we shall choose η' so that $|\eta'|$ is less than each of the two quantities $|a|$ and $\frac{1}{16}|R(4a-1)|$.

From (31) we deduce that

$$|\eta| < |\eta'| \left| \left[1 + \frac{1}{x+1} \left\{ 4a - 1 - \frac{4a}{x+1} + \frac{12(\lambda_2+\epsilon_x)}{x} + \frac{32a(\lambda_2+\epsilon_x)}{x(x+1)} \right.\right.\right.$$
$$\left.\left.\left. + \frac{32(\lambda_2+\epsilon_x)^2}{x^3} + \frac{64a(\lambda_2+\epsilon_x)^2}{x^3(x+1)} \right\} + \frac{2x\eta'}{(x+1)^2} \left\{ 1 + \frac{4(\lambda_2+\epsilon_x)}{x^2} \right\}^2 \right] \right|$$
$$+ \tfrac{1}{2}|x| \left[\frac{2|a|}{|x|} + \frac{1}{|x|} + \frac{2|a|}{|x|^2} \right]^3 [1 - \tfrac{1}{4} - \tfrac{1}{3} - \tfrac{1}{12}]^{-1} + \frac{\Theta_1}{|x|^2}, \tag{34}$$

where Θ_1 is a finite quantity, independent of x, which depends only on $|a|$, β, $|\lambda_2|$, E and E'; this follows from (32) and (33), remembering that $\beta > 16|a|$.

Now put

$$4a - 1 - \frac{4a}{x+1} + \frac{12(\lambda_2+\epsilon_x)}{x} + \frac{32a(\lambda_2+\epsilon_x)}{x(x+1)} + \frac{32(\lambda_2+\epsilon_x)^2}{x^3} + \frac{64a(\lambda_2+\epsilon_x)^2}{x^3(x+1)}$$
$$+ \frac{2x\bar{\eta}_x'}{x+1} \left\{ 1 + \frac{4(\lambda_2+\epsilon_x)}{x^2} \right\}^2 = \xi_x, \tag{35}$$

where $\bar{\eta}_x$ is an arbitrary quantity subject only to the limitations

$$|\bar{\eta}_x'| < |a|, \qquad |\bar{\eta}_x'| < \tfrac{1}{16}|R(4a-1)|.$$

Then, by reason of (32) and (33), $R(\xi_x)$ lies between* the quantities

$$[1 \pm \{\tfrac{2}{9} + \tfrac{2}{16}(1+\tfrac{1}{3})^2\}]\,R(4a-1),$$

i.e., $R(\xi_x)$ lies between $\frac{5}{9}R(4a-1)$ and $\frac{13}{9}R(4a-1)$,

and
$$|\xi_x| < [1 + \tfrac{2}{9} + \tfrac{2}{16}(1+\tfrac{1}{3})^2]\,|4a-1|,$$

i.e.,
$$|\xi_x| < \tfrac{13}{9}|4a-1|.$$

* We write the statement that "c lies between $a+b$ and $a-b$" in the form "c lies between $a \pm b$," for brevity.

Hence the series $\sum_{r=0}^{\infty} \frac{\xi_{x+r}^2}{(x+r+1)^2}$ is absolutely convergent. Accordingly, by Weierstrass' theorem, the product

$$\prod_{r=0}^{\infty}\left[\left(1+\frac{\xi_{x+r}}{x+r+1}\right)\exp\left(\frac{-\xi_{x+r}}{x+r+1}\right)\right]$$

is absolutely convergent for all the values of x under consideration.

And, since all the factors are of finite modulus, it follows that

$$\left|\prod_{r=0}^{s}\left[\left(1+\frac{\xi_{x+r}}{x+r+1}\right)\exp\left(\frac{-\xi_{x+r}}{x+r+1}\right)\right]\right|$$

has a finite upper limit depending only on α, β, E, E'; where s is any positive integer. This upper limit will be called Φ. Also

$$\exp\frac{\xi_{x+r}}{x+r+1}=\exp\frac{\xi_{x+r}-4\alpha+1-4\bar{\eta}'_{x+r}}{x+r+1}\exp\frac{4\alpha-1+4\bar{\eta}'_{x+r}}{x+r+1}.$$

Now the series $\sum_{r=0}^{\infty}\frac{\xi_{x+r}-4\alpha+1-4\bar{\eta}'_{r+r}}{x+r+1}$ is absolutely convergent for all the values of x under consideration; accordingly

$$R\left[\sum_{r=0}^{s}\frac{\xi_{x+r}-4\alpha+1-4\bar{\eta}'_{x+r}}{x+r+1}\right]$$

will have a finite upper limit depending only on α, β, E, E'; s being, as before, any positive integer. This upper limit will be called Φ'.

We now consider the value of

$$R\left[\sum_{r=0}^{s}\frac{4\alpha-1+4\bar{\eta}'_{x+r}}{x+r+1}\right].$$

We first notice that

$R[4\alpha-1+4\bar{\eta}'_{x+r}]$ lies between $(1\pm\frac{1}{4})R(4\alpha-1)$,

and $I[4\alpha-1+4\bar{\eta}'_{x+r}]$ lies between $I(4\alpha-1)\pm\frac{1}{4}R(4\alpha-1)$.

Let $$\frac{1}{x+r+1}=\xi_r+i\upsilon_r.$$

Then all the quantities ξ_r are positive; and all the quantities υ_r have the same sign.

Consequently $$\sum_{r=0}^{s}\left[R(4\alpha-1+4\bar{\eta}'_{x+r}).R\frac{1}{x+r+1}\right]$$

lies between $$(1\pm\tfrac{1}{4})R(4\alpha-1)\sum_{r=0}^{s}\xi_r;$$

and these are two negative quantities since $R(4\alpha-1)$ is negative, while

$$\sum_{r=0}^{s}\left[I(4\alpha-1+4\bar{\eta}'_{x+r}).I\frac{1}{x+r+1}\right]$$

lies between $\quad [I(4\alpha-1)\pm\tfrac{1}{4}R(4\alpha-1)]\sum_{r=0}^{s}\left\{I\frac{1}{x+r+1}\right\}.$

Now if, as we may, we assume

$$|\arg(x+s+2)|<\pi, \qquad |\arg(x+1)|<\pi,$$

we have

$$\frac{d}{dx}\log\Gamma(x+s+2)=\log(x+s+2)+\frac{1}{2(x+s+2)}+\frac{J_{s+1}}{(x+s+2)^2},$$

$$\frac{d}{dx}\log\Gamma(x+1)=\log(x+1)+\frac{1}{2(x+1)}+\frac{J_0}{(x+1)^2},$$

where, for the values of x and s under consideration, $|J_0|$, $|J_{s+1}|$ are less than certain finite quantities depending on β only. These results are due to Mellin.*

But $\quad \sum_{r=0}^{s}\frac{1}{x+r+1}=\frac{d}{dx}[\log\Gamma(x+s+2)-\log\Gamma(x+1)].$

And therefore

$$\sum_{r=0}^{s}I\left(\frac{1}{x+r+1}\right)=I[\log(x+s+2)-\log(x+1)]+I\left[\frac{1}{2(x+s+2)}-\frac{1}{2(x+1)}\right]$$
$$+I\left[\frac{J_{s+1}}{(x+s+2)^2}-\frac{J_0}{(x+1)^2}\right].$$

Now, with the convention which we have made,

$$|I[\log(x+s+2)]-I[\log(x+1)]|<\tfrac{1}{2}\pi,$$

while it is easy to verify that

$$\left|I\left[\frac{1}{2(x+s+2)}-\frac{1}{2(x+1)}\right]\right|<\frac{1}{2\beta},$$

and $\quad \left|I\left[\frac{J_{s+1}}{(x+s+2)^2}-\frac{J_0}{(x+1)^2}\right]\right|<\frac{|J_{s+1}|+|J_0|}{\beta^2}.$

Accordingly

$$\left|\sum_{r=0}^{s}I\left(\frac{1}{x+r+1}\right)\right|<\frac{\pi}{2}+\frac{1}{2\beta}+\frac{|J_{s+1}|+|J_0|}{\beta^2}<\Phi'',$$

where Φ'' is a finite quantity depending on β only.

* Mellin, *Acta Societatis Fennicae*, Vol. XXIX.

And therefore

$$\left|\sum_{r=0}^{s}\{I(4a-1+4\bar{\eta}'_{x+r})\}\left\{I\frac{1}{x+r+1}\right\}\right| < \Phi''[|I(4a-1)|+\tfrac{1}{4}|R(4a-1)|] < \Phi''',$$

where Φ''' is a finite quantity depending only on β and a. Therefore, since $\sum_{r=0}^{s}\{R(4a-1+4\bar{\eta}'_{x+r})\}\left\{R\frac{1}{x+r+1}\right\}$ is negative, we have

$$R\sum_{r=0}^{s}\left\{\frac{4a-1+4\bar{\eta}'_{x+r}}{x+r+1}\right\} < \Phi'''.$$

Hence $$\left|\prod_{r=0}^{s}\left\{1+\frac{\xi_{x+r}}{x+r+1}\right\}\right| < \Phi\left|\exp\sum_{r=0}^{s}\left(\frac{\xi_{x+r}}{x+r+1}\right)\right| < \Phi\exp(\Phi'+\Phi'''),$$

since the modulus of the exponential of a pure imaginary is unity;

i.e., $$\left|\prod_{r=0}^{s}\left\{1+\frac{\xi_{x+r}}{x+r+1}\right\}\right| < \Phi_0, \tag{36}$$

where Φ_0 is a quantity depending only on a, β, E, E'; for convenience, we shall take Φ_0 to be not less than unity.

It is now expedient to impose a further limitation on $|x|$. In the first place we note that it is possible to find a finite quantity Θ, such that

$$3|x|\left[\frac{2|a|}{|x|}+\frac{1}{|x|}+\frac{2|a|}{|x|^2}\right]^3+\frac{\Theta_1}{|x|^2} < \frac{\Theta}{|x|^2}; \tag{37}$$

and we can find a finite quantity Θ', such that when

$$|x|>\beta \quad \text{and} \quad R(x)>-\tfrac{1}{2},$$

$$\sum_{n=0}^{\infty}\frac{1}{|x+n|^2} < \frac{\Theta'}{|x|}, \tag{38}$$

Θ' being a quantity which depends on β only; this follows from the fact that

$$\sum_{n=1}^{\infty}\frac{1}{|x+n|^2} < \sum_{n=1}^{\infty}\frac{1}{|x+\frac{1}{2}|^2+(n-\frac{1}{2})^2} < \frac{\pi\tanh(\pi|x+\frac{1}{2}|)}{|2x+1|}.$$

Also we can find a finite quantity γ, such that, when

$$|x|>\gamma \quad \text{and} \quad R(x)>-\tfrac{1}{2},$$

then $$\frac{\Theta\Theta'\Phi_0}{|x|} < |a| \quad \text{and} \quad \frac{\Theta\Theta'\Phi_0}{|x|} < \tfrac{1}{16}|R(4a-1)|. \tag{39}$$

We shall now suppose that $|x|>\gamma$ *and* $R(x)>-\frac{1}{2}$.

Let us define quantities $\eta_x, \eta_{x+1}, \ldots, \eta_{x+m}$ by the system of equations

$$2+\frac{4(a+\eta_{x+r})}{x+r}=\frac{1}{1-}\left\{\frac{1}{4}+\frac{\lambda_2+\epsilon_{x+r}}{(x+r)^2}\right\}\left\{2+\frac{4(a+\eta_{x+r+1})}{x+r+1}\right\}, \qquad (40)$$

where $r=0, 1, 2, \ldots, m$, and $\eta_{x+m+1}=0$; and *let us assume* that when $r=n, n+1, \ldots, m$, then $|\eta_{x+r+1}|$ is less than each of the quantities $|a|$ and $\frac{1}{16}|R(4a-1)|$.

By (34) and (37), we then have the system of inequalities

$$|\eta_{x+n+r}|<|\eta_{x+n+r+1}|\left|\left[1+\frac{\xi'_{x+n+r}}{x+n+r+1}\right]\right|+\frac{\Theta}{|x+n+r|^2}, \qquad (41)$$

where r now denotes the quantities $0, 1, \ldots, m-n$, and ξ'_x only differs from ξ_x in having $\overline{\eta}'_x$ replaced by η_{x+1}.

From the system of inequalities (41), we deduce that

$$|\eta_{x+n}|<|\eta_{x+m+1}|\prod_{r=0}^{m-n}\left|\left[1+\frac{\xi'_{x+n+r}}{x+n+r+1}\right]\right|+\frac{\Theta}{|x+n|^2}$$
$$+\sum_{r=1}^{m-n}\frac{\Theta}{|x+n+r|^2}\prod_{s=1}^{r}\left|\left[1+\frac{\xi'_{x+n+s}}{x+n+s+1}\right]\right|. \qquad (42)$$

Now, since Φ_0 does not depend on x or n, it follows from (36) that

$$\prod_{s=1}^{r}\left|\left[1+\frac{\xi_{x+n+s}}{x+n+s+1}\right]\right|<\Phi_0.$$

Hence (42) may be written in the form

$$|\eta_{x+n}|<|\eta_{x+m+1}|\,\Phi_0+\Theta\Phi_0\sum_{r=0}^{m-n}\frac{1}{|x+n+r|^2}$$
$$<\frac{\Theta\Phi_0\Theta'}{|x+n|}, \qquad (43)$$

since $$\eta_{x+m+1}=0.$$

That is to say, $|\eta_{x+n}|<|a|$, $|\eta_{x+n}|<\frac{1}{16}|R(4a-1)|$; this follows from the definition of γ.

Therefore all the quantities $|\eta_{x+n}|, |\eta_{x+n+1}|, \ldots, |\eta_{x+m+1}|$, are less than each of the quantities $|a|, \frac{1}{16}|R(4a-1)|$.

Consequently, we can deduce the inequality

$$|\eta_{x+n-1}|<|\eta_{x+n}|\left|\left[1+\frac{\xi'_{x+n-1}}{x+n}\right]\right|+\frac{\Theta}{|x+n-1|^2}.$$

Hence, by reasoning similar to that by which it was deduced that

$$|\eta_{x+n}| < |a|, \qquad |\eta_{x+n}| < \tfrac{1}{16}|R(4a-1)|,$$

we can prove that

$$|\eta_{x+n-1}| < |a|, \qquad |\eta_{x+n-1}| < \tfrac{1}{16}|R(4a-1)|.$$

Hence, by induction, all the quantities $\eta_x, \eta_{x+1}, \ldots, \eta_{x+m}$ are less than each of the quantities $|a|$ and $\frac{1}{16}|R(4a-1)|$.

Also the inequality (43) now holds for all the values $0, 1, \ldots, m$ of n.

Consequently
$$|\eta_x| < \frac{\Theta\Phi_0\Theta'}{|x|},$$

where Θ, Φ_0, Θ' are finite quantities independent of m and x.

We now see that, if $\psi_m(x)$ be defined as the value of the continued fraction,

$$\frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots - \frac{v(x+m)}{1}\left\{2+\frac{4a}{x+m+1}\right\}, \tag{44}$$

where
$$v(x) = \tfrac{1}{4} + \frac{\lambda_2}{x^2} + \frac{\lambda_3}{x^3} + \ldots, \tag{44a}$$

then, when $|x| >$ a certain finite quantity γ, and $R(x) > -\frac{1}{2}$,

$$\psi_m(x) = 2 + \frac{4(a+\eta_x)}{x},$$

where
$$|\eta_x| < \frac{\Theta\Phi_0\Theta'}{|x|}.$$

I.e.,
$$\psi_m(x) = 2 + \frac{4a}{x} + J, \tag{45}$$

where $|x^2 J|$ is finite and has a finite upper limit for the values of x and m under consideration.

We now shew that $\psi_m(x) \to$ a definite limit, $\psi(x)$, as $m \to \infty$. Putting, for brevity,

$$w(x) = v(x)\left\{2+\frac{4a}{x+1}\right\},$$

let the r-th convergent of $\psi_m(x)$ be ${}_mP_r/{}_mQ_r$, taking $w(x+m)$ to be the last element in $\psi_m(x)$.

From (44a) and (45) it follows that the quantities

$$\left| \prod_{t=0}^{m-2} \{4v(x+t)\} \right|,$$

$$\left| \prod_{t=0}^{m} \{\tfrac{1}{2}\psi_{m-t}(x+t)\} \exp\left(\frac{-2\alpha}{x+t}\right) \right|$$

have finite upper limits independent of the values of m and x; let these upper limits be J_1, J_2. The proofs are almost identical with those of the corresponding theorems in § 6, and it seems unnecessary to give them.

Now, by elementary algebra,

$$\psi_m(x)-\psi_{m-1}(x)$$

$$= \frac{(-)^m \left\{ \prod_{t=0}^{m-2} v(x+t) \right\} [v(x+m-1)-w(x+m-1)+w(x+m-1)\,w(x+m)]}{{}_mQ_{m+2} \cdot {}_{m-1}Q_{m+1}}$$

and

$${}_mQ_{m+2} = \prod_{t=0}^{m} \{\psi_{m-t}(x+t)\}^{-1}.$$

We also notice that

$$v(x-1)-w(x-1)+w(x-1)\,w(x) = J_0 x^{-2},$$

where $|J_0|$ has a finite upper limit for values of $|x|$ greater than γ.

Consequently,

$$|\psi_m(x)-\psi_{m-1}(x)| < J_1 J_2^2 |x+m|^{-2} J_3 \left| \prod_{t=0}^{m} \left\{ \exp\left(\frac{4\alpha}{x+t}\right) \right\} \right|,$$

where J_3 is a finite quantity.

I.e.,

$$|\psi_m(x)-\psi_{m-1}(x)| < J_4 |x+m|^{-2} \left| \exp\left\{ 4\alpha \frac{\Gamma'(x+m+1)}{\Gamma(x+m+1)} - 4\alpha \frac{\Gamma'(x)}{\Gamma(x)} \right\} \right|,$$

where J_4 is a finite quantity.

Using Mellin's asymptotic expansion* for the gamma function, we get

$$|\psi_m(x)-\psi_{m-1}(x)| < J_5 \frac{|x^{-4\alpha}|}{|(x+m)^{2-4\alpha}|},$$

where J_5 is finite, and it is assumed that $|\arg(x+m)|$ and $|\arg(x)|$ are less than π.

Consequently,

$$|\psi_m(x)-\psi_n(x)| < |\psi_m(x)-\psi_{m-1}(x)| + \ldots + |\psi_{n+1}(x)-\psi_n(x)|$$

$$< J_5 |x^{-4\alpha}| \sum_{t=n+1}^{m} |(x+t)^{4\alpha-2}|.$$

* *Acta Societatis Fennicae*, Vol. XXIX.

Since $R(4a-1) < 0$, this tends uniformly to zero as m, n tend to infinity. That is to say, $\underset{m=\infty}{\text{Lt}}\, \psi(x)$ is a definite quantity $\psi(x)$, and, by (45), the asymptotic expansion of $\psi(x)$ is

$$\psi(x) = 2 + \frac{4a}{x} + J, \tag{45a}$$

where $|x^2 J|$ is finite and has a finite upper limit for the values of x under consideration.

Now consider the infinite product

$$\chi(x) = \prod_{n=0}^{\infty} [\tfrac{1}{2}\psi(x+n) \exp\{-2a/(x+n)\}],$$

it being assumed that $-R(x)$ is not an indefinitely great positive quantity.

We write this equation in the form

$$\chi(x) = \prod_{n=0}^{m-1} [\tfrac{1}{2}\psi(x+n) \exp\{-2a/(x+n)\}] \times \prod_{n=m}^{\infty} [\tfrac{1}{2}\psi(x+n) \exp\{-2a/(x+n)\}],$$

where m is now the smallest integer which makes

$$|x+m| > \gamma \quad \text{and} \quad R(x+m) > -\tfrac{1}{2}.$$

Then, using (45a), it follows from Weierstrass' theorem that the infinite product converges absolutely.

Now consider the quantity $\frac{1}{2}\psi(x+n) \exp\{-2a/(x+n)\}$, when $0 \leqslant n < m$.

Since $$1/\psi(x+n-1) = 1 - v(x+n-1)\,\psi(x+n),$$

if the function $v(x)$ have all its singularities (in the circle $|x| = \gamma$) isolated, $\psi(x+n)$ will become infinite or zero for certain isolated values of x, but will, in general, be finite. Accordingly, with the exception of these points and the points $x = 0, -1, -2, \ldots$, the function $\chi(x)$ is analytic in the finite part of the x-plane.

If now we write $$\chi(x) = w(x)/\phi(x-1),$$

since $$\psi(x) = \phi(x)/\phi(x-1),$$

we get the difference equation of the first order

$$w(x+1) = 2w(x) \exp\left(\frac{2a}{x}\right),$$

to determine the function $w(x)$.

A solution of this equation is

$$w(x) = \exp\left[(x-1)\log 2 + 2a\frac{\Gamma'(x)}{\Gamma(x)}\right].$$

Hence, in this case, a solution of the difference equation (1) is

$$f(x) = u(x)\exp\left[x\log 2 + 2a\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right]\prod_{n=1}^{\infty}\left[\frac{2}{\psi(x+n)}\exp\left(\frac{2a}{x+n}\right)\right], \quad \text{(III}a\text{)}$$

where $u(x)$ is any solution of the first order difference equation

$$u(x)\,B(x) = u(x-1)\,C(x),$$

and where $\psi(x)$, $v(x)$ are defined by (44) and (44a), and a is that root of the equation

$$a^2 - \tfrac{1}{2}a + \lambda_2 = 0,$$

which is such that $R(a) < \frac{1}{4}$; it being supposed that λ_2 is not a real quantity equal to or greater than $\frac{1}{16}$.

[*Note added March 17th*, 1909.—It has been assumed in the above investigation that $a \neq 0$; if $a = 0$, then $\lambda_2 = 0$, and the investigation has to be modified accordingly; the result is that, if

$$v(x) = \tfrac{1}{4} + \frac{\lambda_{h+1}}{x^{h+1}} + \frac{\lambda_{h+2}}{x^{h+2}} + \ldots,$$

where h is an integer greater than unity, and if

$$\psi(x) = \operatorname*{Lt}_{m=\infty}\left[\frac{1}{1} - \frac{v(x)}{1} - \frac{v(x+1)}{1} - \ldots - \frac{v(x+m)}{1}\left\{2 + \frac{8\lambda_{h+1}}{h(x+m+1)^h}\right\}\right],$$

then when $R(x) > -\frac{1}{2}$ and $|x|$ is greater than a certain finite quantity γ,

$$\psi(x) = 2 + \frac{8\lambda_{h+1}}{hx^h} + \frac{J}{x^{h+1}},$$

where $|J|$ is less than a finite quantity depending on γ. The solution of the difference equation is accordingly obtained by putting $a = 0$ in the result (IIIa) above.]

8. It might be remarked that all the preceding analysis is valid if the series for $v(x)$ be not a convergent series, but an asymptotic expansion valid for large values of $|x|$ when $R(x) > -\frac{1}{2}$. In § 9, however, we shall need an expansion of $v(x)$ in power series valid when $R(x) < \frac{1}{2}$.

And it may be proved that if the asymptotic expansion of $v(x)$ is of the form

$$v(x) = \lambda_0 + \frac{\lambda_1}{x} + \frac{\lambda_2}{x^2} + \ldots, \quad \text{when} \quad R(x) > -\tfrac{1}{2},$$

and of the form

$$v(x) = \lambda_0' + \frac{\lambda_1'}{x} + \frac{\lambda_2'}{x^2} + \ldots, \quad \text{when} \quad R(x) < \tfrac{1}{2},$$

then, for all values of n, $\lambda_n = \lambda_n'$, and for sufficiently large values of $|x|$, the series for $v(x)$ is absolutely convergent.* This is the reason for the restriction (i.) of § 1.

It is perhaps necessary to point out that, if in each of the cases (I), (II), (III) and (IIIa), we had assumed that $\psi(x)$ could be expanded as a power series in x^{-1}, we could obtain the coefficients in this power series by ordinary algebra;† there is, however, no reason to suppose that $\psi(x)$ can be expressed as a convergent series, and, indeed, it may be shewn by general reasoning that the power series which is obtained in this manner is not convergent except in special cases.

It is not improbable that the power series gives a complete asymptotic expansion for $\psi(x)$, though I have not attempted to prove this; there is, of course, no need to obtain more than the asymptotic expressions which have been obtained, so far as the theory of linear difference equations is concerned.

An instance of the fact that the assumption that $\psi(x)$ can be expanded in a power series in x^{-1} is an illegitimate assumption is afforded by the failure of the investigation of § 7, when $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, and λ_2 is a real quantity greater than or equal to $\frac{1}{16}$.

9. It is possible to obtain solutions of the difference equation (1) in a somewhat different form to those solutions already given.

In some particular cases, these second solutions appear to be essentially different from those already obtained; but whether in all cases independent solutions can be obtained by this method I am not prepared to say.

In the original equation (1) put

$$x = -y, \quad f(x) = F_1(y),$$

$$A(x) = A_1(y), \quad B(x) = B_1(y), \quad C(x) = C_1(y).$$

* Bromwich, *Theory of Infinite Series*, pp. 335, 336.

† This is the method by which the values of the asymptotic expressions of §§ 4–7 were first obtained.

The equation (1) then becomes

$$C_1(y)\,F_1(y+1)-B_1(y)\,F_1(y)+A_1(y)\,F_1(y-1)=0. \qquad (46)$$

Let $$v_2(y)=\frac{A_1(y+1)\,C_1(y)}{B_1(y+1)\,B_1(y)}=\frac{A(x-1)\,C(x)}{B(x-1)\,B(x)}=v(x-1),$$

so that $v_2(y)$ can be expanded in the form

$$v_2(y)=\lambda_0'-\frac{\lambda_1'}{y}+\frac{\lambda_2'}{y^2}-\ldots,$$

when $$|y|>\mu+1.$$

Comparing equation (46) with equation (1) we obtain the following results by results already proved.

Let $u_1(x)$ be any solution of the first order difference equation

$$u_1(x)\,B(x)=u_1(x+1)\,A(x).$$

(IV) If $\lambda^0=\lambda_1'=0$; *i.e.*, if $\lambda_0=\lambda_1=0$, and if

$$\psi_1(x)=\operatorname*{Lt}_{m=\infty}\left[\frac{1}{1}-\frac{v(x-1)}{1}-\frac{v(x-2)}{1}-\ldots-\frac{v(x-m-1)}{1}\times\{1+v(x-m-2)\}\right],$$

a solution of the difference equation (1) is

$$f_1(x)=u_1(x)\prod_{n=1}^{\infty}\left[\frac{1}{\psi_1(x-n)}\right],$$

(V) If $\lambda_0'=0$, $\lambda_1'\neq 0$; *i.e.*, if $\lambda_0=0$, $\lambda_1\neq 0$, and if

$$\psi_1(x)=\operatorname*{Lt}_{m=\infty}\left[\frac{1}{1}-\frac{v(x-1)}{1}-\frac{v(x-2)}{1}-\ldots-\frac{v(x-m-1)}{1}\times\{1+v(x-m-2)\}\right],$$

a solution of the difference equation (1) is

$$f_1(x)=u_1(x)\exp\left\{\lambda_1\frac{\Gamma'(1-x)}{\Gamma(1-x)}\right\}\prod_{n=1}^{\infty}\left[\frac{1}{\psi_1(x-n)}\exp\left(\frac{\lambda_1}{x-n}\right)\right].$$

(VI) If $\lambda_0'\neq 0$, nor a real quantity greater than or equal to $\frac{1}{4}$; *i.e.*, if $\lambda_0\neq 0$, nor a real quantity greater than or equal to $\frac{1}{4}$; and if

$$\psi_1(x)=\operatorname*{Lt}_{m=\infty}\left\{\frac{1}{1}-\frac{v(x-1)}{1}-\frac{v(x-2)}{1}-\ldots-\frac{v(x-m-1)}{a_1}\times\left\{1+\frac{\lambda_1}{a_1(a_1-a_2)(x-m-1)}\right\}\right],$$

with the definitions of a_1, a_2 given in § 6, a solution of the difference equation (1) is

$$f_1(x) = u_1(x) \exp\left[x \log a_1 - \frac{\lambda_1}{a_1(a_1-a_2)} \frac{\Gamma'(1-x)}{\Gamma(1-x)} \times \prod_{n=1}^{\infty} \left[\frac{1}{a_1 \psi_1(x-n)} \exp \frac{\lambda_1}{a_1(a_1-a_2)(x-n)}\right]\right].$$

(VIa) If $\lambda_0' = \frac{1}{4}$, $\lambda_1' = 0$, $\lambda_2' \neq$ a real quantity greater than or equal to $\frac{1}{16}$; *i.e.*, if $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, $\lambda_2 \neq$ a real quantity greater than or equal to $\frac{1}{16}$; and if

$$\psi_1(x) = \operatorname*{Lt}_{m=\infty} \left[\frac{1}{1} - \frac{v(x-1)}{1} - \frac{v(x-2)}{1} - \ldots - \frac{v(x-m-1)}{1}\left\{2 - \frac{4a}{x-m-1}\right\}\right],$$

with the definition of a given in § 7, a solution of the difference equation (1) is

$$f_1(x) = u_1(x) \exp\left[-x \log 2 + 2a \frac{\Gamma'(1-x)}{\Gamma(1-x)}\right] \prod_{n=1}^{\infty} \left[\frac{2}{\psi(x-n)} \exp\left(\frac{-2a}{x-n}\right)\right].$$

10. From the point of view of the theory of linear differential equations, it is desirable to have an upper limit for the value of the solution of the modified difference equation (2a), viz.,

$$\phi(x-1) = \phi(x) - v(x)\, \phi(x+1),$$

when $|x|$ is large and $R(x) < -\frac{1}{2}$.

The analysis which gives a satisfactory result is very similar to a portion of the paper by Poincaré, already quoted.

Let the singularities of $v(x)$ be $x_0, x_1, x_2, \ldots, x_r, \ldots$.

Suppose that we agree to consider only those values of x which satisfy the inequality

$$|x+m-x_r| > \delta,$$

where δ is an arbitrary pre-assigned finite non-zero quantity (thus we might take $\delta = 10^{-6}$); and where m, r are any positive integers or zero.

Then, for all the values of x under consideration, it is possible to find a finite quantity τ, such that

$$|v(x)| < \tau.$$

Let p be the smallest integer, such that

$$|x+p| > \gamma, \qquad R(x+p) > -\tfrac{1}{2},$$

the quantity γ being defined for the various forms of $v(x)$ in §§ 4–7.

We then have $|\phi(x)| \leqslant |\phi(x+1)| + \tau|\phi(x+2)|$,

$$|\phi(x+1)| \leqslant |\phi(x+2)| + \tau|\phi(x+3)|,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$|\phi(x+p-1)| \leqslant |\phi(x+p)| + \tau|\phi(x+p+1)|.$$

From these we deduce that

$$|\phi(x)| \leqslant \frac{\tau_2|\phi(x+p)| - |\phi(x+p+1)|}{\tau_2-\tau_1}\tau_1^p + \frac{\tau_1|\phi(x+p)| - |\phi(x+p+1)|}{\tau_1-\tau_2}\tau_2^p,$$

where $\tau_1 = \frac{1}{2}\{1+(1+4\tau)^{\frac{1}{2}}\}$, $\tau_2 = \frac{1}{2}\{1-(1+4\tau)^{\frac{1}{2}}\}$.

so that $|\tau_2| < |\tau_1|$ and $\tau_1 > 1$.

Now, if $\phi(x+p)$, $\phi(x+p+1)$ have the values which have been obtained for them in §§ 4–7 respectively, it is possible to find a finite quantity ϖ, such that

$$|\phi(x+p)| < \varpi, \quad |\phi(x+p+1)| < \varpi,$$

and consequently

$$|\phi(x)| < \varpi\tau_1^p\left[\frac{1+\tau_2}{\tau_1-\tau_2} + \frac{\tau_1+1}{\tau_1-\tau_2}\left(\frac{\tau_2}{\tau_1}\right)^p\right]$$

$$< \varpi\tau_1^p\left[\frac{1+\tau_2}{\tau_1-\tau_2} + \frac{1+\tau_1}{\tau_1-\tau_2}\right].$$

That is to say, it is possible to find finite quantities A and τ, such that

$$|\phi(x)| < A\left[\tfrac{1}{2}\{1+(1+4\tau)^{\frac{1}{2}}\}\right]^p.$$

And since $p < \frac{1}{2} - R(x)$,

$$|\phi(x)| < A\left|\exp\left\{\left(\tfrac{1}{2}-x\right)\log\frac{1+(1+4\tau)^{\frac{1}{2}}}{2}\right\}\right|,$$

the real value of the logarithm being taken.

This upper limit proves to be of use in many applications of the theory.

In conclusion, it may be stated that the particular difference equations considered in § 6 and § 7 arise in the theories of certain very general linear differential equations of the second and third orders respectively. These differential equations will form the subject of subsequent papers.

THE SOLUTION OF A CERTAIN TRANSCENDENTAL EQUATION

By G. N. WATSON.

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1. It is proposed to investigate, in this paper, the reduction of a transcendental equation (involving functions, certain properties of which are known) to an equation which is, in effect, an algebraic equation in $\cot \pi x$, x being the variable in the original equation.

This result is of some importance in connection with the solution of certain types of linear differential equations.

In a previous paper,* the solution of the linear difference equation of the second order

$$A(x)f(x+1)-B(x)f(x)+C(x)f(x-1) = 0,$$

was obtained in the following circumstances:

The function $$v(x) = \frac{A(x)\,C(x+1)}{B(x)\,B(x+1)}$$

was constructed, and it was assumed that when $|x|$ was sufficiently large $v(x)$ could be expanded in the convergent series

$$\lambda_0+\lambda_1 x^{-1}+\lambda_2 x^{-2}+\ldots,$$

and that, either (i) λ_0 was not a real quantity greater than or equal to $\frac{1}{4}$; or (ii) that $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, and λ_2 was not a real quantity greater than or

* *Supra*, pp. 125–161.

equal to $\frac{1}{16}$; and in each of these cases it was shewn that a definite meaning could be attached to the infinite continued fractions

$$\psi(x) = \frac{1}{1-} \frac{v(x)}{1-} \frac{v(x+1)}{1-} \cdots,$$

$$\psi_1(x) = \frac{1}{1-} \frac{v(x-1)}{1-} \frac{v(x-2)}{1-} \cdots.$$

In fact, $\psi(x)$ and $\psi_1(x)$ were defined as the limits of certain finite continued fractions, the forms of which varied somewhat, according to the values of λ_0, λ_1, λ_2.

The transcendental equation, the solution of which is required, is with this notation

$$1 - v(x)\,\psi(x+1)\,\psi_1(x) = 0. \tag{1}$$

As the solution of the equation is required only when $A(x)$, $B(x)$, $C(x)$ are polynomials, it will be assumed that

$$B(x) \equiv (x+\epsilon_1)(x+\epsilon_2)\ldots(x+\epsilon_p). \tag{2}$$

The solution requires two investigations according to the value of λ_0; these are given in Parts I and II of this paper respectively, and it will be shewn that the equation (1) is equivalent to an equation in $\cot \pi x$ of degree p or $p-2$, according as λ_0, λ_1, λ_2 satisfy the first or the second set of conditions stated above.

PART I.

2. The case when λ_0 is not a real quantity greater than or equal to $\frac{1}{4}$.

Let α_1, α_2 be the roots of the quadratic $z^2 - z + \lambda_0 = 0$, so chosen that $|\alpha_1| > |\alpha_2|$.

Then it has been proved* that the infinite product

$$\prod_{n=1}^{\infty} \left[\frac{1}{\alpha_1 \psi(x+n)} \exp\left\{ \frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x+n)} \right\} \right]$$

$$\times \prod_{n=0}^{\infty} \left[\frac{1}{\alpha_1 \psi_1(x-n)} \exp\left\{ \frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x-n)} \right\} \right]$$

is convergent, unless one of the factors in the product happens to be infinite.

* *Supra*, pp. 146, 160.

Denoting the product by $F(x)$, we have

$$\frac{F(x+1)}{F(x)} = \frac{a_1\psi(x+1)\exp[-\lambda_1/\{a_1(a_1-a_2)(x+1)\}]}{a_1\psi_1(x+1)\exp[-\lambda_1/\{a_1(a_1-a_2)(x+1)\}]} = \frac{\psi(x+1)}{\psi_1(x+1)}.$$

Also

$$\frac{1-v(x)\,\psi_1(x)\,\psi(x+1)}{1-v(x+1)\,\psi_1(x+1)\,\psi(x+2)}$$

$$= \left[1-\frac{\psi_1(x+1)-1}{\psi_1(x+1)}\,\psi(x+1)\right] \Big/ \left[1-\psi_1(x+1)\,\frac{\psi(x+1)-1}{\psi(x+1)}\right]$$

$$= \psi(x+1)/\psi_1(x+1).$$

Consequently

$$[1-v(x)\,\psi_1(x)\,\psi(x+1)]\,F(x) = [1-v(x+1)\,\psi_1(x+1)\,\psi(x+2)]\,F(x+1).$$

That is to say, $[1-v(x)\,\psi_1(x)\,\psi(x+1)]\,F(x)$

is a periodic function of x with period unity.

Now put
$$\Phi(x) = F(x)\exp\left\{\frac{-\lambda_1}{a_1(a_1-a_2)}\,\pi\cot\pi x\right\}, \tag{3A}$$

$$\Theta(x) = \{1-v(x)\,\psi(x+1)\,\psi_1(x)\}\,\Phi(x). \tag{3B}$$

Then $\Theta(x)$ is a periodic function of x with period unity; and the roots of the equation (1) are the roots of the equation

$$\Theta(x) = 0, \tag{4A}$$

omitting the roots of the equation

$$\Phi(x) = 0. \tag{4B}$$

On writing $\Phi(x)$ in the form

$$\frac{1}{a_1\psi(x+m)}\prod_{n=1}^{\infty}{}'\left[\frac{1}{a_1\psi(x+n)}\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)(x+n)}\right\}\right]$$

$$\times\prod_{n=0}^{\infty}\left[\frac{1}{a_1\psi_1(x-n)}\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)(x-n)}\right\}\right]$$

$$\times\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)}\left(\frac{1}{x+m}-\pi\cot\pi x\right)\right\},$$

the $'$ denoting that the term in $x+m$ is omitted from the product, we see that $\Phi(x)$ has not a zero or infinity at the points -1, -2, -3, ... due to the presence of the exponential terms; treating the other infinite product involved in the definition of $F(x)$ in a similar manner, we see that

neither $\Phi(x)$ nor $\Theta(x)$ has a zero or infinity at any of the points $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ due to the presence of the exponential terms.

Hence $\Phi(x)$ has no zeros except at the infinities of

$$\psi(x+1),\ \psi(x+2),\ \ldots,\ \psi_1(x),\ \psi_1(x-1),\ \psi_1(x-2),\ \ldots.$$

But, since

$$\psi(x+m-1)\,\psi(x+m) = [\{\psi(x+m)\}^{-1} - v(x+m-1)]^{-1},$$

$$\psi_1(x-m+1)\,\psi_1(x-m) = [\{\psi_1(x-m)\}^{-1} - v(x-m)]^{-1},$$

we see that even though $\psi(x+m)$ or $\psi_1(x-m)$ be infinite, then $\Phi(x)$ is still finite, unless, simultaneously, $v(x+m-1)$ or $v(x-m)$ respectively is zero; or, unless $\psi_1(x)$ or $\psi(x+1)$ is infinite.

But, in general, the zeros of $\psi(x)$ and those of $v(x-1)$ are quite distinct, as are also the zeros of $\psi_1(x)$ and those of $v(x)$.*

Hence, in general, the only zeros of $\Phi(x)$ are at the infinities of $\psi_1(x)$ and of $\psi(x+1)$.

We now consider the singularities of $\Theta(x)$.

From equations (3A) and (3B) and from the definition of $F(x)$ it follows that the only singularities of $\Theta(x)$ are at the poles of $v(x)$ and at the zeros of $\psi(x+2),\ \psi(x+3),\ \ldots,\ \psi_1(x-1),\ \psi_1(x-2),\ \ldots$; and there may be an essential singularity at infinity.

But, since

$$[\psi(x+m)]^{-1}[\psi(x+m+1)]^{-1} = v(x+m)[\psi(x+m)-1]^{-1},$$

$$[\psi_1(x-m)]^{-1}[\psi_1(x-m-1]^{-1} = v(x-m-1)[\psi_1(x-m)-1]^{-1},$$

we see that, if $\psi(x+m)$ or $\psi_1(x-m)$ is zero, then $\Theta(x)$ is still finite, unless $v(x+m)$ or $v(x-m-1)$ respectively is infinite.

Consequently the only singularities of $\Theta(x)$ in the finite part of the plane are at the singularities of $v(x \pm m)$, where m is any positive integer or zero.

---

* If the zeros of $v(x)$ be at the points $\zeta_1, \zeta_1, \ldots, \zeta_q$, where $q \leqslant 2p$, then the equations $\psi_1(\zeta_1) = 0,\ \psi_1(\zeta_2) = 0,\ \ldots,\ \psi_1(\zeta_q) = 0$ are *transcendental* equations giving relations between $\zeta_1, \zeta_2, \ldots, \zeta_q, \epsilon_1, \epsilon_2, \ldots, \epsilon_p$, and it is evident that none of such equations are *in general* satisfied; and, similarly, *in general*, none of the equations $\psi(\zeta_1+1) = 0,\ \psi(\zeta_2+1) = 0,\ \ldots,\ \psi(\zeta_q+1) = 0$ are satisfied. It is to be noted that $v(x)$ does not occur in the expression for $\psi_1(x)$, nor $v(x-1)$ in the expression for $\psi(x)$.

Now ..., $v(x-2)$, $v(x-1)$, $v(x)$, $v(x+1)$, ... have poles at the points

$$-\epsilon_1 \pm m, \quad -\epsilon_2 \pm m, \quad \ldots, \quad -\epsilon_p \pm m,$$

where m is any positive integer or zero.

We shall assume for the present* that none of the quantities $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ differ by integers.

It is then possible to shew that

$$\operatorname*{Lt}_{x=-\epsilon_1} [(x+\epsilon_1)\,\Theta(x)]$$

is finite, and its value may be found.

We notice that $v(x)$ has simple poles at $x = -\epsilon_1$, $x = -\epsilon_1 - 1$, and that

$$[\psi(x)]^{-1} = 1 - v(x)\,\psi(x+1),$$

$$[\psi_1(x)]^{-1} = 1 - v(x-1)\,\psi_1(x-1),$$

always.

From these results, we get

$$\begin{aligned}\operatorname*{Lt}_{x=-\epsilon_1} (x+\epsilon_1)\,\Theta(x) = &\left[\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)}\left(\pi\cot\pi\epsilon_1 - \frac{1}{\epsilon_1}\right)\right\}\right] \\ &\times \prod_{n=1}^{\infty}\left[\frac{1}{a_1\psi(n-\epsilon_1)}\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)(n-\epsilon_1)}\right\}\right] \\ &\times \prod_{n=1}^{\infty}\left[\frac{1}{a_1\psi_1(-\epsilon_1-n)}\exp\left\{\frac{\lambda_1}{a_1(a_1-a_2)(-\epsilon_1-n)}\right\}\right] \\ &\times \operatorname*{Lt}_{x=-\epsilon_1}\left[\frac{(x+\epsilon_1)\{1-v(x)\,\psi_1(x)\,\psi(x+1)\}}{a_1\psi_1(x)}\right].\end{aligned}$$

The first factor and the two infinite products in this expression are finite, provided that $\epsilon_1-\epsilon_2$, $\epsilon_1-\epsilon_3$, ..., $\epsilon_1-\epsilon_p$ are not integers; and the last factor may be written in the form

$$-\frac{\psi_1(-\epsilon_1-1)}{a_1}\operatorname*{Lt}_{x=-\epsilon_1}\{(x+\epsilon_1)\,v(x-1)\} - \frac{\psi(1-\epsilon_1)}{a_1}\operatorname*{Lt}_{x=-\epsilon_1}\{(x+\epsilon_1)\,v(x)\},$$

and is consequently finite.

Let $$\operatorname*{Lt}_{x=-\epsilon_1} [(x+\epsilon_1)\,\Theta(x)] = \theta\{\epsilon_1;\ \epsilon_2, \epsilon_3, \ldots, \epsilon_p\}. \tag{5}$$

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* If any pairs do differ by integers, the algebra is considerably more elaborate, as will be seen later (§ 3).

Then $$\Theta(x)-\theta\{\epsilon_1;\ \epsilon_2,\ \epsilon_3,\ \ldots,\ \epsilon_p\}\,\pi\cot\pi(x+\epsilon_1)$$

has no pole at any of the points $-\epsilon_1 \pm m$, where m is any positive integer or zero, since $\Theta(x)$ is a periodic function of x with period unity. And consequently

$$\Theta(x)-\sum_{r=1}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\ \epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\,\pi\cot\pi(x+\epsilon_r)\right]$$

has no poles in the finite part of the x-plane.

Consider the behaviour of this last function when $I(x)$ becomes indefinitely great and positive, while $0 \leqslant R(x) \leqslant 1$.

It is evident that $\cot\pi(x+\epsilon_r)$ tends to $-i$.

It has been proved* that when $|x|$ is greater than a finite quantity γ and $R(x) > -\frac{1}{2}$,

$$\psi(x) = \frac{1}{\alpha_1}\left\{1+\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)\,x}+\frac{J}{x^2}\right\}, \tag{6A}$$

where $|J|$ is less than a finite quantity depending on γ.

The corresponding result for $\psi_1(x)$ is that when

$$|x-1|>\gamma \quad \text{and} \quad R(x)<\tfrac{3}{2},$$

$$\psi_1(x) = \frac{1}{\alpha_1}\left\{1+\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)\,x}+\frac{J'}{x^2}\right\}. \tag{6B}$$

Consequently, when $|I(x)|>\gamma$ and $0 \leqslant R(x) \leqslant 1$, we may write

$$\prod_{n=1}^{\infty}\left[\frac{1}{\alpha_1\psi(x+n)}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x+n)}\right\}\right]$$

and

$$\prod_{n=0}^{\infty}\left[\frac{1}{\alpha_1\psi_1(x-n)}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x-n)}\right\}\right],$$

in the forms

$$\prod_{n=1}^{\infty}\left[1+\frac{\eta_{x+n}}{(x+n)^2}\right] \quad \text{and} \quad \prod_{n=0}^{\infty}\left[1+\frac{\eta'_{x-n}}{(x-n)^2}\right],$$

respectively, where $|\eta_{x+n}|$, $|\eta'_{x-n}|$ are less than finite quantities depending on γ.

Hence, since $\sum_{n=0}^{\infty}\frac{1}{(x\pm n)^2}$ is absolutely and uniformly convergent for

* *Supra*, p. 145.

all values of x which are not integers, and $\sum_{n=0}^{\infty} \frac{1}{|x \pm n|^2}$ tends to zero* as $I(x)$ tends to infinity, while $0 \leqslant R(x) \leqslant 1$, we see that

$$\prod_{n=1}^{\infty} \left[\frac{1}{a_1 \psi(x+n)} \exp\left\{ \frac{\lambda_1}{a_1(a_1-a_2)(x+n)} \right\} \right]$$

and

$$\prod_{n=0}^{\infty} \left[\frac{1}{a_1 \psi_1(x-n)} \exp\left\{ \frac{\lambda_1}{a_1(a_1-a_2)(x-n)} \right\} \right],$$

tend to unity as $I(x)$ tends to infinity, while $0 \leqslant R(x) \leqslant 1$.

Consequently, by reason of (3A), (3B), (5A), and (5B), $\Theta(x)$ tends to

$$\left(1 - \frac{\lambda_0}{a_1^2}\right) \exp\left\{ \frac{\lambda_1 \pi i}{a_1(a_1-a_2)} \right\}.$$

Similarly, when $I(x)$ tends to $-\infty$, while $0 \leqslant R(x) \leqslant 1$, $\cot \pi(x+\epsilon_r)$ tends to $+i$, and $\Theta(x)$ tends to

$$\left(1 - \frac{\lambda_0}{a_1^2}\right) \exp\left\{ \frac{-\lambda_1 \pi i}{a_1(a_1-a_2)} \right\}.$$

Hence $$\Theta(x) - \sum_{r=1}^{p} \left[\theta\{\epsilon_r ; \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\} \pi \cot \pi(x+\epsilon_r) \right],$$

is never infinite† in the infinite strip of the x-plane given by

$$0 \leqslant R(x) \leqslant 1 ;$$

hence, from the periodicity of $\Theta(x)$ and of $\cot \pi(x+\epsilon_r)$, we see that

$$\Theta(x) - \sum_{r=1}^{p} \left[\theta\{\epsilon_r ; \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\} \pi \cot \pi(x+\epsilon_r) \right]$$

is never infinite for *any* value of x; and hence, by Liouville's theorem,

$$\Theta(x) - \sum_{r=1}^{p} \left[\theta\{\epsilon_r ; \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\} \pi \cot \pi(x+\epsilon_r) \right]$$

is a pure constant.

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* When $0 \leqslant R(x) \leqslant 1$,

$$\sum_{n=0}^{\infty} \left| \frac{1}{(x+n)^2} \right| < \sum_{n=0}^{\infty} \frac{1}{|x|^2+n^2} < \frac{1}{2|x|} \left[\pi \coth \pi |x| + \frac{1}{|x|} \right],$$

and this tends to zero as $|x|$ tends to infinity. On the uniform convergence of infinite products, a recent paper by Hardy may be consulted (*Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 40–48).

† It is especially to be noticed how this argument fails for such a periodic function as $\sin^2 \pi x$, which becomes indefinitely great as $I(x)$ tends to $\pm\infty$.

Make $I(x)$ tend to $+\infty$, and we see that this constant is equal to

$$\left(1-\frac{\lambda_0}{a_1^2}\right)\exp\left\{\frac{\lambda_1\pi i}{a_1(a_1-a_2)}\right\}+\pi i\sum_{r=1}^{p}\left[\theta\left\{\epsilon_r;\ \epsilon_{r+1},\ \epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\right\}\right];$$

making $I(x)$ tend to $-\infty$, we see that it is also equal to

$$\left(1-\frac{\lambda_0}{a_1^2}\right)\exp\left\{\frac{-\lambda_1\pi i}{a_1(a_1-a_2)}\right\}-\pi i\sum_{r=1}^{p}\left[\theta\left\{\epsilon_r;\ \epsilon_{r+1},\ \epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\right\}\right].$$

From these two values for the constant, we deduce that

$$\Theta(x)\equiv\left(1-\frac{a_2}{a_1}\right)\cos\frac{\lambda_1\pi}{a_1(a_1-a_2)}+\sum_{r=1}^{p}\left[\theta\left\{\epsilon_r;\epsilon_{r+1},\epsilon_{r+2},\ldots,\epsilon_{r-1}\right\}\pi\cot\pi(x+\epsilon_r)\right], \tag{7A}$$

and incidentally we notice that

$$\left(1-\frac{a_2}{a_1}\right)\sin\frac{\lambda_1\pi}{a_1(a_1-a_2)}=-\pi\sum_{r=1}^{p}\left[\theta\left\{\epsilon_r;\ \epsilon_{r+1},\ \epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\right\}\right]. \tag{7B}$$

From (7A) we see that the equation $\Theta(x)=0$ is equivalent to an equation in $\cot\pi x$ of degree p, with coefficients which may be regarded as known.

This is the result which is required for applications of this theory to the theory of linear differential equations.

If one or more of the quantities $\theta\left\{\epsilon_r;\ \epsilon_{r+1},\ \ldots,\ \epsilon_{r-1}\right\}$ is zero, the equation $\Theta(x)=0$ is equivalent to an equation in $\cot\pi x$ of degree *less* than p; in applications of this theory, it can, however, be shewn that in these circumstances satisfactory modifications can be made without any difficulty.

It is to be noticed that $\cot\pi x=i$ and $\cot\pi x=-i$ are not roots of $\Theta(x)=0$; for, if either were a root, then

$$\left(1-\frac{a_2}{a_1}\right)\exp\left\{\frac{\pm\lambda_1\pi i}{a_1(a_1-a_2)}\right\}=0,$$

which is obviously not the case since $a_1\neq a_2$.

3. The modifications to be made in these results, when a pair of the quantities $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ differ by an integer, may be stated; if more than one pair differ by integers, the results cannot be expressed in so simple a form.

Suppose that $\epsilon_2-\epsilon_1$ is an integer or zero.

Then $\Theta(x)$ has poles at the points

$$-\epsilon_1 \pm n, \ -\epsilon_3 \pm n, \ \ldots, \ -\epsilon_p \pm n,$$

the poles being simple except at $-\epsilon_1 \pm n$, n being any positive integer or zero; and it may be shewn, by methods similar to those of § 2, that $(x+\epsilon_1)^2\Theta(x)$ has a finite limit when $x=-\epsilon_1$, and the actual value of the limit may be written down; *e.g.*, when $\epsilon_2-\epsilon_1$ is an integer, such that $\epsilon_2 > \epsilon_1+1$, the limit is

$$\exp\left[\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)}\left\{\pi\cot\pi\epsilon_1-\frac{1}{\epsilon_1}-\frac{1}{\epsilon_1+1}-\ldots-\frac{1}{\epsilon_2}\right]\right.$$

$$\times\prod_{n=1}^{\infty}\left[\frac{1}{\alpha_1\psi(n-\epsilon_1)}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(n-\epsilon_1)}\right\}\right]$$

$$\times\prod_{n=1}^{\infty}\left[\frac{1}{\alpha_1\psi_1(-n-\epsilon_2)}\exp\left\{\frac{-\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(n+\epsilon_2)}\right\}\right]$$

$$\times\prod_{n=1}^{k-1}\left[\frac{1}{\alpha_1\psi_1(-\epsilon_1-n)}\right]\frac{1}{\alpha_1^2}\operatorname*{Lt}_{x=-\epsilon_2}\left[(x+\epsilon_2)\,v(x-1)\right]$$

$$\times\operatorname*{Lt}_{x=-\epsilon_1}\left[(x+\epsilon_1)\,v(x-1)\psi_1(-1-\epsilon_1)+(x+\epsilon_1)\,v(x)\psi(1-\epsilon_1)\right]\psi_1(-\epsilon_2-1),$$

where $$\epsilon_2-\epsilon_1=k.$$

When $k=1$, 0, -1, or a negative integer <-1, the corresponding value of the limit can easily be written down; we shall call this limit $\theta'(\epsilon_1, \epsilon_2)$.

Then $$\Theta(x)-\theta'(\epsilon_1, \epsilon_2)\left\{\pi\cot\pi(x+\epsilon_1)\right\}^2$$

has only simple poles in the finite part of the plane.

Consequently the function

$$\Theta(x)-\theta'(\epsilon_1, \epsilon_2)\left\{\pi\cot\pi(x+\epsilon_1)\right\}^2-A\pi\cot\pi(x+\epsilon_1)$$

$$-\sum_{r=3}^{p}\left[\theta\left\{\epsilon_r;\ \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\right\}\pi\cot\pi(x+\epsilon_r)\right]$$

has no poles in the finite part of the plane if the constant A be suitably chosen, and the function θ is that defined in (5).

This function is, moreover, a periodic function of x. If we make $I(x)$ large and positive while $0 \leqslant R(x) \leqslant 1$, we find that it approaches a finite

limit, viz.,

$$\left(1-\frac{\lambda_0}{a_1^2}\right)\exp\left\{\frac{\lambda_1\pi i}{a_1(a_1-a_2)}\right\}+\pi^2\theta'(\epsilon_1,\epsilon_2)+A\pi i$$
$$+\pi i\sum_{r=3}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\right].$$

If, however, we make $I(x)$ large and negative, while $0\leqslant R(x)\leqslant 1$, we find that the function approaches the limit

$$\left(1-\frac{\lambda_0}{a_1^2}\right)\exp\left\{\frac{-\lambda_1\pi i}{a_1(a_1-a_2)}\right\}+\pi^2\theta'(\epsilon_1,\epsilon_2)-A\pi i$$
$$-\pi i\sum_{r=3}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\right].$$

Hence, by Liouville's theorem, the function is a constant, and the function is always equal to its value when $I(x)=\pm\infty$ and $0\leqslant R(x)\leqslant 1$.

Consequently,

$$\Theta(x)-\theta'(\epsilon_1,\epsilon_2)\{\pi\cot\pi(x+\epsilon_1)\}^2-A\{\pi\cot\pi(x+\epsilon_1)\}$$
$$-\sum_{r=3}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\,\pi\cot\pi(x+\epsilon_r)\right]$$
$$=\left(1-\frac{a_2}{a_1}\right)\cos\left\{\frac{\lambda_1\pi}{a_1(a_1-a_2)}\right\}+\pi^2\theta'(\epsilon_1,\epsilon_2),$$

and $\pi A+\pi\sum_{r=3}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\right]+\left(1-\frac{a_2}{a_1}\right)\sin\left\{\frac{\lambda_1\pi}{a_1(a_1-a_2)}\right\}=0.$

Consequently, we get

$$\Theta(x)=\theta'(\epsilon_1,\epsilon_2)\{\pi^2+\pi^2\cot^2\pi(x+\epsilon_1)\}$$
$$-\left(1-\frac{a_2}{a_1}\right)\sin\left\{\frac{\lambda_1\pi}{a_1(a_1-a_2)}\right\}\cot\pi(x+\epsilon_1)+\left(1-\frac{a_2}{a_1}\right)\cos\frac{\lambda_1\pi}{a_1(a_1-a_2)}$$
$$+\sum_{r=3}^{p}\left[\theta\{\epsilon_r;\ \epsilon_{r+1},\epsilon_{r+2},\ \ldots,\ \epsilon_{r-1}\}\{\pi\cot\pi(x+\epsilon_r)-\pi\cot\pi(x+\epsilon_1)\}\right].$$

Accordingly the equation $\Theta(x)=0$ is still equivalent to an equation of degree p in $\cot\pi x$.

4. We now return to the main problem; we have to exclude from our consideration those solutions of the equation $\Theta(x)=0$ which make $\psi(x+1)$ or $\psi_1(x)$ infinite, if any such solutions exist.

If $\psi(x+1)$ or $\psi_1(x)$ is infinite, then

$$\Theta(x) = -\frac{v(x)}{a_1^2}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)}\left(\frac{1}{x}+\frac{1}{x+1}-\pi\cot\pi x\right)\right\}$$
$$\times \prod_{n=2}^{\infty}\left[\frac{1}{\alpha_1\psi(x+n)}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x+n)}\right\}\right]$$
$$\times \prod_{n=1}^{\infty}\left[\frac{1}{\alpha_1\psi_1(x-n)}\exp\left\{\frac{\lambda_1}{\alpha_1(\alpha_1-\alpha_2)(x-n)}\right\}\right],$$

and this vanishes only when

$$v(x) = 0 \quad \text{or} \quad [\psi(x+2)]^{-1} = 0 \quad \text{or} \quad [\psi_1(x-1)]^{-1} = 0,$$

or when both $[\psi(x+m+1)]^{-1}$ and $v(x+m)$ vanish, or when both $[\psi_1(x-m)]^{-1}$ and $v(x-m)$ vanish, m being any positive integer greater than unity.

Now, in general, $v(x)$ has $2p$ zeros, which are not distinguishable one from another, and no pair differ by integers. Hence, in general, $v(x)$ and $\Theta(x)$ have no common zeros; if they have common zeros, some transcendental relation (which is not an identity) must exist between the zeros of $v(x)$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$; and further, the simultaneous equations

$$v(x+m) = 0, \quad [\psi(x+m+1)]^{-1} = 0,$$

or

$$v(x-m) = 0, \quad [\psi_1(x-m)]^{-1} = 0,$$

or any other pair which might be written down would also give transcendental relations [between $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ and the zeros of $v(x)$] which are not, in general, satisfied.

Hence, in general, the equation

$$1-v(x)\psi(x+1)\psi_1(x) = 0$$

is equivalent to an equation of degree p in $\cot\pi x$; and the coefficients of this equation in $\cot\pi x$ can be regarded as known for any given form of $v(x)$ which satisfies the required conditions.

PART II.

5. The case when $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, and λ_2 is not a real quantity greater than or equal to $\frac{1}{16}$.

Let α be that root of the equation $z^2-\frac{1}{2}z+\lambda_2 = 0$ which satisfies the inequality $R(4\alpha-1) < 0$.

* We cannot take more than p zeros of $\Theta(x)$ such that no pair differ by integers.

Then it has been proved* that the infinite product

$$\prod_{n=1}^{\infty}\left[\frac{2}{\psi(x+n)}\exp\left(\frac{2\alpha}{x+n}\right)\right]\prod_{n=0}^{\infty}\left[\frac{2}{\psi_1(x-n)}\exp\left(\frac{-2\alpha}{x-n}\right)\right]$$

is convergent, unless one of the factors in the product happens to be infinite.

Denoting the product by $F_0(x)$, we have

$$\frac{F_0(x+1)}{F_0(x)}=\frac{\psi(x+1)}{\psi_1(x+1)}\exp\left(\frac{-4\alpha}{x+1}\right).$$

Also

$$\frac{1-v(x)\psi_1(x)\psi(x+1)}{1-v(x+1)\psi_1(x+1)\psi(x+2)}$$

$$=\left[1-\frac{\psi_1(x+1)-1}{\psi_1(x+1)}\psi(x+1)\right]\Big/\left[1-\psi_1(x+1)\frac{\psi(x+1)-1}{\psi(x+1)}\right]$$

$$=\psi(x+1)/\psi_1(x+1).$$

And hence

$$[1-v(x)\psi_1(x)\psi(x+1)]F_0(x)\exp\left[2\alpha\left\{\frac{\Gamma'(x+1)}{\Gamma(x+1)}+\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}\right]$$

is a periodic function of x with period unity.

Now put $$\Phi_0(x)=F_0(x)\exp\left[2\alpha\left\{\frac{\Gamma'(x+1)}{\Gamma(x+1)}+\frac{\Gamma'(-x)}{\Gamma(-x)}\right\}\right], \tag{8A}$$

$$\Theta_0(x)=\{1-v(x)\psi(x+1)\psi_1(x)\}\Phi_0(x). \tag{8B}$$

Then $\Theta_0(x)$ is a periodic function of x with period unity; and the roots of the equation (1) are the roots of the equation

$$\Theta_0(x)=0, \tag{9A}$$

omitting the roots of the equation

$$\Phi_0(x)=0. \tag{9B}$$

By arguments similar to those employed in Part I, it may be shewn that $\Phi_0(x)$ has not a singularity at any of the points $\ldots, -2, -1, 0, 1, 2, \ldots$ due to the presence of the exponential terms in the infinite products; and that, in general, the only zeros of $\Phi_0(x)$ are at the infinities of $\psi_1(x)$ and $\psi(x+1)$.

We now consider the singularities of $\Theta_0(x)$.

* *Supra*, pp. 156, 160.

From equations (8A) and (8B), and from the definition of $F_0(x)$, it follows that the only singularities of $\Theta_0(x)$ are at the poles of $v(x)$ and at the zeros of $\psi(x+2)$, $\psi(x+3)$, ..., $\psi_1(x-1)$, $\psi_1(x-2)$, ..., and there may be an essential singularity at infinity.

Just as in Part I of this paper, it may now be shewn that the only singularities of $\Theta_0(x)$ are at the points

$$-\epsilon_1 \pm m, \ -\epsilon_2 \pm m, \ \ldots, \ \epsilon_p \pm m,$$

where m is any positive integer or zero.

We shall assume for the present* that none of the quantities $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ differ by integers. It is then possible to shew that

$$\operatorname*{Lt}_{x=-\epsilon_1} [(x+\epsilon_1)\,\Theta_0(x)]$$

is finite; and its value may be determined.

Let the value of this limit be $\theta_0\{\epsilon_1;\ \epsilon_2, \epsilon_3, \ldots, \epsilon_p\}$. Then

$$\Theta_0(x) - \theta_0\{\epsilon_1;\ \epsilon_2, \epsilon_3, \ldots, \epsilon_p\}\ \pi \cot \pi(x+\epsilon_1)$$

has no pole at any of the points $-\epsilon_1 \pm m$, where m is any positive integer or zero; this follows from the periodicity of $\Theta_0(x)$.

Consequently,

$$\Theta_0(x) - \sum_{r=1}^{p} [\theta_0\{\epsilon_r;\ \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\}\ \pi \cot \pi(x+\epsilon_r)]$$

has no singularity in the finite part of the x plane.

Consider the behaviour of this last function when $I(x)$ becomes indefinitely great and positive while $0 \leqslant R(x) \leqslant 1$.

It is evident that $\cot \pi(x+\epsilon_r)$ tends to $-i$.

It has been proved† that when $|x|$ is greater than a finite quantity γ, and $R(x) > -\frac{1}{2}$,

$$\psi(x) = 2 + \frac{4a}{x} + \frac{J}{x^2},$$

where $|J|$ is less than a finite quantity depending on γ.

The corresponding result for $\psi_1(x)$ is that when

$$|x-1| > \gamma \quad \text{and} \quad R(x) < \tfrac{3}{2},$$

$$\psi_1(x) = 2 - \frac{4a}{x} + \frac{J'}{x^2}.$$

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* See p. 166.

† *Supra*, p. 156.

Consequently, when $|I(x)|>\gamma$ and $0\leqslant R(x)\leqslant 1$, we may write

$$\prod_{n=1}^{\infty}\left[\frac{2}{\psi(x+n)}\exp\left(\frac{2\alpha}{x+n}\right)\right] \quad \text{and} \quad \prod_{n=0}^{\infty}\left[\frac{2}{\psi_1(x-n)}\exp\left(\frac{-2\alpha}{x-n}\right)\right]$$

in the forms

$$\prod_{n=1}^{\infty}\left[1+\frac{\eta_{x+n}}{(x+n)^2}\right] \quad \text{and} \quad \prod_{n=0}^{\infty}\left[1+\frac{\eta'_{x-n}}{(x-n)^2}\right],$$

respectively, where $|\eta_{x+n}|$, $|\eta'_{x-n}|$ are less than finite quantities depending on γ.

Hence, since $\sum_{n=0}^{\infty}\frac{1}{(x\pm n)^2}$ is absolutely and uniformly convergent for all values of x which are not integers, and $\sum_{n=0}^{\infty}\frac{1}{|x\pm n|^2}$ tends to zero as $I(x)$ tends to infinity, while $0\leqslant R(x)\leqslant 1$, we see that both

$$\prod_{n=1}^{\infty}\left[\frac{2}{\psi(x+n)}\exp\left(\frac{2\alpha}{x+n}\right)\right] \quad \text{and} \quad \prod_{n=0}^{\infty}\left[\frac{2}{\psi_1(x-n)}\exp\left(\frac{-2\alpha}{x-n}\right)\right]$$

tend to unity as $I(x)$ tends to $+\infty$ or to $-\infty$, while $0\leqslant R(x)\leqslant 1$.

We have now to consider the behaviour of

$$[1-v(x)\psi_1(x)\psi(x+1)]\exp\left\{2\alpha\frac{\Gamma'(x+1)}{\Gamma(x+1)}+2\alpha\frac{\Gamma'(-x)}{\Gamma(-x)}\right\},$$

as $I(x)$ tends to infinity.

Since
$$v(x)=\tfrac{1}{4}+\frac{\lambda_2+\xi_x}{x^2},$$

where ξ_x tends to zero as $|x|$ tends to infinity, we get

$$1-v(x)\psi_1(x)\psi(x+1)=\frac{2\alpha-4\alpha^2}{x(x+1)}-\frac{4(\lambda_2+\xi_x)}{x^2}+\frac{2J''}{(x+1)^2}+\frac{2J'}{x^2}+\frac{K}{x^3}, \quad (10)$$

where
$$\psi(x+1)=2+\frac{4\alpha}{x+1}+\frac{J''}{(x+1)^2},$$

and $|K|$ is less than a finite quantity depending on α, λ_2, γ, and the upper limit of $|\xi_x|$ when $|x|>\gamma$.

Also, by Mellin's formula for the asymptotic expansion of the gamma-function,*

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)}=\log(x+1)+\frac{1}{2(x+1)}+\frac{J_0}{(x+1)^2},$$

$$\frac{\Gamma'(-x)}{\Gamma(-x)}=\log(-x)-\frac{1}{2x}+\frac{J'_0}{x^2},$$

* Mellin, *Acta Societatis Fennicae*, Vol. XXIX.

where $|\arg(x+1)|$, $|\arg(-x)|$ are less than π, and J_0, J_0' are less than finite quantities depending on γ.

From these two formulæ it may be shewn that, when $0 \leqslant R(x) \leqslant 1$ and $|I(x)| > \gamma$, the expression

$$|x|^{-4\alpha} \left| \exp\left\{ 2\alpha \frac{\Gamma'(x+1)}{\Gamma(x+1)} + 2\alpha \frac{\Gamma'(-x)}{\Gamma(-x)} \right\} \right|$$

is finite, and approaches a finite (non-zero) limit as $I(x)$ tends to $+\infty$ and a different finite (non-zero) limit as $I(x)$ tends to $-\infty$.

Hence, since $R(4\alpha-2)$ is negative, we see, from these two results, and from (10), that

$$[1-v(x)\psi_1(x)\psi(x+1)] \exp\left\{ 2\alpha \frac{\Gamma'(x+1)}{\Gamma(x+1)} + 2\alpha \frac{\Gamma'(-x)}{\Gamma(-x)} \right\}$$

tends to zero as $I(x)$ tends to $\pm\infty$, while $0 \leqslant R(x) \leqslant 1$. Collecting our results, we see that $\Theta_0(x)$ tends to zero as $I(x)$ tends to $\pm\infty$.

Thus, as in the corresponding investigation in § 2,

$$\Theta_0(x) - \sum_{r=1}^{p} [\theta_0\{\epsilon_r;\ \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\} \pi \cot \pi(x+\epsilon_r)]$$

is a periodic function of x, with period unity, which is never infinite in the infinite strip of the plane given by $0 \leqslant R(x) \leqslant 1$; and hence by Liouville's theorem* it is a pure constant.

Making $I(x)$ tend to $\pm\infty$, we find that this constant is equal to each of the quantities

$$\pm \pi i \sum_{r=1}^{p} [\theta_0\{\epsilon_r;\ \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_{r-1}\}],$$

and hence we see that

$$\Theta_0(x) \equiv \sum_{r=1}^{p} [\theta_0\{\epsilon_r;\ \epsilon_{r+1}, \ldots, \epsilon_{r-1}\} \pi \cot \pi(x+\epsilon_r), \tag{11A}$$

and also that

$$\sum_{r=1}^{p} \theta_0\{\epsilon_r;\ \epsilon_{r+1}, \ldots, \epsilon_{r-1}\} = 0. \tag{11B}$$

From (11A) we see that $\Theta_0(x)$ is equal to a polynomial of degree p in $\cot \pi x$, divided by $\prod_{r=1}^{p} (\cot \pi x + \cot \pi \epsilon_r)$; and on account of the condition (11B), this polynomial is divisible by $1+\cot^2 \pi x$.

That is to say,

$$\Theta_0(x) \equiv \frac{\operatorname{cosec}^2 \pi x \times \text{a polynomial of degree } p-2 \text{ in } \cot \pi x}{\prod_{r=1}^{p} (\cot \pi x + \cot \pi \epsilon_r)};$$

* See foot-note, p. 168, above.

and the coefficients in this polynomial of degree $p-2$ may be regarded as known.

If two of the quantities $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ differ by integers, we may, by analysis similar to that of § 3, prove that the equation $\Theta_0(x) = 0$ is still equivalent to an equation of degree $p-2$ in $\cot \pi x$; we observe that $\operatorname{cosec} \pi x$ does not vanish for any finite value of x.

6. The arguments of § 4 may be applied, *mutatis mutandis*, to shew that, *in general*, the values of x which make $\Theta_0(x) = 0$ do not make $\psi(x+1)$ or $\psi_1(x)$ infinite; so that, in general, when $\lambda_0 = \frac{1}{4}$, $\lambda_1 = 0$, and λ_2 is not a real quantity greater than or equal to $\frac{1}{16}$, the equation

$$1 - v(x)\,\psi(x+1)\,\psi_1(x) = 0$$

is equivalent to an equation of degree $p-2$ in $\cot \pi x$.

THE EQUATIONS OF ELECTRODYNAMICS AND THE NULL INFLUENCE OF THE EARTH'S MOTION ON OPTICAL AND ELECTRICAL PHENOMENA

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Introduction.

The fundamental equations of the electromagnetic field are usually taken to be those given by Profs. Larmor and Lorentz, deduced in both cases from the hypothesis that matter is formed of discrete electrons. For matter in bulk an averaging process is used in order to obtain the necessary equations. This process introduces new vectors, viz., the electric polarization, the magnetization, and the current of conduction. The scheme of equations is then completed by the assumption of relations connecting these vectors and the two fundamental vectors—electric force and magnetic induction.*

In the following paper the relations connecting these two classes of vectors are deduced from the hypothesis that optical and electrical phenomena are independent of the direction of the motion of the Earth through space. All experiments which have been carried out tend to justify this hypothesis, and their number has been added to recently in an experiment by Prof. Trouton and Mr. Rankine.†

The outcome of this hypothesis is that it must be possible to reduce the equations of electrodynamics for moving media to the same form as those of the well known Maxwellian equations for matter at rest. For matter considered as formed of discrete electrons, this was accomplished by Larmor.‡

In the case of media in bulk, the special case of a non-magnetizable dielectric was treated by H. A. Lorentz,§ in 1904, in an investigation as to the possibility of a positive result for the Michelson-Morley experiment arising in the case of propagation through a dielectric.

* Cf. Larmor, *Aether and Matter*, p. 112.

† *Proc. Roy. Soc.*, A, Vol. lxxx, 1907–8, p. 420.

‡ *Aether and Matter*, Chap. xi.

§ *Proc. Acad. of Sciences of Amsterdam*, 1904, p. 809.

If we distinguish thus between the *equations of the electromagnetic field* and the equations connecting the subsidiary and the fundamental vectors, which latter we shall refer to as the *equations of connection*, we shall assume the ordinary equations of the field, but not the equations of connection except in so far as they hold good for matter at rest, the problem being to deduce from the above hypothesis the equations of connection for media in motion.

The investigations contained in this paper were practically completed before the author knew of a paper on the same subject by the late H. Minkowski*; the relation of the work in this paper to that of Minkowski is discussed below.

1. *The Equations of the Electromagnetic Field.*

If we take axes x, y, z fixed in space and a time t, the general equations of the electromagnetic field for media in bulk moving with a constant velocity v are, using the vector notation,†

$$\operatorname{curl} H = \frac{1}{c}\left\{C + \frac{\partial D}{\partial t} + \rho v + \operatorname{curl}[P, v]\right\},$$

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial B}{\partial t},$$

$$\operatorname{div} D = \rho, \qquad \operatorname{div} B = 0.$$

Here P, C, M are the electric polarization, the conduction current, and the magnetization respectively, while B, E, and ρ are the magnetic induction, the electric force, and the true electric density; also H and D are defined by

$$H \equiv B - M, \qquad D \equiv E + P.$$

The units are those used by Lorentz, and c is the velocity of radiation. The operator

$$\operatorname{div} A \equiv \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z,$$

the vector

$$\operatorname{curl} A \equiv \frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y, \quad \frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z, \quad \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x,$$

where A is any vector.

Also, if A and B are two vectors, $[A, B]$ is a vector whose components are $A_y B_z - B_y A_z$, $A_z B_x - B_z A_x$, $A_x B_y - B_x A_y$, while the symbol

$$(A, B) \equiv A_x B_x + A_y B_y + A_z B_z.$$

* *Göttinger Nachrichten*, 1908.

† For these equations see Larmor, *Aether and Matter*, Section II, § 73, or H. A. Lorentz, *Ency. der Math. Wiss.*, V, 14, § 33.

If we refer the equations to axes x', y', z' moving with the matter and a time t' $(= t)$, we have

$$x = x' + v_x t', \quad \ldots,$$

whence

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad \ldots,$$

and

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + (v\nabla) \equiv \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z},$$

and the equations become

$$\text{curl}' H' = \frac{1}{c}\left\{C + \frac{\partial D}{\partial t'}\right\}, \tag{1}$$

$$\text{curl}' E' = -\frac{1}{c}\frac{\partial B}{\partial t'}, \tag{2}$$

$$\text{div}' D = \rho, \qquad \text{div}' B = 0, \tag{3, 4}$$

where the new vectors H' and E' are defined by

$$H' \equiv H - \frac{1}{c}[v, E], \qquad E' \equiv E + \frac{1}{c}[v, B].$$

The operators div′ and curl′ refer to the moving axes x', y', z' just as div and curl referred to the fixed axes.

The problem before us is to consider the possibility of reducing the general equations of the electromagnetic field to a scheme of equations which shall have the same form as the ordinary Maxwellian scheme for matter at rest.

Starting from the equations referred to the moving axes x', y', z', let us first apply to them the transformation

$$t'' = t' - \frac{1}{c^2}(k, x'), \qquad x'' = x', \qquad y'' = y', \qquad z'' = z', \tag{A}$$

where k is a constant vector, having components k_x, k_y, k_z as yet undetermined, whence

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t''}, \qquad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x''} - \frac{k_x}{c^2}\frac{\partial}{\partial t''}, \qquad \ldots.$$

The equations then become

$$\text{curl}'' H' - \frac{1}{c^2}\left[k, \frac{\partial H'}{\partial t''}\right] = \frac{1}{c}\left\{\frac{\partial D}{\partial t''} + C\right\}, \tag{1a}$$

$$\text{curl}'' E' - \frac{1}{c^2}\left[k, \frac{\partial E'}{\partial t''}\right] = -\frac{1}{c}\frac{\partial B}{\partial t''}, \tag{2a}$$

$$\text{div}'' D - \frac{1}{c^2}\left(k, \frac{\partial D}{\partial t''}\right) = \rho, \tag{3a}$$

$$\text{div}'' B - \frac{1}{c^2}\left(k, \frac{\partial B}{\partial t''}\right) = 0. \tag{4a}$$

On substituting in the fourth of these equations for $\partial B/\partial t''$ from the second, we find

$$\operatorname{div}'' B + \frac{1}{c}(k, \operatorname{curl}'' E') = 0$$

or

$$\operatorname{div}'' \left\{ B - \frac{1}{c}[k, E'] \right\} = 0, \tag{4b}$$

whilst equation (2a) may be written

$$\operatorname{curl}'' E' = -\frac{1}{c}\frac{\partial}{\partial t''}\left\{ B - \frac{1}{c}[k, E'] \right\}. \tag{2b}$$

Similarly equation (3a) becomes, on substituting for $\partial D/\partial t''$ from (1a),

$$\operatorname{div}'' D - \frac{1}{c}(k, \operatorname{curl}'' H') = \rho - \frac{(k, C)}{c^2}$$

or

$$\operatorname{div}'' \left\{ D + \frac{1}{c}[k, H'] \right\} = \rho - \frac{(k, C)}{c^2}, \tag{3b}$$

whilst equation (1a) can be written

$$\operatorname{curl}'' H' = \frac{1}{c^2}\left\{ C + \frac{\partial}{\partial t''}\left\{ D + \frac{1}{c}[k, H'] \right\} \right\}. \tag{1b}$$

2. *A Special Case of the Transformation.*

In the scheme of equations (1b), ..., (4b), put

$$B' = B - \frac{1}{c}[k, E'],$$

$$D' = D + \frac{1}{c}[k, H'],$$

$$\rho' = \rho - \frac{1}{c^2}(k, C),$$

and the scheme transforms into

$$\operatorname{curl}'' H' = \frac{1}{c}\left\{ C + \frac{\partial D'}{\partial t''} \right\}, \tag{1c}$$

$$\operatorname{curl}'' E' = -\frac{1}{c}\frac{\partial B'}{\partial t''}, \tag{2c}$$

$$\operatorname{div}'' D' = \rho', \qquad \operatorname{div}'' B' = 0. \tag{3c, 4c}$$

As these equations are of the same form as the Maxwellian equations of the electromagnetic field for matter at rest, the problem specified in § 1 has been solved. In order to complete the scheme of electrodynamical

equations for matter at rest, we must add to the above equations of the field the equations of connection in the form assumed for matter at rest, viz.,

$$D' = K_0 E', \qquad B' = \mu_0 H', \qquad C = \frac{E'}{\sigma_0}.$$

Hence, for the original scheme of equations referred to fixed axes, it follows that

$$D + \frac{1}{c}[k, H'] = K_0 E',$$

$$B - \frac{1}{c}[k, E'] = \mu_0 H', \qquad C = \frac{E'}{\sigma_0}.$$

In the case of a non-magnetizable medium $B = H$, so that the second of these last equations gives us in this case

$$H - \frac{1}{c}[k, E'] = \mu_0 H'$$

or
$$H - \frac{1}{c}[k, E] - \frac{1}{c^2}[k, [v, H]] = \mu_0 \left\{H - \frac{1}{c}[v, E]\right\}$$

or
$$H - \frac{1}{c}[k, E] - \frac{1}{c^2} v(k, H) + \frac{1}{c^2} H(k, v) = \mu_0 \left\{H - \frac{1}{c}[v, E]\right\}.$$

Now let us choose our axes so that the velocity v is parallel to the axis of x', *i.e.*, parallel to the axis of x'', and the above vector equation, written out in full, becomes

$$H_x - \frac{1}{c}\{k_y E_z - k_z E_y\} - \frac{1}{c^2}\bar{v}(k, H) + \frac{1}{c^2} H_x k_x \bar{v} = \mu_{0x} H_x,$$

$$H_y - \frac{1}{c}\{k_z E_x - k_x E_z\} + \frac{1}{c^2} H_y k_x \bar{v} = \mu_{0y} H_y + \frac{\mu_{0y}}{c}\bar{v} E_z,$$

$$H_z - \frac{1}{c}\{k_x E_y - k_y E_x\} + \frac{1}{c^2} H_z k_x \bar{v} = \mu_{0z} H_z - \frac{\mu_{0z}}{c}\bar{v} E_y,$$

which equations can only be satisfied by putting

$$k_y = k_z = 0, \quad \mu_{0x} = 1, \quad \mu_{0y} = \mu_{0z} = \left(1 - \frac{\bar{v}^2}{c^2}\right)^{-1} \equiv \epsilon, \quad \text{and} \quad k_x = \epsilon\bar{v},$$

where $\bar{v}$ is the absolute magnitude of the velocity of the matter, and is no longer a vector.

Similarly, from the equation connecting D and E', we should find for the free ether, where $D = E$ and $B = H$, that

$$K_{0x} = 1, \qquad K_{0y} = K_{0z} = \epsilon.$$

As ϵ involves $\bar{v}$, which ought to disappear from the equations for the free ether, this special case of the transformation is not valid for our purpose.

3. *The General Transformation.*

The failure of the transformation (A) suggests the possibility that the transformation is not sufficiently general and that it should be extended in a perfectly general manner, the constants of the transformation being afterwards determined so as to satisfy the demands of particular cases.

Starting from the equations (1c), ..., (4c) of the last paragraph, let us now take the velocity v to be parallel to the axis of x', and therefore parallel to the axis of x'', and put

$$\left.\begin{aligned} t_1 &= lt'' \\ x_1 &= a_t t''+a_x x''+a_y y''+a_z z'' \\ y_1 &= b_t t''+b_x x''+b_y y''+b_z z'' \\ z_1 &= c_t t''+c_x x''+''c_y y''+c_z z'' \end{aligned}\right\}, \tag{B}$$

where the various coefficients are constants.

Combining the transformations (A) and (B), it is immediately apparent that they give the most general linear transformation from the moving axes x', y', z' to the axes x_1, y_1, z_1, and from a time t' to a time t_1.

The equations (1c), ..., (4c) will then be transformed into equations in which the independent variables of space and time are x_1, y_1, z_1, and t_1 respectively.

For example, the first of the equations (2c) becomes, written out in full,

$$\left(a_y\frac{\partial}{\partial x_1}+b_y\frac{\partial}{\partial y_1}+c_y\frac{\partial}{\partial z_1}\right)E'_z-\left(a_z\frac{\partial}{\partial x_1}+b_z\frac{\partial}{\partial y_1}+c_z\frac{\partial}{\partial z_1}\right)E'_y$$
$$+\frac{1}{c}\left(l\frac{\partial}{\partial t_1}+a_t\frac{\partial}{\partial x_1}+b_t\frac{\partial}{\partial y_1}+c_t\frac{\partial}{\partial z_1}\right)B'_x=0.$$

If we therefore put

$$\left.\begin{aligned} b_y E'_z-b_z E'_y+\frac{b_t}{c}B'_x &\equiv E_{1z}\nu_x \\ c_y E'_z-c_z E'_y+\frac{c_t}{c}B'_x &\equiv -E_{1y}\nu_x \\ \frac{l}{c}B'_x &\equiv \frac{B_{1x}}{c}\nu_x \\ a_y E'_z-a_z E'_y+\frac{a_t}{c}B'_x &\equiv 0 \end{aligned}\right\},$$

the above equation transforms into

$$\frac{\partial}{\partial y_1} E_{1z} - \frac{\partial}{\partial z_1} E_{1y} = -\frac{1}{c} \frac{\partial B_{1x}}{\partial t_1},$$

which is of the form required for matter at rest.

Similarly the other two equations of the vector equation (2c) can be brought into the proper form for matter at rest by substitutions of the above type.

These substitutions, twelve in number, can be written in the abbreviated forms, using the vector notation where possible,

$$\left.\begin{aligned} lB' &= \{\nu_x, \nu_y, \nu_z\} B_1 \\ [a, E'] + \frac{a_t}{c} B' &= \{0, -\nu_y E_{1z}, \nu_z E_{1y}\} \\ [b, E'] + \frac{b_t}{c} B' &= \{\nu_x E_{1z}, 0, -\nu_z E_{1x}\} \\ [c, E'] + \frac{c_t}{c} B' &= \{-\nu_x E_{1y}, \nu_y E_{1x}, 0\} \end{aligned}\right\}. \tag{5}$$

In an exactly similar way the vector equation (1c) will transform into the vector equation

$$\operatorname{curl}_1 H_1 = \frac{1}{c} \left\{ C_1 + \frac{\partial D_1}{\partial t_1} \right\} \tag{1d}$$

by means of the substitutions

$$\left.\begin{aligned} lD' &= \{\lambda_x, \lambda_y, \lambda_z\} D_1 \\ [a, H'] + \frac{a_t}{c} D' &= \{0, -\lambda_y H_{1z}, \lambda_z H_{1y}\} \\ [b, H'] + \frac{b_t}{c} D' &= \{\lambda_x H_{1z}, 0, -\lambda_z H_{1x}\} \\ [c, H'] + \frac{c_t}{c} D' &= \{-\lambda_x H_{1y}, \lambda_y H_{1x}, 0\} \end{aligned}\right\}. \tag{6}$$

and

$$C_1 = \left\{ \frac{1}{\lambda_x}, \frac{1}{\lambda_y}, \frac{1}{\lambda_z} \right\} C$$

As a preliminary to determining the constants of the transformation, we notice that equations (5) and (6) give two values for each of the components of E_1 and H_1, which must necessarily agree with one another.

For example, taking H_{1z}, we have, from (6),

$$H_{1z} = -\frac{1}{\lambda_y} \left(a_z H'_x - a_x H'_z + \frac{a_t}{c} D'_y \right),$$

and also

$$= \frac{1}{\lambda_x} \left(b_y H'_z - b_z H'_y + \frac{b_t}{c} D'_x \right),$$

which can only agree if

$$a_t = b_t = a_z = b_z = 0 \quad \text{and} \quad a_x\lambda_x = b_y\lambda_y.$$

Similarly, taking H_{1y}, we should find that

$$c_t = a_t = a_y = c_y = 0 \quad \text{and} \quad \lambda_z c_z = a_x\lambda_x;$$

and, taking H_{1x}, $\quad c_t = b_t = b_x = c_x = 0,$

and $$\lambda_y b_y = \lambda_z c_z.$$

We therefore immediately find that

$$a_t = b_t = c_t = 0,$$

$$b_x = c_x = a_y = c_y = a_z = b_z = 0,$$

and $$\lambda_x a_x = \lambda_y b_y = \lambda_z c_z \ (\equiv \xi).$$

The equations (5) and (6) only differ as regards the constants by the substitution of ν for λ, so that we obtain exactly similar results from a consideration of the different values of E_1, save that

$$\nu_x a_x = \nu_y b_y = \nu_z c_z \ (\equiv \eta\xi),$$

so that $$\frac{\lambda_x}{\nu_x} = \frac{\lambda_y}{\nu_y} = \frac{\lambda_z}{\nu_z} = \frac{1}{\eta}.$$

Equations (5) and (6) therefore take the simpler forms

$$\left.\begin{aligned} lB' &= \eta B_1\{\lambda_x, \lambda_y, \lambda_z\} \\ lD' &= D_1\{\lambda_x, \lambda_y, \lambda_z\} \\ \xi\{\lambda_x, \lambda_y, \lambda_z\} H' &= H_1\lambda_x\lambda_y\lambda_z \\ \xi\{\lambda_x, \lambda_y, \lambda_z\} E' &= E_1\eta^2\lambda_x\lambda_y\lambda_z \end{aligned}\right\}. \qquad (7)$$

Further, equations (3c) and (4c) now transform into

$$\operatorname{div}_1 D_1 = \rho_1, \quad \operatorname{div}_1 B_1 = 0, \qquad (3d, 4d)$$

where $$\rho_1 = \frac{l\rho'}{\xi}.$$

To complete the scheme of electrodynamical equations for matter at rest, we must add to the above equations of the field the equations of connection in the form

$$D_1 = K_0 E_1, \quad B_1 = \mu_0 H_1, \quad C_1 = \frac{E_1}{\sigma_0},$$

and it now only remains to find the nine constants λ_x, λ_y, λ_z, η, ξ, l,

k_x, k_y, k_z, the last three of which enter into the definitions of B', D', and ρ' at the beginning of § 2.

4. *The Determination of the Constants of Transformation.*

In the special case of no conduction current, we must have

$$\iiint \mathrm{div}_1 D_1 dx_1 dy_1 dz_1 = \iiint \mathrm{div}' D\, dx'\, dy'\, dz',$$

since motion of a material medium cannot alter the charge per unit volume.

This gives $$\xi^2 l = \lambda_x \lambda_y \lambda_z,$$

by using equation (3*d*), and the original equation

$$\mathrm{div}' D = \rho.$$

From the equation $$D_1 = K_0 E_1,$$

we deduce, by means of the equation (7),

$$D' = K_0 \frac{\xi}{l\eta^2 \lambda_x \lambda_y \lambda_z} \{\lambda_x^2, \lambda_y^2, \lambda_z^2\} E'. \tag{8}$$

Since $$D' = D + \frac{1}{c}[k, H'],$$

and $$E' = E + \frac{1}{c}[v, B],$$

this gives us the relation connecting the electric displacement D and the vectors E, H' and B.

The above equation (8) must hold good for the free ether, *i.e.*, when $D = E$, and $B = H$. We must therefore have

$$E + \frac{1}{c}[k, H'] = \frac{K_0 \xi}{l\eta^2 \lambda_x \lambda_y \lambda_z} \{\lambda_x^2, \lambda_y^2, \lambda_z^2\} E'.$$

Now $$H' = H - \frac{1}{c}[v, E],$$

thus $$E + \frac{1}{c}[k, H] - \frac{1}{c^2}[k, [v, E]]$$

$$= \frac{K_0 \xi}{l\eta^2 \lambda_x \lambda_y \lambda_z} \{\lambda_x^2, \lambda_y^2, \lambda_z^2\} \left\{E + \frac{1}{c}[v, H]\right\},$$

or $$E+\frac{1}{c}[k, H]-\frac{1}{c^2}v(k, E)+\frac{E}{c^2}(k, v)$$

$$=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\{\lambda_x^2, \lambda_y^2, \lambda_z^2\}\left\{E+\frac{1}{c}[v, H]\right\}.$$

Writing out this vector equation in full, we have

$$E_x+\frac{1}{c}\{k_yH_z-k_zH_y\}=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\lambda_x^2E_x,$$

$$E_y\left\{1+\frac{k_x\bar{v}}{c^2}\right\}+\frac{1}{c}\{k_zH_x-k_xH_z\}=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\lambda_y^2\left\{E_y-\frac{\bar{v}}{c}H_z\right\},$$

$$E_z\left\{1+\frac{k_x\bar{v}}{c^2}\right\}+\frac{1}{c}\{k_xH_y-k_yH_x\}=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\lambda_z^2\left\{E_z+\frac{\bar{v}}{c}H_y\right\}.$$

We must therefore have

$$k_y=k_z=0, \qquad K_0\xi\lambda_x^2=l\eta^2\lambda_x\lambda_y\lambda_z,$$

$$\left\{1+\frac{k_x\bar{v}}{c^2}\right\}=\frac{K_0\xi\lambda_y^2}{l\eta^2\lambda_x\lambda_y\lambda_z}=\frac{K_0\xi\lambda_z^2}{l\eta^2\lambda_x\lambda_y\lambda_z},$$

and $$k_x=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\lambda_y^2\bar{v}=\frac{K_0\xi}{l\eta^2\lambda_x\lambda_y\lambda_z}\lambda_z^2\bar{v},$$

which can be satisfied by putting $K_0=1$,

$$k_x=\frac{\xi}{l\eta^2}\frac{\bar{v}}{\lambda_x}=\bar{v}\left(1-\frac{\bar{v}^2}{c^2}\right)^{-1}=\bar{v}\epsilon,$$

$$\lambda_y=\lambda_z=\lambda_x\epsilon^{\frac{1}{2}}.$$

Thus the constant K_0 for the free ether is unity, which is consistent with the original equations of the electromagnetic field from which we set out, which are based on the hypothesis of a stagnant ether.

From the equation $$B_1=\mu_0H_1,$$

we deduce from the equations (7), that

$$B'=\frac{\mu_0\eta\xi}{l\lambda_x\lambda_y\lambda}\{\lambda_x^2, \lambda_y^2, \lambda_z^2\}H'. \qquad (9)$$

Since $$B' = B - \frac{1}{c}[k, E'],$$

and $$H' = H - \frac{1}{c}[v, E],$$

this gives the relation connecting the magnetic induction B with the electric and magnetic forces E and H.

This equation (9) must hold good in the case of a non-magnetizable medium, *i.e.*, when $B = H$. The equation may then be written in the form

$$H - \frac{1}{c}[k, E'] = \frac{\mu_0 \xi\eta}{l\lambda_x\lambda_y\lambda_z} \{\lambda_x^2, \lambda_y^2, \lambda_z^2\} H',$$

or, since $$E' = E + \frac{1}{c}[v, H],$$

$$H - \frac{1}{c}[k, E] - \frac{1}{c^2} v(k, H) + \frac{H}{c^2}(k, v)$$

$$= \frac{\mu_0 \xi\eta}{l\lambda_x\lambda_y\lambda_z} \{\lambda_x^2, \lambda_y^2, \lambda_z^2\} \left\{H - \frac{1}{c}[v, E]\right\}.$$

On writing out this vector equation in full, we should find that it can be satisfied by putting

$$\mu_0 = 1, \quad k_x = \frac{\xi\eta}{l\lambda x}\bar{v} = \bar{v}\left(1 - \frac{\bar{v}^2}{c^2}\right)^{-1} = \bar{v}\epsilon,$$

$$\lambda_y = \lambda_z = \lambda_x \epsilon^{\frac{1}{2}}.$$

The various formulæ connecting the constants of the transformation can now be brought together into the scheme

$$k_x = \bar{v}\epsilon, \quad k_y = 0, \quad k_z = 0,$$

$$\lambda_y = \lambda_z = \lambda_x \epsilon^{\frac{1}{2}}, \quad \frac{\xi\eta}{l\lambda_x} = \frac{\xi}{l\eta^2\lambda_x} = \epsilon,$$

and $$\xi^2 l = \lambda_x\lambda_y\lambda_z.$$

It follows then that $\eta = 1$, whence, expressing everything in terms of ϵ and $\epsilon^{\frac{1}{2}}l\ (\equiv j)$, we have

$$k_x = \bar{v}\epsilon, \quad \lambda_x = \epsilon^{-\frac{1}{2}} j^3, \quad \lambda_y = \lambda_z = j^3, \quad l = j\epsilon^{-\frac{1}{2}}, \quad \text{and} \quad \xi = j^4.$$

Thus the transformation is fully determined except as to the constant j.

5. *Final Scheme of the Transformation.*

In Lorentz's paper in 1904, where the case of a non-magnetizable dielectric is considered, this same indeterminateness occurs, and it is

there shown that, in order to get consistent results for the longitudinal electromagnetic mass, the value of the constant j must be taken to be unity.

The transformation is thus finally expressible in the following form:—

The equations referred to axes moving with the matter along the direction of the axis of x' can be reduced to the form for axes at rest, by the transformation

$$t_1 = \epsilon^{-\frac{1}{2}} \left(t' - \frac{\epsilon}{c^2} \bar{v} x' \right), \quad x_1 = \epsilon^{\frac{1}{2}} x', \quad y_1 = y', \quad z_1 = z',$$

$$H' = \{H_{1x},\ \epsilon^{-\frac{1}{2}} H_{1y},\ \epsilon^{-\frac{1}{2}} H_{1z}\},$$

$$E' = \{E_{1x},\ \epsilon^{-\frac{1}{2}} E_{1y},\ \epsilon^{-\frac{1}{2}} E_{1z}\},$$

$$B - \frac{\epsilon}{c}[v, E'] = \{B_{1x},\ \epsilon^{\frac{1}{2}} B_{1y},\ \epsilon^{\frac{1}{2}} B_{1z}\},$$

$$D + \frac{\epsilon}{c}[v, H'] = \{D_{1x},\ \epsilon^{\frac{1}{2}} D_{1y},\ \epsilon^{\frac{1}{2}} D_{1z}\},$$

$$C = \{\epsilon^{-\frac{1}{2}} C_{1x},\ C_{1y},\ C_{1z}\}.$$

The equations become

$$\operatorname{curl}_1 E_1 = -\frac{1}{c} \frac{\partial B_1}{\partial t_1},$$

$$\operatorname{curl}_1 H_1 = \frac{1}{c} \left\{ C_1 + \frac{\partial D_1}{\partial t_1} \right\},$$

$$\operatorname{div}_1 B_1 = 0, \qquad \operatorname{div}_1 D_1 = \rho_1,$$

where

$$\rho_1 = \left\{ \rho - \frac{\epsilon \bar{v} C_x}{c^2} \right\} \epsilon^{-\frac{1}{2}}.$$

Further, the relations between the electric displacement, the magnetic induction, the conduction current, and the electric and magnetic forces for the case of matter in motion are

$$D + \frac{\epsilon}{c}[v, H'] = K_0 \{E'_x,\ \epsilon E'_y,\ \epsilon E'_z\},$$

$$B - \frac{\epsilon}{c}[v, E'] = \mu_0 \{H'_x,\ \epsilon H'_y,\ \epsilon H'_z\},$$

$$C = \frac{1}{\sigma_0} \{E'_x \epsilon^{-\frac{1}{2}},\ E'_y \epsilon^{\frac{1}{2}},\ E'_z \epsilon^{\frac{1}{2}}\}.$$

Since $$E' = E + \frac{1}{c}[v, B],$$

the second of the above relations can be written in the form

$$B - \frac{\epsilon}{c}[v, E] - \frac{\epsilon}{c^2} v(v, B) + \epsilon \frac{v^2}{c^2} B = \mu_0 \{H'_x, \epsilon H'_y, \epsilon H'_z\},$$

or $$B - \epsilon\{H - H'\} - \frac{\epsilon}{c^2} v(v, B) + \epsilon \frac{v^2}{c^2} B = \mu_0 \{H'_x, \epsilon H'_y, \epsilon H'_z\}.$$

Therefore, writing out in full this vector equation, which, it must be remembered, only holds good for a velocity whose components are $\bar{v}$, 0, 0, we have

$$\left.\begin{aligned} B_x &= \mu_0 H'_x = \mu_0 H_x \\ B_y &= H_y + (\mu_0 - 1) H'_y \\ B_z &= H_z + (\mu_0 - 1) H'_z \end{aligned}\right\}. \tag{10}$$

Since $$H' = H - \frac{1}{c}[v, E],$$

this scheme of equations expresses B in terms of H and E for the case of matter in motion.

No modification of the equation of connection between the magnetic induction and the electric and magnetic forces is used by Lorentz for the case of matter in motion, but the above equation is the same as that used by G. T. Walker in his work on *Aberration and the Electromagnetic Field.*

For the relation connecting D with H and E we have, in a similar way,

$$D + \frac{\epsilon}{c}[v, H] - \frac{\epsilon}{c^2} v(v, E) + \epsilon \frac{v^2}{c^2} E = K_0 \{E'_x, \epsilon E'_y, \epsilon E'_z\},$$

or $$D + \frac{\epsilon}{c}[v, H - B] - \frac{\epsilon}{c^2} v(v, E) + \epsilon \frac{v^2}{c^2} E$$

$$= K_0 \{E'_x, \epsilon E'_y, \epsilon E'_z\} - \frac{\epsilon}{c}[v, B]$$

$$= K_0 \{E'_x, \epsilon E'_y, \epsilon E'_z\} - \epsilon\{E' - E\}.$$

Writing this vector equation out in full, the components of the vector v being $\bar{v}$, 0, 0, we have

$$\left.\begin{aligned} D_x &= K_0 E'_x = K_0 E_x \\ D_y &= E_y + \epsilon(K_0 - 1) E'_y - \frac{\epsilon}{c} \bar{v}\{B_z - H_z\} \\ D_z &= E_z + \epsilon(K_0 - 1) E'_z + \frac{\epsilon}{c} \bar{v}\{B_y - H_y\} \end{aligned}\right\}. \tag{11}$$

The equation of connection in this case, as given by Larmor and Lorentz, is

$$D = E + (K_0 - 1) E',$$

which differs essentially from the above form.

For the case of a non-magnetizable medium $B = H$, and the two equations of connection agree up to terms of the first order in $\bar{v}/c$.

Finally, the relation connecting C with E and B is

$$C = \frac{1}{\sigma_0} \{E'_x \epsilon^{-\frac{1}{2}},\ \epsilon^{\frac{1}{2}} E'_y,\ \epsilon^{\frac{1}{2}} E'_z\},$$

which is the same as that ordinarily given, up to terms of the first order in $\bar{v}/c$.

6. *Comparison with other Theories.*

Cohn accepts the equations of the field as given here, and puts forward as the equations of connection*

$$D = K_0 E' - \frac{1}{c}[v, H'],$$

$$C = \frac{1}{\sigma_0} E',$$

$$B = \mu_0 H' + \frac{1}{c}[v, E'].$$

On comparing these with the equations as given in this paper, it is easily seen that the difference between the two sets of equations is of the second order, owing to the factor ϵ which enters into our equations.

An objection to the above equations is that for the free ether, *i.e.*, for $B = H$, and $D = E$, μ_0 and K_0 do not become unity; nor can the equations be satisfied by assuming aelotropy in μ_0 and K_0, as is the case in the equations in § 2, which differ from those of Cohn in the constant k, which latter constant we determined there so as to make the equations consistent in the case of the free ether.

The differences between the equations of connection given in this paper and the equations ordinarily given have been pointed out in § 5. Here it must be remembered that the equations of connection given by Larmor and Lorentz for the case of matter in motion did not pretend to

* *Göttinger Nachrichten*, 1901, Vol. 1; *Ann. der Physik*, Vol. 29, 1902.

be correct up to the second order in $\bar{v}/c$, so that a difference in this order is to be expected.*

Besides differences of the second order, the equations (11) differ from the equations of Larmor and Lorentz by a term of the first order, agreeing with them up to the first order in the case of non-magnetizable bodies.

A system of equations of a somewhat similar type have recently been given by the late H. Minkowski,† a demonstration of which has also been given by Einstein and Laub.‡ The method used in these papers seems to be essentially different from that employed here. Einstein and Laub start from the Maxwellian scheme of equations for matter at rest, and then apply to them the equations of transformation which belong to the Lorentz-Einstein theory of relativity, thereby arriving at the equations of electrodynamics for matter in motion.

The equations thus reached differ from the equations of Larmor and Lorentz, not only as regards the equations of connection, but also as regards the equations of the electromagnetic field.

For the sake of comparison, we give here at length Minkowski's equations.

Referred to axes *fixed in space*, the equations of the field are

$$\operatorname{curl} H = \frac{1}{c}\left\{\frac{\partial D}{\partial t} + C + \rho v\right\},$$

$$\operatorname{curl} E = \frac{1}{c}\frac{\partial B}{\partial t},$$

$$\operatorname{div} D = \rho, \quad \operatorname{div} B = 0,$$

whilst the equations of connection are

$$D + \frac{1}{c}[v, H] = K_0\left\{E + \frac{1}{c}[v, B]\right\},$$

$$B - \frac{1}{c}[v, E] = \mu_0\left\{H - \frac{1}{c}[v, D]\right\},$$

$$C = \frac{1}{\sigma_0}\left\{E'_x\epsilon^{-\frac{1}{2}},\ E'_y\epsilon^{\frac{1}{2}},\ E'_z\epsilon^{\frac{1}{2}}\right\}.$$

Comparing the equations of the field with those given at the beginning of § 1, the difference lies in the first equation, where the term curl $[P, v]$ is included in the first equation in § 1. As regards the equations of con-

* Cp. *e.g.*, Larmor, *Aether and Matter*, p. 113.

† *Göttinger Nachrichten*, 1908.

‡ *Annalen der Physik*, 1908, Vol. 26, p. 552.

nection, the first two of Minkowski's equations differ from those given in this paper.

The differences seem to lie in the fact that a non-magnetizable medium is defined by Larmor and Lorentz as a medium in which $B = H$, which condition has been used in § 4, in finding the constants of the general transformation; according to Minkowski, however, non-magnetizable media are defined by $\mu_0 = 1$, so that for such a medium B is a subsidiary vector defined as equal to

$$H - \frac{1}{c}[v, D-E],$$

whence B is no longer equal to H, unless at the same time $D = E$, *i.e.*, except in the case of the free ether.

The first two of Minkowski's equations of connection are, in fact, two simultaneous equations giving B and D in terms of E and H.

Solving them, we find*

$$D + \frac{K_0\mu_0}{c^2}[v, [v, D]] = K_0\left\{E + \frac{1}{c^2}[v, [v, E]]\right\} + \frac{(K_0\mu_0-1)}{c}[v, H],$$

$$B + \frac{K_0\mu_0}{c^2}[v, [v, D]] = \mu_0\left\{H + \frac{1}{c^2}[v, [v, H]]\right\} - \frac{(K_0\mu_0-1)}{c}[v, E].$$

Taking as a particular case a direction z perpendicular to the velocity $\bar{v}$, these equations read

$$D_z\left\{1 - K_0\mu_0\frac{\bar{v}^2}{c^2}\right\} = K_0\left(1 - \frac{\bar{v}^2}{c^2}\right)E_z + \frac{\bar{v}}{c}(K_0\mu_0-1)H_y,$$

$$B_z\left\{1 - K_0\mu_0\frac{\bar{v}^2}{c^2}\right\} = \mu_0\left(1 - \frac{\bar{v}^2}{c^2}\right)H_z - \frac{\bar{v}}{c}(K_0\mu_0-1)E_y.$$

Comparing these with the corresponding equations as given in § 5, which may be written

$$D_z = E_z\left\{1 + \epsilon(K_0-1) + \frac{\epsilon\bar{v}^2}{c^2}K_0(\mu_0-1)\right\} + \frac{\epsilon\bar{v}}{c}H_y(K_0\mu_0-1),$$

$$B_z = \mu_0 H_z - \frac{\bar{v}}{c}(\mu_0-1)E_y,$$

the differences are at once apparent.

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* Cf. Einstein and Laub, *loc. cit.*, p. 539.

Since $$\epsilon = \left(1-\frac{\bar{v}^2}{c^2}\right)^{-1},$$

we see that the equations for D_z are the same to the first order, viz.,

$$D_z = K_0 E_z + \frac{\bar{v}}{c} H_y (K_0\mu_0 - 1).$$

The equations for B_z are, however, different, even up to this order, being respectively

$$B_z = \mu_0 H_z - \frac{\bar{v}}{c} (K_0\mu_0 - 1) E_y,$$

and $$B_z = \mu_0 H_z - \frac{\bar{v}}{c} (\mu_0 - 1) E_y.$$

This difference, of course, depends on the different conceptions of magnetic media, as stated above, since, according to Minkowski, a non-magnetic medium possesses a magnetic induction B, which is not equal to the magnetic force H, differing therefrom by a quantity of the first order in $\bar{v}/c$.

ON A CLASS OF CONDITIONALLY CONVERGENT INFINITE PRODUCTS

By J. E. LITTLEWOOD.

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1. In a paper recently published in these *Proceedings*,† Mr. Hardy raises the problem: *to find a product* $\Pi(1+a_n)$, *such that* Σa_n^k *is always convergent* [*i.e.*, for positive integral values of k], *but never absolutely, and whose convergence, divergence, or oscillation is capable of proof.*

Mr. Hardy shows that the product $\Pi(1+a_n e^{n\theta i})$ is divergent when θ/π is rational, where a_n is a positive function of n, which tends steadily to zero as $n \to \infty$, and which is such that Σa_n^k is divergent for all positive integral values of k.‡ The question naturally arises: Is such a product ever convergent when θ/π is irrational? This question will be seen to admit of a comparatively simple answer, *i.e.*, an answer not cumbered with elaborate restrictions as to the nature of a_n. It will be shown that, *if* a_n *be any positive decreasing function of* n *which tends to zero as* $n \to \infty$, *the product* $\Pi(1+a_n e^{n\theta i})$ *is convergent for a certain class of irrational values of* θ/π, *which class is independent of the* a*'s, and includes all algebraic numbers.*

2. It is well known that, if $\lim_{n\to\infty} |u_n| = 0$, the product

$$\prod_{n=1}^{\infty} \left[(1+u_n)\exp\left(-u_n+\frac{u_n^2}{2}-\frac{u_n^3}{3}+\ldots\pm\frac{u_n^n}{n}\right)\right]$$

is convergent, so that the same product, when taken from $n=1$ to ν,

* Shortly after the paper was communicated, and at the suggestion of the Council, I altered the original title, and gave a more explicit account of Abel's lemma, which is used in § 2. Some time later I discovered that the original discussion corresponding to §§ 2 and 3 was incomplete. The part of the paper then rewritten (October 24th) consists of § 3 and the part of § 2 which follows the result (3) of that article.

† Ser. 2, Vol. 7, p. 40, "On the Continuity or Discontinuity of a Function defined by an Infinite Product." See, in particular, pp. 47, 48.

‡ The simplest example of such a function is $\Pi(1+e^{n\theta_n}/\log n)$.

tends to a definite limit as $\nu \to \infty$. Taking $u_n = a_n e^{n\theta_\iota}$, we see then that

$$\prod_{n=1}^{\infty} (1+a_n e^{n\theta_\iota})$$

is convergent, provided

$$\prod_{n=1}^{\nu} \exp\left[\sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{nm\theta_\iota}\right]$$

tends to a definite non-zero limit, or (on taking the logarithm) provided

$$f(\nu) = \sum_{n=1}^{\nu} \sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{nm\theta_\iota} \tag{1}$$

tends to a definite limit as $\nu \to \infty$.

Let $$\theta_m = 2\pi\left[\frac{m\theta}{2\pi}\right],$$

where $[x]$ denotes the difference between x and the nearest integer. Then, if θ/π is irrational, we evidently have

$$0 < |\theta_m| < \pi. \tag{2}$$

Since $m\theta$ differs from θ_m by a multiple of 2π, we obtain from (1)

$$f(\nu) = \sum_{n=1}^{\nu} \sum_{m=1}^{n} \frac{(-)^m}{m} a_n^m e^{n\theta_m\iota}.$$

If in this expression we change the order of summation, we obtain

$$\left.\begin{aligned} f(\nu) &= \sum_{m=1}^{\nu} \sum_{n=m}^{\nu} \frac{(-)^m}{m} a_n^m e^{n\theta_m\iota} \\ &= \sum_{m=1}^{\nu} \frac{(-)^m}{m} \sigma_m(\nu) \end{aligned}\right\} \tag{3}$$

where $$\left.\sigma_m(\nu) = \sum_{n=m}^{\nu} a_n^m e^{n\theta_m\iota}\right\}.$$

A necessary and sufficient condition that a function $f(\nu)$ of ν should tend to a definite limit is that, for an arbitrarily small positive ϵ, and for all values of $\nu' > \nu$, we have

$$|f(\nu') - f(\nu)| < \epsilon, \quad \text{when} \quad \nu > N,$$

where N depends on ϵ, but not on ν'.

In the present case we have

$$f(\nu') - f(\nu) = \sum_{m=1}^{\nu} \frac{(-)^m}{m} \{\sigma_m(\nu') - \sigma_m(\nu)\} + \sum_{m=\nu+1}^{\nu'} \frac{(-)^m}{m} \sigma_m(\nu'). \tag{4}$$

Now the expressions

$$\sigma_m(\nu') - \sigma_m(\nu) = \sum_{n=\nu+1}^{\nu'} a_n^m e^{n\theta_m \iota},$$

and

$$\sigma_m(\nu') = \sum_{n=m}^{\nu'} a_n^m e^{n\theta_m \iota},$$

are sums of the type considered in the theorem known as Abel's lemma. The theorem is as follows.*

If the sequence $(v_1, v_2, \ldots)$ of positive terms never increases, then

$$\left| \sum_{n=1}^{p} a_n v_n \right| \leqslant H v_1,$$

where H is the upper limit of the expressions

$$|a_1|, \ |a_1+a_2|, \ |a_1+a_2+a_3|, \ \ldots, \ |a_1+a_2+\ldots+a_p|.$$

In $\sigma_m(\nu') - \sigma_m(\nu)$, we may take

$$(a_{\nu+1}^m, a_{\nu+2}^m, \ldots, a_{\nu'}^m), \quad (e^{(\nu+1)\theta_m \iota}, e^{(\nu+2)\theta_m \iota}, \ldots, e^{\nu'\theta_m \iota}),$$

respectively for the sequences of v's and a's, and for $\sigma_m(\nu')$, we have the sequences

$$(a_m^m, a_{m+1}^m, \ldots, a_{\nu'}^m), \quad (e^{m\theta_m \iota}, e^{(m+1)\theta_m \iota}, \ldots, e^{\nu'\theta_m \iota}).$$

Now the sum $\sum_{n=p}^{q} e^{n\theta_m \iota}$ of any number of consecutive terms of the sequence $e^{\theta_m \iota}, e^{2\theta_m \iota}, e^{3\theta_m \iota}, \ldots$, has a modulus less than $2\pi |\theta_m|^{-1}$.† Hence the number H corresponding to each of the expressions $\sigma_m(\nu') - \sigma_m(\nu)$, $\sigma_m(\nu')$, is less than $2\pi |\theta_m|^{-1}$. By the theorem, then, we have

$$|\sigma_m(\nu') - \sigma_m(\nu)| < 2\pi a_{\nu+1}^m |\theta_m|^{-1},$$

$$|\sigma_m(\nu')| < 2\pi a_m^m |\theta_m|^{-1}.$$

* The theorem in Abel's original form is given in Bromwich's "Infinite Series," pp. 54, 55. The theorem for complex a's given above follows by a trifling modification of the argument: it is also a particular case of the result to be found at the bottom of p. 205.

† We have $$\left| \sum_{n=p}^{q} e^{n\omega_m} \right| = \left| \frac{e^{(q+1)\omega_m} - e^{p\omega_m}}{1-e^{\omega_m}} \right| < \frac{2}{|1-e^{\omega_m}|}.$$

Now, if $|\theta_m| < \frac{1}{2}\pi$, then

$$|1-e^{\omega_m}|^{-1} < |\sin\theta_m|^{-1} < [|\theta_m|/(\tfrac{1}{2}\pi)]^{-1} < \tfrac{1}{2}\pi |\theta_m|^{-1};$$

and, if $\frac{1}{2}\pi \leqslant |\theta_m| < \pi$, then

$$|1-e^{\omega_m}|^{-1} \leqslant |1-\cos\theta_m|^{-1} \leqslant 1 < \pi |\theta_m|^{-1}.$$

It follows that in any case $$\left| \sum_{n=p}^{q} e^{n\omega_m} \right| < 2\pi |\theta_m|^{-1}.$$

Hence, from (4),

$$|f(\nu')-f(\nu)| < 2\pi \sum_{m=1}^{\nu} \frac{1}{m}\, a_{\nu+1}^m\, |\theta_m|^{-1} + 2\pi \sum_{m=\nu+1}^{\nu'} \frac{1}{m}\, a_m^m\, |\theta_m|^{-1}$$

$$< 2\pi \sum_{m=1}^{\nu} a_\nu^m\, |\theta_m|^{-1} + 2\pi \sum_{m=\nu+1}^{\nu'} a_\nu^m\, |\theta_m|^{-1}$$

$$\left(\text{since } \frac{1}{m} \leqslant 1,\ a_{\nu+1} \leqslant a_\nu, \text{ and } a_m \leqslant a_\nu, \text{ when } m > \nu\right)$$

$$< 2\pi \sum_{m=1}^{\nu'} a_\nu^m\, |\theta_m|^{-1}. \tag{5}$$

3. Let us now suppose that for *all* values of m we have

$$|\theta_m| > K^{-m},$$

where K is some positive constant. Then, from (5) above,

$$|f(\nu')-f(\nu)| < 2\pi \sum_{m=1}^{\nu'} (Ka_\nu)^m. \tag{1}$$

Let ν be chosen so large that

$$Ka_n < \tfrac{1}{2}.$$

Then we shall have, taking the geometric series in (1) to infinity,

$$|f(\nu')-f(\nu)| < 4\pi a_\nu.$$

This last expression is independent of ν', and may be made less than ϵ by choosing ν sufficiently large. The function $f(\nu)$ will consequently tend to a definite limit.

4. We conclude, then, that $\Pi(1+a_n e^{n\theta i})$ is convergent, provided we can find a constant K, such that

$$|\theta_m| > K^{-m}, \quad \text{when} \quad m \geqslant 1.$$

This condition will certainly be fulfilled if θ/π is an algebraic number. For it is well known that, if x is an algebraic number, and r/m an arithmetical fraction differing from x by less than (say) unity, then

$$|x-r/m| > hm^{-q},$$

where q is the degree of the equation with rational coefficients of which x is a root, and where h depends only on x.* Hence, in the case we are considering, if

$$m\theta/(2\pi) = \theta_m/(2\pi)+r,$$

* See, for example, Borel, *Leçons sur la Théorie des Fonctions*, p. 27.

so that r/m differs from $\theta/(2\pi)$ by less than unity, we have

$$|\theta/(2\pi)-r/m| > hm^{-q},$$

or

$$|\theta_m| > hm^{1-q}.$$

We shall then have, for all values of $m \geqslant 1$,

$$|\theta_m| > K_1^{-1} e^{-(m-1)},$$

where K_1 is a constant, for the expression

$$e^{-(m-1)}/(hm^{1-q})$$

has, for positive values of m, a finite maximum value when $m = q-1$. If now we choose K to be the greater of e and K_1, we have

$$|\theta_m| > K^{-m}.$$

ON GAUSS'S THEOREM CONCERNING THE SURFACE INTEGRAL OF NORMAL FORCE IN THE THEORY OF ATTRACTIONS

By J. G. Leathem.

[Received April 30th, 1909.—Read May 13th, 1909.]

1. The proof of Gauss's theorem* in the theory of attractions, as given in most text-books, involves the view that the attracting or repelling substance (whether matter or electricity) is made up of discrete particles of dimensions which are either absolutely zero or negligibly small in comparison with other distances in the contemplated configuration. It is shewn that the contribution of a particle of mass m to the surface integral of normal force is zero if the particle is definitely outside the space enclosed by the surface, and $4\pi\gamma m$ if the particle is definitely inside that space. (Here γ denotes the constant of proportionality in the law of inverse square.)

A particle of absolutely zero dimensions is necessarily in the enclosed region, outside the enclosed region, or in the surface which constitutes the boundary. In the last case it is easy to shew that the contribution of the particle to the surface integral of normal force is generally $2\pi\gamma m$, but may have some other value if the particle is at a singular point of the surface. There is therefore no difficulty in verifying, for point particles, the complete theorem as stated originally by Gauss, namely so as to include particles of no dimensions situated in the boundary.

When the particles considered have size, the ordinary argument applies, with as much precision as there is in our knowledge of the truth of the Newtonian Law, to such as are at a distance from the boundary great in comparison with their own linear dimensions. But in the case of particles, some of whose points are at a distance from the boundary which is not great in comparison with their linear dimensions, two difficulties arise. In the first place the law of the inverse square may fail adequately to represent the field of force at such close proximity to the particle; and, in the second place, in the absence of information

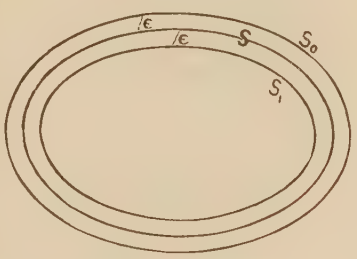
* C. F. Gauss, "Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte," *Ges. Werke*, Bd. v, s. 224.

as to the size and structure of a particle, it may be uncertain whether the particle is or is not wholly on one side of the boundary surface, and, if it is cut by the surface, how its contribution to the integral is to be reckoned.

The mathematical theory of attractions, however it may appear fundamentally and originally to have treated of particles, has by modern convention been in great measure transferred to another field of investigation; and, doubtless because the precision of a limit appeals more strongly to the pure mathematician than the apparent laxity of what is merely a close approximation, the most familiar propositions of the theory are enunciated as applying to an ideal attracting substance which is not made up of discrete particles, *i.e.*, is not of molecular structure, but is a continuum. The transition to such a substance from the particles originally discussed is made by dividing space into volume elements, and treating the continuous matter in a volume element as a particle. Clearly this is justifiable so long as it is recognized that the matter in a volume-element constitutes a particle whose dimensions are not zero, and which therefore only comes under the elementary reasoning which leads to the Gauss theorem when it is at a distance from the boundary great compared with its linear dimensions.

Thus the transition, usually assumed without discussion, from the Gauss theorem for particles to the corresponding theorem for continuous matter, is quite safe provided the surface S over which the integral is taken does not cut through the continuous matter. But the transition is not obviously safe, and requires special proof, for a surface that intersects the matter; for in this case some of the volume elements which are treated as particles must necessarily be actually in contact with the surface of integration.

2. In order to see what additional proof is required in this case, let us draw two surfaces parallel to the surface S at a small distance ϵ from it, the one S_1 inside it, the other S_0 outside it.



The normal force N at any point of the surface S may be regarded as made up of three parts, namely: (i) N_1, the part due to all the matter inside S_1, (ii) N_0, the part due to all the matter outside S_0, (iii) N', the part due to all the matter in the space between S_1 and S_0.

For any selected value of ϵ, the volume elements which take the place of particles can always be chosen so small that their linear dimensions are as small as we please in comparison with ϵ. Hence the ordinary elementary reasoning is valid for all the matter M_1 inside S_1, and for all the matter M_0 outside S_0. Hence, for the surface S,

$$\int N_1 dS = 4\pi\gamma M_1, \qquad \int N_0 dS = 0.$$

Thus $$\int N dS = \int N_1 dS + \int N_0 dS + \int N' dS = 4\pi\gamma M_1 + \int N' dS.$$

If now we pass to the limit, for $\epsilon \to 0$, it is obvious that $\operatorname{Lim} M_1 = M$, where M is the mass of all the matter inside S.* Hence

$$\int N dS = 4\pi\gamma M + \operatorname*{Lim}_{\epsilon \to 0} \int N' dS ;$$

so the Gauss theorem is true if, and only if,

$$\operatorname*{Lim}_{\epsilon \to 0} \int N' dS = 0.$$

3. A class of cases in which this limit is zero can easily be specified. For if, for any selected value of ϵ, there is a maximum value or superior limit to the values of $|N'|$ at points on S, say n, then

$$\left| \int N' dS \right| < nS,$$

where S is the complete area of the closed surface. And if, further,

$$\operatorname*{Lim}_{\epsilon \to 0} n = 0,$$

then clearly $$\operatorname*{Lim}_{\epsilon \to 0} \int N' dS = 0.$$

We shall see that these conditions are satisfied in ordinary cases of matter so distributed that the volume-density is everywhere finite.

Volume-Density.—If the distribution of continuous matter between S_1 and S_0 has everywhere finite volume-density, we know† that the force-intensity at every point in that region is definite, and so N' is definite at every point of S. Consequently the superior limit or maximum value n exists. It remains to ascertain whether $n \to 0$ as $\epsilon \to 0$. The first step towards this is to shew that in all ordinary cases $N' \to 0$ as $\epsilon \to 0$. When this has been established, if n is a maximum value of $|N'|$, the convergence of all values of N' to zero necessarily involves

* M does not include matter distributed with finite surface density *in* the surface S.

† Leathem, *Volume and Surface Integrals used in Physics*, Cambridge Tracts, No. 1, § 23.

also the convergence of n to zero. But if n is a superior limit to $|N'|$ without being a maximum value, we can be sure of the convergence of n to zero only if the convergence of N' to zero is uniform for the values of N' corresponding to all points of the surface S.

4. If we assume that the part of the material distribution contained between S_1 and S_0 consists only of finite volume-density which, though not necessarily continuous, is a function of position free from other analytical peculiarities, then singularities in the function N' can only arise from peculiarities in the shape of the surface S and in the direction of the normal along which N' is the component of force. At an ordinary conical point, or at a point on the intersection of two sheets of the surface S, the direction of the normal is indeterminate and so also is the value of N'; nevertheless there are definite directions to which the normal tends as to a limit, and corresponding values which are limits of the function N' without being values of the function. One of the limits of N' at such a point might be the superior limit n and yet not be a maximum value. This sort of case must not be omitted from the present discussion, for one of the closed surfaces most frequently employed in applications of the Gauss theorem is a cylinder with flat ends, *i.e.*, a surface of three sheets with two nodal lines.

The study of the influence of peculiarities of the surface S can be avoided if we express our test in terms of F', the resultant force-intensity due to the matter between S_0 and S_1, instead of N' its normal component. For $N' \leqslant F'$, and, if f be the maximum value or superior limit of F',

$$\left| \int N' \, dS \right| < fS,$$

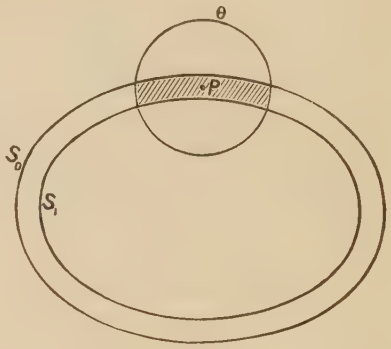
and the Gauss theorem holds provided $f \to 0$ as $\epsilon \to 0$.

For the simple kinds of material distribution contemplated, F' is definite and continuous everywhere between S_1 and S_0, so there is no possibility of f being a superior limit without being a maximum value. And therefore, if we can shew that $F' \to 0$ for every point on S, we are thereby assured that $f \to 0$. The convergence of F' to zero is proved by shewing that the component G' of F' in any arbitrarily selected direction tends to zero.

5. We aim, then, at shewing that the component in any direction of the intensity of force at a point, due to a distribution of given volume-density contained between two parallel surfaces, tends to zero as the surfaces tend to coincidence. This result would be obvious if the point were definitely outside the attracting distribution. But it is not obvious

in the present instance since the point is inside the distribution, nor would it be obvious if the point, though outside the distribution, were at a distance from its boundary which tended to vanishing.

Let P be the point at which F' is estimated. With P as centre, describe a sphere of radius θ. Then G' at P is the sum of two terms, G_1' due to that part of the matter between S_1 and S_0 which is exterior to the sphere, and G_2' due to that part of the matter between S_1 and S_0 which is interior to the sphere.



Now in consequence of the convergence of the integral representing the component of force in any direction due to a volume distribution, it is always possible to choose θ so small that G_2' shall be less than any assigned small quantity $\frac{1}{2}\omega$. And when θ has been selected we notice that P is definitely outside the matter which gives rise to G_1', so that G_1' tends to zero as the quantity of matter tends to zero. Thus we can always choose ϵ so small that G_1' shall be less than $\frac{1}{2}\omega$. Consequently we have been able to choose ϵ so small that

$$G_1' + G_2' = G' < \omega.$$

Thus

$$\lim_{\epsilon \to 0} G' = 0.$$

This is all we require for a proof of the Gauss theorem in the case in which there is no distribution of matter between S_0 and S_1 save such as has finite volume-density. We have seen that, since $G' \to 0$,

$$f \to 0 \quad \text{and} \quad \int N' dS \to 0.$$

So we conclude that when the surface S cuts only through matter of finite volume-density the Gauss theorem is valid.

6. *Surface-Density.*—Let us now consider the case in which the distribution of matter between S_1 and S_0 includes a surface distribution, say on a surface Ω, of finite surface-density, which means of course infinite volume-density. In general the surface Ω will cut the surface S in a curve. A particular case, requiring special consideration, is that in which the surfaces S and Ω coincide over a definite area.

At the outset it is important to remember that the force-intensity at a point of a surface distribution such as Ω is represented by a semi-

convergent integral,* and is therefore not completely defined. By a suitable convention the definition may be completed, and in such a way that the force-intensity is finite. Adopting some such convention, we have N' finite at points on the curve of intersection of Ω and S; we have already seen that N' is finite at all other points of S, so that there is a finite superior limit or maximum value n of the values of $|N'|$ for all the points of S.

7. It is to be remarked that the particular convention adopted in order to give definiteness to N' does not generally affect the value of $\int N'\,dS$. If the surface Ω cuts the surface S in a curve, then the elements of area associated with dubious values of N' in the surface-integral form a narrow strip on S along the curve of intersection of the two surfaces, and in the process of integration the area of this strip tends to zero, making its contribution to the surface integral also zero. On the other hand if the surface Ω coincides, over an area A, with the surface S, N' at points of A is the component of force-intensity normal to the former surface as well as to the latter. Now all the indeterminateness or semi-convergence of force due to a surface distribution is in the tangential component; the normal component is definite, and so there is no indeterminateness in the value of $\int N'\,dS$.

Incidentally we recall the fact† that the normal force-intensity at a point in a surface containing surface density σ differs from the limit of the normal force for a point approaching the surface but not in it by $2\pi\gamma\sigma$. Consequently the surface-integral $\int N'\,dS$ is less when the surface of integration coincides with the surface on which the surface-density resides than if the former surface were just outside the latter by the amount $2\pi\gamma\int\sigma\,dS$ taken over the common area A.

8. In the more general case, in which Ω intersects S, having seen that $\int N'\,dS$ is definite we must examine what is the limit of its value as $\epsilon \to 0$. This we do by considering the component G' in any direction of the force-intensity at a point P in the curve of intersection of Ω and S, due to that part of the surface Ω which is intercepted between S_1 and S_0, and by investigating whether, as $\epsilon \to 0$, G' tends to infinity or to a finite or zero limit.

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* Leathem, *l.c.*, § 30. † Leathem, *l.c.*, § 33.

The problem is strictly analogous to that already discussed for volume-density, being, namely, that of examining the limit of a component of attraction at any point in a distribution of finite surface-density, as the distribution is diminished to zero in one of its dimensions without change of surface-density. The result is, however, quite different from that obtained for volume-density.

Let us begin by attempting to apply the method already used for volume-density, with such changes as are demanded by the changed circumstances. The attempt will be unsuccessful, but a study of the reason for its failure will help to make clear the nature of the difficulty which has to be faced. P is a point in the strip of breadth $2\epsilon \operatorname{cosec} \chi$ cut by S_0 and S_1 on the surface Ω, where χ is the angle of intersection of S and Ω. Suppose the ambiguity arising out of the semi-convergence of the surface-integral representing G' to have been removed by selecting some special shape for the vanishing cavity round P. Describe round P, in the surface Ω, a curve θ of the special shape selected. The component of force at P due to the whole strip of Ω contained between S_1 and S_0 consists of two parts, namely, G_1' due to the part of the strip outside θ, and G_2' due to the part of the strip inside θ. On account of the convergence of the integral representing force due to a surface distribution, it is possible to choose the dimensions of θ so small that $|G_2'|$ is less than any selected small quantity $\frac{1}{2}\omega$. And when θ is chosen and fixed, P is definitely outside the distribution that gives rise to G_1', so clearly G_1' tends to zero as the distribution that gives rise to it tends to vanishing. Accordingly we can choose ϵ so small that $|G_1'| < \frac{1}{2}\omega$. Hence we have been able to choose ϵ so small that $G_1' + G_2' = G' < \omega$. Hence, apparently,

$$\operatorname*{Lim}_{\epsilon \to 0} G' = 0.$$

9. This reasoning, however, is not sound, and the result obtained is false. Semi-convergence is equivalent to convergence associated with a vanishing cavity about P of a definite shape, and implies that the limit of the attraction component (defined by means of such a vanishing cavity) of a portion of the distribution enclosed by a curve θ of the same shape about P is zero, as the linear dimensions of θ tend to zero. Now when the dimensions of θ have been selected and fixed in the above argument, the subsequent selection of ϵ may give us a strip (as in the diagram of § 5) which is narrower than θ, so that the whole of θ is not occupied by matter. Thus the actual matter within θ is bounded, not by θ, but by a curve of different shape, and the reasons for regarding the corresponding attraction at P as less than $\frac{1}{2}\omega$ cease to be applicable.

If we had absolute convergence instead of semi-convergence it would be quite another matter. For then we should not be tied to any particular shape of cavity or of boundary, and the fact of the boundary's becoming something else instead of θ would not invalidate the assumed inequality. We should, in fact, be entitled then to say that the attraction at P due to any area surrounding P and lying entirely within θ is less than $\frac{1}{2}\omega$. Now the component of force normal to Ω is represented by an absolutely convergent integral, and so its limit is zero as the breadth of the strip tends to vanishing. But it is not so with the component in any other direction.

Consider, for example, a tangential component of force at P. For different shapes of vanishing cavity the values of this force are different. The difference between the values for two selected shapes of cavity is due to the area bounded internally by one cavity and externally by the other, or rather to the limit of this area as it vanishes. Hence we infer the important fact that the difference of the values of the force corresponding to different cavities is independent of the size and shape of the outer boundary of the surface distribution. It is simply a function of the shapes of the two cavities, proportional of course to the surface-density at P. Now since these values of a tangential component corresponding to different cavities have definite differences independent of the outer boundary of the distribution, their limits, as the outer boundary tends to any limiting form, must have the same definite differences; one may vanish, but certainly not all. [In the case of a plane surface of uniform density the one which vanishes would correspond to a cavity similar and similarly situated to the limiting form of the outer boundary, with P for centre of similitude.] Thus the result apparently obtained in § 8 could not be true.

10. At this stage it is natural to note the probability that the exclusion of infinitely great differences between the values of the force for different cavities implies a restriction on the nature of the contemplated cavities and their mode of vanishing. And we therefore consider for a moment the question of what kinds of passage to limit make the semi-convergent force-integrals converge to definite values.

An answer, though not a complete one, may be founded upon a well known theorem already made use of without explicit quotation at the end of § 9. The theorem is that an annular plane lamina of uniform surface-density, whose inner and outer boundaries are similar and similarly situated curves, exerts no tangential attraction at the centre of similitude. Applying this result to the part of a uniform disc contained between two

successive positions of a contour which is diminishing in size without change of shape round a fixed centre of similitude P, we find that the shrinkage of the contour makes no difference in the value of the tangential force at P due to the part of the disc outside the contour; accordingly the limit of this force is definite for vanishing of the cavity. If we have to do with a surface-density which is not uniform, residing in a curved surface, the errors involved in neglecting the variability of the surface-density and the curvature of the surface can be rendered as small as we please by taking a sufficiently small contour, provided the curvature of the surface is definite and the surface density continuous at P. Hence the reasoning for a plane uniform surface can be rendered applicable to the more general case, and we arrive at the following result :—The semi-convergent integral representing tangential force at any point P of a surface distribution is rendered convergent by selecting a cavity of definite shape, and diminishing to zero, without change of shape, the scale of the geometrical configuration consisting of the cavity and the point P.

11. A wider range of cases of convergence might be obtained by study of the form of the integral representing the component of attraction in the direction of the axis of x, at an origin situated in a plane disc of uniform surface-density σ, occupying part of the plane $z = 0$. The integral is, in the notation of polar coordinates,

$$\iint \gamma\sigma r^{-1} \cos\theta \, dr \, d\theta,$$

and if the external boundary and the boundary of the cavity are respectively $r = F(\theta)$ and $r = f(\theta)$, this reduces to

$$r\sigma \int_0^{2\pi} \{\log F(\theta) - \log f(\theta)\} \cos\theta \, d\theta.$$

The danger of an infinity here arises out of the tendency of $f(\theta)$ to zero, as the inner boundary closes in round the origin. But if the form of $f(\theta)$ is such that we can write it $\eta\phi(\theta)$, where η is independent of θ and tends to zero with the vanishing of the cavity, while $\phi(\theta)$ is neither zero nor infinite for any value of θ, we may put

$$\log f(\theta) = \log \eta + \log \phi(\theta),$$

and the only term threatening an infinity is now

$$\gamma\sigma \log \eta \int_0^{2\pi} \cos\theta \, d\theta,$$

which vanishes for all values of η different from zero, and therefore has the limit zero for $\eta \to 0$.

If $\phi(\theta)$ is independent of η we have the case discussed otherwise in § 10. If $\phi(\theta)$ involves η, but in such a way that its order of magnitude is not determined by that of η, we have a type of convergence more general than that of § 10, but not of a seriously different character.

We note here that the disappearance of the term which threatened infinity is bound up with one special restriction on the cavity contemplated, namely that the contour of the cavity, *i.e.*, the nearer edge of the matter, must completely surround the point at which the attraction is estimated, otherwise the integrals of $\cos\theta$ and $\sin\theta$ would not vanish. This explains the fact that at a point on the edge of a disc the attraction is infinite.

Another feature of the cavities now under discussion is that the distances from the origin of the various points of the edge of the cavity all become small of the same order as the cavity closes in.

12. The class of cavity just described, (which it will be convenient to call class α,) does not, of course, represent the only mode of making the force integrals converge to definite values. It is to be expected that there are other classes of vanishing cavity capable of producing convergence, and of these an important example is that which comprises such as are symmetrical about two axes at right angles through the point P. For the contours of such cavities, both

$$\int \log f(\theta) \cos\theta \, d\theta \quad \text{and} \quad \int \log f(\theta) \sin\theta \, d\theta$$

vanish, so that both components of tangential force are definite. In this case the value of each force-component is independent of the shape of the cavity, so that the vanishing of the cavity need not be effected by keeping the shape unaltered and gradually diminishing the scale ; for it is permissible, while the linear dimensions are diminished, to keep changing the shape, so long as the symmetry conditions are never infringed.

For example, the cavity might be a rectangular slit whose length, though tending to zero, tends to be infinitely great in comparison with the breadth ; here the distances of the various points of the edge of the cavity do not all become small of the same order.

The class of symmetrical cavities we may call class β.

13. In the present application, in order to be able to employ the foregoing principles of semi-convergence, we shall suppose that the surface density in Ω is a continuous function at the points where Ω meets S, which includes the supposition that the material distribution in Ω does not terminate on the surface S, but passes through it. We postulate, further, that the force shall always be reckoned for a cavity of the class leading to finite values. And now we see that, since the differences of the values of the force for different cavities of this class are definite and independent of ϵ, one value of F' may tend to zero as $\epsilon \to 0$, but certainly not all; but, if one has a definite limit for $\epsilon \to 0$, then also all the others have definite limits.

14. Now let us return to the argument in § 8 by which the attempt was made to prove that a force component G', specified by a cavity θ of assigned shape, tends to zero with ϵ. The reasoning broke down because the boundary of the portion whose attraction was denoted by G_2' ceased to be the curve θ. In the construction by which it was proposed to make $|G'|$ less than an arbitrarily assigned small quantity ω, the dimensions of θ were first selected, and *afterwards* a value of ϵ was chosen such that for it and all smaller values $|G_1'| < \frac{1}{2}\omega$. This order of choice of the dimensions of θ and ϵ implies that the effective contour of the portion whose attraction is G_2' is a quadrilateral on the surface Ω bounded by parallel curves on the surfaces on S_1 and S_0 and by two opposite portions of the curve θ; and the breadth of the quadrilateral between the former curves may be taken as tending to infinite smallness in comparison with the length from one to the other of the opposite portions of the curve θ.

The limit form of this quadrilateral contour is simply the kind of rectangle quoted above as an example of a cavity of the class β. True it is not plane, not rectilineal, and not right-angled. But, given simple analytical circumstances, small errors may always be considered separately, and therefore when the curves and surfaces concerned are free from geometrical singularity the errors due to neglect of their curvatures may always be rendered as small as we please by making ϵ and the dimensions of θ sufficiently small. And when the quadrilateral may be as narrow as we please in comparison with its length, the contributions of its short and relatively distant ends to the contour integral may be made as small (relatively to the whole) as we please. Thus the possible obliquity of the ends is unimportant, and the contour may be regarded as tending ultimately to the narrow rectangular form.

Thus the reasoning which failed to prove that G' tends to zero with ϵ, when calculated for a cavity of the shape θ, does actually prove that G'

tends to zero with ϵ when calculated for a particular cavity of the class β, and therefore also for all cavities of the class β.

Now some cavities of the class β belong also to the class α; therefore G' tends to zero with ϵ when calculated for some cavities of the class α. Therefore G' tends to a definite limit, as $\epsilon \to 0$, when calculated for any selected cavity of the class α.

15. This is just the result which we require in order to demonstrate the vanishing of $\lim_{\epsilon \to 0} \int N' dS$, when there is present a surface distribution of finite surface-density. For N' tends to zero at all points of S except those on a certain curve, the intersection of S and Ω; and non-zero values, provided they are definite, confined merely to a curve on the surface of integration, do not contribute to the value of the integral. Now we have shewn that, at points on the curve, N' calculated for any cavity of the class already discussed has a finite limit for $\epsilon \to 0$. Hence

$$\lim_{\epsilon \to 0} \int N' dS = 0.$$

16. [*Added June 24th*, 1909.]—The case in which the angle χ of intersection of the surfaces S and Ω vanishes at a point P is not covered by the preceding reasoning, and it is desirable to give a brief outline of the manner in which it may be discussed. In this case the surfaces Ω and S touch at the point P, and it is desired to prove that at P

$$\lim_{\epsilon \to 0} N' = 0.$$

The portion of Ω contained between S_1 and S_0 is not now a long narrow strip, and we must examine the shape of that part of its boundary which is nearest to P. Let us suppose, (noting that the supposition excludes geometrical singularities in Ω and S,) that with P as origin and suitably chosen axes the part of S near P is approximately

$$2z = ax^2 + 2hxy + by^2,$$

and the equation to the part of Ω near P is approximately

$$2z = a'x^2 + 2h'xy + b'y^2;$$

then the equations of the parts of S_1 and S_0 near to P are (to a sufficient approximation for the present purpose)

$$2z = \pm 2\epsilon + ax^2 + 2hxy + by^2.$$

Hence the projections on the tangent plane at P of the parts of the curves of intersection of Ω with S_1 and S_0 which are near to P are given approximately by the equations

$$\pm 2\epsilon = (a'-a)x^2+2(h'-h)xy+(b'-b)y^2,$$

and are, in fact, curves analogous to the indicatrix of Ω and no less definite in character. These may be called "quasi-indicatrices."

If the surface Ω does not cross the surface S at P, *i.e.*, if

$$(a'-a)(b'-b) > (h'-h)^2,$$

one of the quasi-indicatrices is imaginary, the other a real ellipse with P as centre. This ellipse is an approximation to the boundary of the part of Ω contained between S_1 and S_0, and tends to the same limit form closing in round P for $\epsilon \to 0$. The convergence of N (absolute in this case) is a guarantee that the limit, for $\epsilon \to 0$, of the corresponding N' is zero.

If $(a'-a)(b'-b) < (h'-h)^2$, so that the surfaces S and Ω cross one another at P, the two quasi-indicatrices are conjugate hyperbolas with P for centre, their vertices tending to coincidence with P as $\epsilon \to 0$. These hyperbolas are an approximation to the part of the boundary near P of the part of Ω contained between S_1 and S_0. Now describe round P, in the surface Ω, a curve θ which (since N' is absolutely convergent at P) may be taken to be a circle. The diagram would be an oblique curvilinear cross inside a circle. N' may be split up into N_1' due to the part of Ω between S_1 and S_0 but outside θ, and N_2' due to the part of Ω between S_1 and S_0 and inside θ, namely the cross-shaped area. Having selected any small quantity ω we can, on account of the absolute convergence of N', choose θ so small that $N_2' < \frac{1}{2}\omega$; and then, since when θ is once chosen P is definitely outside the distribution that gives rise to N_1', we can choose ϵ so small that $N_1' < \frac{1}{2}\omega$; hence we have been able to choose ϵ so small that $N' < \omega$. Thus $\operatorname{Lim}_{\epsilon \to 0} N' = 0$.

Therefore the contact of Ω and S at P does not disturb the value of

$$\operatorname{Lim}_{\epsilon \to 0} \int N' \, dS.$$

17. Thus, both for distributions of finite volume-density and for those which include finite surface-density, the validity of the Gauss theorem has been established.

ON THE BEHAVIOUR AT THE POLES OF A SERIES OF LEGENDRE'S FUNCTIONS REPRESENTING A FUNCTION WITH INFINITE DISCONTINUITIES

By F. J. W. WHIPPLE.

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1. In his paper "On the Representation of a Function by a Series of Legendre's Functions" (*Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 24), Dr. Hobson has shewn that, if a function $f(x)$ is of the form

$$A(x-\xi)^{-k}+\phi(x),$$

in the neighbourhood of a point ξ in the interior of $(-1, 1)$ and $k \geqslant \frac{1}{2}$. Then the series corresponding to $f(x)$ does not converge to $f(1-0)$, $f(-1+0)$ at the points 1 and -1.

The question, what is the nature of the sum of the series at those points, is of special interest in the case $k = \frac{1}{2}$, as infinite discontinuities of this type occur in the electrical problem of the distribution of electricity near the sharp edge of a conductor and in the corresponding problems in hydrodynamics and elasticity.

2. Let us write $x = \cos\theta$, $\xi = \cos\gamma$; then the simplest form of the function under discussion may be written

$$\left.\begin{aligned} f(x) &= \frac{C}{\sqrt{(\gamma-\theta)}}+\phi(x), \quad \theta<\gamma \\ f(x) &= \quad 0 \quad +\phi(x), \quad \gamma<\theta<\pi \end{aligned}\right\}, \tag{1}$$

where $\phi(x)$ is a function with limited total fluctuation between $x = \pm 1$.

The series to be investigated is

$$\Sigma \frac{2n+1}{2} P_n(1) \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{2}$$

Dr. Hobson's analysis (*loc. cit.*, § 4) shews that the part of the sum of

n terms which depends on the value of $f(x)$ near $\theta = \gamma$, is

$$\sqrt{\frac{2n}{\pi}} \int_{\gamma-\eta_1}^{\gamma+\eta_2} \sin\{(n+1)\theta - \tfrac{1}{4}\pi\} \frac{f(\cos\theta)}{\sqrt{(2\tan\frac{1}{2}\theta)}}\, d\theta + \tau_1(n), \tag{3}$$

where τ with any suffix is used for a function which tends to zero as n tends to infinity. In the special case the expression (3) reduces to

$$\sqrt{\frac{2n}{\pi}} \int_{\gamma-\eta_1}^{\gamma} \frac{1}{\sqrt{(2\tan\frac{1}{2}\theta)}} \sin\{(n+1)\,\theta - \tfrac{1}{4}\pi\} \frac{C}{\sqrt{(\gamma-\theta)}}\, d\theta + \tau_2(n), \tag{4}$$

or, if

$$(n+1)(\gamma-\theta) = v,$$

to

$$\sqrt{\frac{1}{\pi\tan\frac{1}{2}\gamma}}\, C \int_0^{(n+1)\eta_1} \sin\{(n+1)\,\gamma - \tfrac{1}{4}\pi - v\} \frac{dv}{\sqrt{v}} + \tau_3(n), \tag{5}$$

or

$$\sqrt{\frac{1}{\pi\tan\frac{1}{2}\gamma}}\, C \left[\sin\{(n+1)\gamma - \tfrac{1}{4}\pi\} \int_0^{\infty} \cos v \frac{dv}{\sqrt{v}} - \cos\{(n+1)\,\gamma - \tfrac{1}{4}\pi\} \int_0^{\infty} \sin v \frac{dv}{\sqrt{v}}\right] + \tau_4(n), \tag{6}$$

or

$$-\frac{C}{\sqrt{\tan\frac{1}{2}\gamma}} \cos(n+1)\,\gamma + \tau_4(n), \tag{7}$$

since

$$\int_0^{\infty} \cos v \frac{dv}{\sqrt{v}} = \int_0^{\infty} \sin v \frac{dv}{\sqrt{v}} = \sqrt{(\tfrac{1}{2}\pi)}.$$

3. Hence it appears that at the pole $\theta = 0$, the series which represents $f(\cos\theta)$ is not convergent, but oscillating. The sum of n terms differs by a term which vanishes when n is great from

$$f(1) - \frac{C}{\sqrt{\tan\frac{1}{2}\gamma}} \cos(n+1)\,\gamma. \tag{8}$$

If the discontinuity is on the far side of $\theta = \gamma$, *i.e.*, if

$$\left.\begin{aligned} f(x) &= \frac{C}{\sqrt{(\theta-\gamma)}} + \phi(x) \quad (\pi > \theta > \gamma) \\ f(x) &= \quad 0 \quad + \phi(x) \quad (\gamma > \theta > 0) \end{aligned}\right\}, \tag{9}$$

similar analysis shews that at the point $\theta = 0$, the series takes the form

$$f(1) + \frac{C}{\sqrt{\tan\frac{1}{2}\gamma}} \sin(n+1)\,\gamma. \tag{10}$$

4. By writing $\pi-\theta$ instead of θ we can find the corresponding results for the other pole.

If the condition (1) holds, then at the pole $\theta = \pi$ the series leads to

$$f(-1)+\frac{C}{\sqrt{\cot\frac{1}{2}\gamma}}\sin(n+1)(\pi-\gamma), \tag{11}$$

and on the other hand with condition (9), we get

$$f(-1)-\frac{C}{\sqrt{\cot\frac{1}{2}\gamma}}\cos(n+1)(\pi-\gamma). \tag{12}$$

5. It will be noticed in each case that the mean value of the sum of the series is $f(\pm 1)$.

The effects of a finite number of discontinuities of the proposed form can evidently be added, so that if there are k discontinuities, and if the function $f(x)$ is of the type

$$\left.\begin{aligned} f(x) &= \frac{C_p}{\sqrt{(\theta-\gamma_p)}}+\phi(x) \quad [\tfrac{1}{2}(\gamma_{p+1}+\gamma_p)>\theta>\gamma_p] \\ f(x) &= \frac{D_p}{\sqrt{(\gamma_p-\theta)}}+\phi(x) \quad [\gamma_p>\theta>\tfrac{1}{2}(\gamma_p+\gamma_{p-1})] \end{aligned}\right\}, \tag{13}$$

and if $f(x)$ has a Lebesgue integral throughout the range excluding the points of discontinuity, then the series will have for its sum, when $\theta = 0$,

$$f(1)+\Sigma\,\frac{C_p\sin(n+1)\gamma_p-D_p\cos(n+1)\gamma_p}{\sqrt{\tan\frac{1}{2}\gamma_p}}. \tag{14}$$

It follows that the sum of the series according to Cesàro's definition is $f(1)$.

6. *Electrified Bowl.*— These results can be illustrated by the consideration of the distribution of electricity on the surface of an insulated bowl. This problem was solved in finite terms by Lord Kelvin in 1846.* If the potential of the bowl is unity, then the potential at any other

* W. Thomson (Lord Kelvin) published his results in 1846. The details of his analysis were published in 1869 (*Papers on Electricity and Magnetism*, p. 178). He found the density at any point, but not the potential.

[*Added February 5th*, 1910.]—The solution was completed by N. M. Ferrers (*Quarterly Journal*, Vol. XVII, p. 97). Ferrers obtained our equations in the order 16, 22, 19, 15. The direct method of finding 15 is to notice that each term is a solution of Laplace's equation, the first being the potential of a circular disc filling the aperture of the bowl, and the second term the potential of the inverse of that disc under the influence of a charge at the centre of inversion.

point is given by

$$V = \frac{1}{\pi} \text{ arc sin } \frac{2c}{r_1+r_2} + \frac{1}{\pi} \frac{a}{r} \text{ arc sin } \left(\frac{r}{a} \frac{2c}{r_1+r_2} \right), \tag{15}$$

where $2c$ is the diameter of the aperture, r_1 and r_2 are distances from the nearest and furthest points of the aperture, r is the distance from the centre of the sphere of which the bowl is a segment, and a is the radius of that sphere. The inverse sines are subject to the convention that the former is greater than $\frac{1}{2}\pi$ for points within the bowl, whilst the latter is greater than $\frac{1}{2}\pi$ for points between the bowl and the inverse of the aperture with respect to the centre of the sphere.

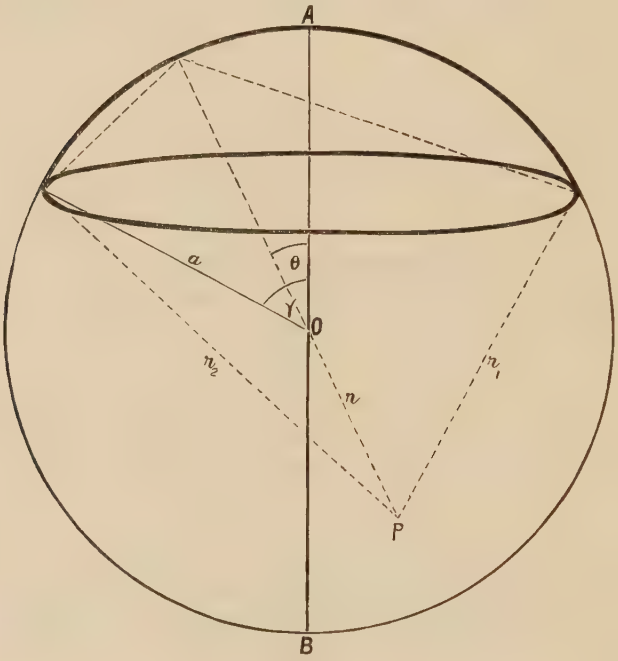

Fig. 1.

The sum of the internal and external densities is ρ, where

$$4\pi^2 a\rho = \pi - 2 \text{ arc sin } \frac{2c}{r_1+r_2} + 2\sqrt{\left(\frac{(2c)^2}{(r_1+r_2)^2-4c^2} \right)},$$

or, if θ is the angular distance from the crown of the bowl, and γ is the

angular radius of the bowl,

$$4\pi^2 a\rho = 2 \arccos\left(\frac{\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\theta}\right) + 2\frac{\cos\frac{1}{2}\gamma}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\gamma)}}. \tag{16}$$

The density at the crown of the bowl is given by

$$4\pi^2 a\rho = \gamma + 2\cot\tfrac{1}{2}\gamma. \tag{17}$$

The density at points near the edge of the bowl is very great: at a point whose angular distance from the rim is ξ,

$$4\pi^2 a\rho = \frac{2}{\sqrt{(\xi\tan\frac{1}{2}\gamma)}}. \tag{18}$$

It will be noticed that ρ is a function of the type suggested in § 2. Of course, the density is zero at all points for which $\theta > \gamma$.

7. *Electrified Bowl. Solution in Spherical Harmonics.*—The simplest way to transform the expression for the potential into a series of zonal harmonics is to deal with points on the axis. For a point on the axis at a distance z from the centre, the formula (15) may be written

$$V = \frac{1}{\pi}\arctan\frac{a\sin\gamma}{a\cos\gamma - z} + \frac{1}{\pi}\frac{a}{z}\arctan\frac{z\sin\gamma}{a - z\cos\gamma}$$

$$= \frac{1}{\pi}\left[\gamma + \frac{z}{a}\sin\gamma + \frac{z^2}{a^2}\frac{\sin 2\gamma}{2} + \frac{z^3}{a^3}\frac{\sin 3\gamma}{3} + \ldots\right.$$

$$\left. + \sin\gamma + \frac{z}{a}\frac{\sin 2\gamma}{2} + \frac{z^2}{a^2}\frac{\sin 3\gamma}{3} + \frac{z^3}{a^3}\frac{\sin 4\gamma}{4} + \ldots\right]. \tag{19}$$

Hence, since V_i is solenoidal at all internal points, we can write for the internal potential

$$V_i = \frac{1}{\pi}\left[(\gamma + \sin\gamma) + \sum_{n=1}^{\infty} P_n(\cos\theta)\frac{r^n}{a^n}\left(\frac{\sin n\gamma}{n} + \frac{\sin(n+1)\gamma}{n+1}\right)\right]. \tag{20}$$

The potential is continuous at the surface of the sphere, so the external potential satisfies the equation

$$V_0 = \frac{1}{\pi}\left[(\gamma + \sin\gamma)\frac{a}{r} + \sum_{n=1}^{\infty} P_n(\cos\theta)\frac{a^{n+1}}{r^{n+1}}\left(\frac{\sin n\gamma}{n} + \frac{\sin(n+1)\gamma}{n+1}\right)\right]. \tag{21}$$

From these formulæ we see that the density ρ will, if the series is con-

vergent, satisfy the equation

$$4\pi^2 a\rho = (\gamma+\sin\gamma)+\Sigma(2n+1)P_n(\cos\theta)\left(\frac{\sin n\gamma}{n}+\frac{\sin(n+1)\gamma}{n+1}\right). \quad (22)$$

We shall verify that this series has a discontinuity of the form (1) when $\theta=\gamma$, and shall discuss its behaviour at $\theta=0$ and $\theta=\pi$.

8. Using $\sigma(n,\epsilon)$ with any suffix to denote a function of n which for all values of ϵ remains finite when n is increased indefinitely, we see that (22) is equivalent, when $\theta=\gamma-\epsilon$, to

$$4\pi^2 a\rho = 2\Sigma[\sin n\gamma+\sin(n+1)\gamma]P_n[\cos(\gamma-\epsilon)]+\sigma_1(n,\epsilon).$$

The known approximation to the value of P_n reduces this series to

$$4\pi^2 a\rho = 4\Sigma\sin(n+\tfrac{1}{2})\gamma\cos\tfrac{1}{2}\gamma\sqrt{\left(\frac{2}{n\pi\sin\gamma}\right)}\cos\{(n+\tfrac{1}{2})(\gamma-\epsilon)-\tfrac{1}{4}\pi\}+\sigma_2(n,\epsilon)$$

$$=\frac{2}{\sqrt{(\pi\tan\frac{1}{2}\gamma)}}\left[\Sigma\frac{\sin\{(2n+1)(\gamma-\tfrac{1}{2}\epsilon)-\tfrac{1}{4}\pi\}}{\sqrt{n}}+\Sigma\frac{\sin\{(n+\frac{1}{2})\epsilon+\frac{1}{4}\pi\}}{\sqrt{n}}\right]$$
$$+\sigma_2(n,\epsilon). \quad (23)$$

The first Σ in (23) represents a series which is finite when ϵ is small. In the second Σ we note that $\sum_1^\infty \frac{\sin(n+\frac{1}{2})\epsilon}{\sqrt{n}}$ differs by a finite quantity from

$$\frac{1}{\sqrt{\epsilon}}\int_0^\infty\frac{\sin v\,dv}{\sqrt{v}} \quad \text{or} \quad \sqrt{\left(\frac{\pi}{2\epsilon}\right)},$$

so that $\sum_1^n \frac{\sin\{(n+\frac{1}{2})\epsilon+\frac{1}{4}\pi\}}{\sqrt{n}} = \sqrt{\left(\frac{\pi}{2\epsilon}\right)}(\cos\tfrac{1}{4}\pi+\sin\tfrac{1}{4}\pi)+\sigma_3(n,\epsilon)$

$$=\sqrt{\left(\frac{\pi}{\epsilon}\right)}+\sigma_3(n,\epsilon).$$

Thus (22) has been reduced to

$$4\pi^2 a\rho = \frac{2}{\sqrt{(\epsilon\tan\frac{1}{2}\gamma)}}+\text{a term which is finite when } \epsilon \text{ is small.}$$

For points just off the rim $\theta=\gamma+\epsilon$, and the analysis proceeds as before, but since

$$\sum_1^\infty\frac{\sin\{-(n+\frac{1}{2})\epsilon+\frac{1}{4}\pi\}}{\sqrt{n}} \doteqdot \sqrt{(\tfrac{1}{2}\pi)}(-\cos\tfrac{1}{4}\pi+\sin\tfrac{1}{4}\pi),$$

$4\pi^2 a\rho$ is finite or as the problem indicates zero.

The value of $4\pi^2 a\rho$ is accordingly of the type proposed in (1). The diagram (Fig. 2) shews the nature of the discontinuity of

$$y = \int_0^\infty \frac{\sin(\lambda\epsilon + \frac{1}{4}\pi)}{\sqrt{\lambda}}\, d\lambda.$$

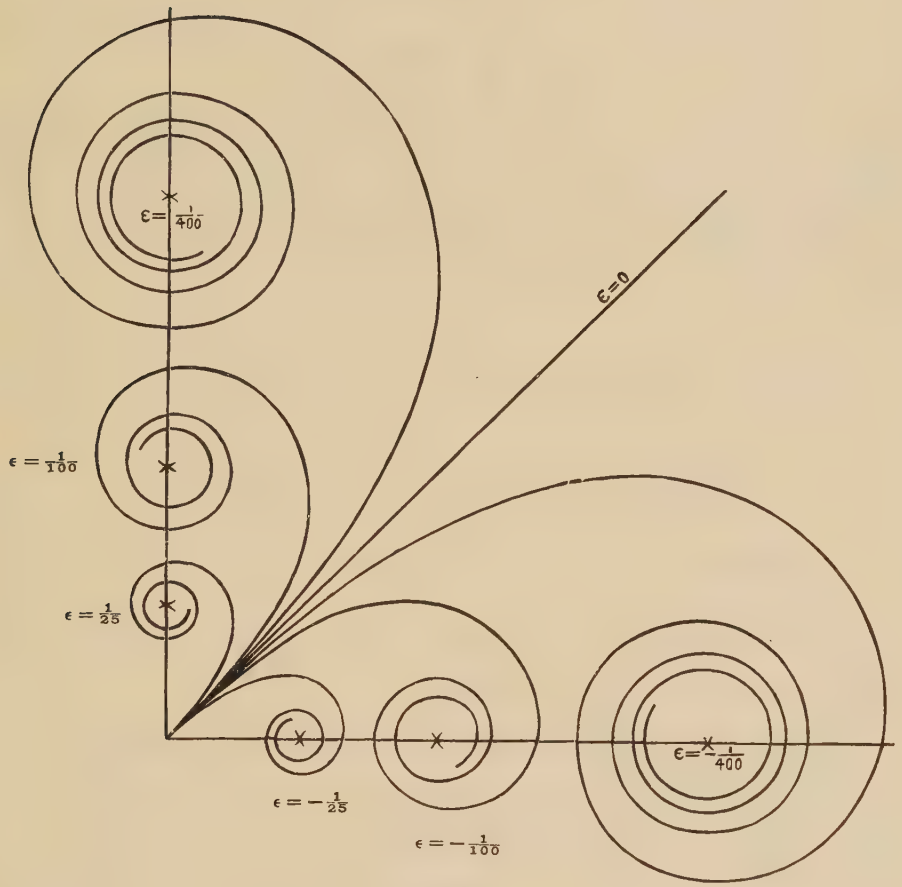

FIG. 2.

Each of the spirals is made by taking elements of length proportional to $\lambda^{-\frac{1}{2}}$ inclined at such angles to the axis of x that the ordinate of the focus of the spiral may be y.

It will be seen that when ϵ is very small and positive y is very great; when ϵ is negative the spiral turns in the other sense and y vanishes. The discontinuity of the series (22) has exactly the same character.

9. Now, when $\theta = 0$, the sum of $(n+1)$ terms of the series (22) for $4\pi^2 a\rho$ is

$$\gamma + \sin\gamma + \sum_1^n (2n+1)\left(\frac{\sin n\gamma}{n} + \frac{\sin(n+1)\gamma}{n+1}\right)$$

$$= \gamma + 4\sum_1^n \sin n\gamma + \frac{2n+1}{n+1}\sin(n+1)\gamma$$

$$= \gamma + 2\cot\tfrac{1}{2}\gamma - 2\cot\tfrac{1}{2}\gamma\cos(n+1)\gamma - \frac{\sin(n+1)\gamma}{n+1}. \qquad (24)$$

The mean value of this expression is $\gamma + 2\cot\frac{1}{2}\gamma$, which agrees with the value of the density at the crown of the bowl found in (17), and the oscillating term is $-C\sqrt{(\cot\frac{1}{2}\gamma)}\cos(n+1)\gamma$, where C has the value $2\sqrt{(\cot\frac{1}{2}\gamma)}$ found in the last paragraph. This result is in agreement with the formula (8).

10. Again, when $\theta = \pi$, the sum of $(n+1)$ terms of the series (22), for $4\pi^2 a\rho$, is

$$\gamma + \sin\gamma + \sum_1^n (-)^n (2n+1)\left(\frac{\sin n\gamma}{n} + \frac{\sin(n+1)\gamma}{n+1}\right)$$

$$= \gamma - 2\sum_1^{n+1} (-)^{r-1}\frac{\sin r\gamma}{r} + 2(-)^n \sin(n+1)\gamma$$

$$= 2\sin(n+1)(\pi-\gamma) + \tau(n), \qquad (25)$$

where $\tau(n)$ tends to zero as n increases.

The mean value of this expression is zero, as it should be, since there is no density off the bowl.

The oscillating term is $C\sqrt{(\tan\frac{1}{2}\gamma)}\sin(n+1)(\pi-\gamma)$, where C has the value $2\sqrt{(\cot\frac{1}{2}\gamma)}$ of § 8, and this result is in accordance with the formula (11).

The complete agreement between the special case and the general theory is accordingly established.

Note I.—An alternative method of obtaining the results (20), (21), (22) may be based on the use of Mehler's integrals

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\gamma}{\sqrt{(\cos^2\frac{1}{2}\gamma - \cos^2\frac{1}{2}\theta)}}\,d\gamma = \frac{1}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\gamma}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\gamma)}}\,d\gamma.$$

In the following table the function defined between 0 and π in the first two columns is expanded in the third column in a series of Legendre's functions.

	$0 < \gamma < \theta < \pi$	$0 < \theta < \gamma < \pi$	
I	$\dfrac{1}{\sqrt{(\cos^2\frac{1}{2}\gamma - \cos^2\frac{1}{2}\theta)}}$	0	$2\sum_0^\infty P_n(\cos\theta)\cos(n+\frac{1}{2})\gamma$
II	0	$\dfrac{1}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\gamma)}}$	$2\sum_0^\infty P_n(\cos\theta)\sin(n+\frac{1}{2})\gamma$
III	$\dfrac{\cos\frac{1}{2}\gamma}{\sqrt{(\cos^2\frac{1}{2}\gamma - \cos^2\frac{1}{2}\theta)}}$	0	$\sum_0^\infty P_n(\cos\theta)[\cos n\gamma + \cos(n+1)\gamma]$
IV	0	$\dfrac{\cos\frac{1}{2}\gamma}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\gamma)}}$	$\sum_0^\infty P_n(\cos\theta)[\sin n\gamma + \sin(n+1)\gamma]$
V	$2\arcsin\left(\dfrac{\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\theta}\right)$	π	$(\gamma+\sin\gamma)+\sum_1^\infty P_n(\cos\theta)\left(\dfrac{\sin n\gamma}{n}+\dfrac{\sin(n+1)\gamma}{n+1}\right)$
VI	0	$\dfrac{\sin\frac{1}{2}\gamma}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\gamma)}}$	$\sum_0^\infty P_n(\cos\theta)[\cos n\gamma - \cos(n+1)\gamma]$
VII	0	$2\arccos\left(\dfrac{\cos\frac{1}{2}\gamma}{\cot\frac{1}{2}\theta}\right)$	$(\gamma-\sin\gamma)+\sum_1^\infty P_n(\cos\theta)\left(\dfrac{\sin n\gamma}{n}-\dfrac{\sin(n+1)\gamma}{n+1}\right)$
VIII	0	$\left\{\begin{array}{l} 2\arccos\left(\dfrac{\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\theta}\right) \\ +2\cos\frac{1}{2}\gamma/\sqrt{(\cos^2\frac{1}{2}\theta-\cos^2\frac{1}{2}\gamma)}\end{array}\right.$	$(\gamma+\sin\gamma)+\sum_1^\infty (2n+1)P_n(\cos\theta)\left(\dfrac{\sin n\gamma}{n}+\dfrac{\sin(n+1)\gamma}{n+1}\right)$
IX	$\mathrm{dn}^{-1}(\cos\frac{1}{2}\gamma)\bmod(\sin\frac{1}{2}\theta)$	K modulus $(\sin\frac{1}{2}\theta)$	$2\Sigma P_n(\cos\theta)\dfrac{\sin(n+\frac{1}{2})\gamma}{2n+1}$
X	K modulus $(\cos\frac{1}{2}\theta)$	$\mathrm{dn}^{-1}(\sin\frac{1}{2}\gamma)\bmod(\cos\frac{1}{2}\theta)$	$2\Sigma P_n(\cos\theta)\dfrac{\cos(n+\frac{1}{2})\gamma}{2n+1}$

Note II.—It is somewhat remarkable that the results of the last note may be used with θ and γ interchanged to give the solution of the two-dimensional problem analogous to that of the bowl. If r, θ are the polar coordinates in two dimensions, then the potential due to the charged insulated strip $r = a$, $\theta < \gamma$ is

$$V_0 = -\log\frac{r}{a} + \sum_1^\infty \frac{\cos n\theta}{n}\left(\frac{a}{r}\right)^n [P_n(\cos\gamma) + P_{n-1}(\cos\gamma)],$$

$$V_i = \sum_1^\infty \frac{\cos n\theta}{n}\left(\frac{r}{a}\right)^n [P_n(\cos\gamma) + P_{n-1}(\cos\gamma)].$$

When there are two strips both charged positively, one between $\theta = \pm\gamma$,

and the other between $\theta = \pi \pm \gamma$, then the potential is given by

$$V_0 = -\log\frac{r}{a} + \sum_1^\infty \frac{\cos 2n\theta}{2n}\left(\frac{a}{r}\right)^{2n}[P_n(\cos 2\gamma)+P_{n-1}(\cos 2\gamma)],$$

$$V_i = \sum_1^\infty \frac{\cos 2n\theta}{2n}\left(\frac{r}{a}\right)^{2n}[P_n(\cos 2\gamma)+P_{n-1}(\cos 2\gamma)].$$

When there are two strips with equal and opposite charges

$$V_0 = \sum_0^\infty \frac{\cos(2n+1)\theta}{2n+1}\left(\frac{a}{r}\right)^{2n+1} P_n(\cos 2\gamma),$$

$$V_i = \sum_0^\infty \frac{\cos(2n+1)\theta}{2n+1}\left(\frac{r}{a}\right)^{2n+1} P_n(\cos 2\gamma).$$

The elliptic functions of Note I (IX and X) arise naturally when this last problem is solved by transforming to the half plane and using Schwartz's method. The corresponding problem which arises when two caps on opposite poles of a sphere are charged does not seem to be soluble.

THE TRANSFORMATION OF THE ELECTRODYNAMICAL EQUATIONS

By H. BATEMAN.

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6. The Invariants of a Spherical Wave Transformation.
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1. *Introduction.*

The numerous experimental and theoretical investigations that have followed Bradley's discovery of aberration have led finally to the enunciation of a principle of relativity. This principle which is being widely used as a working hypothesis is gradually attaining a position analogous to that held by the second law of thermodynamics,* inasmuch as it gives a satisfactory account of the experimental results and is not in direct contradiction with any known facts.†

The mathematical analysis on which the principle is based was introduced by Lorentz in 1892. It consists of a certain space time transformation of the coordinates in which the fundamental equations of the electrodynamical field are unaltered in form.

The principal features of the transformation are the introduction of a local time, and the hypothesis that a body moving through the æther with a constant velocity v suffers a contraction $\sqrt{(1-v^2/c^2)} : 1$ in the direction of motion: c being the velocity of light. With the help of this hypothesis

* Cf. H. A. Bumstead, *Amer. Jour. of Science*, Vol. XXVI (1908).

† The principle has received additional support from some recent experiments made by Dr. Bucherer, *Phys. Zeitschr.*, Sept., 1908, p. 755.

a satisfactory explanation of the results of the Michelson-Morley experiment can be given on the supposition of a stationary æther. The results of the experiment seem at first sight to indicate that the æther is carried along with the earth, but on such a supposition a satisfactory explanation of aberration would still be wanting.

The theory of Lorentz's transformation has been developed very considerably by Einstein, Planck, and Minkowski. The transformation has become a powerful instrument of research, inasmuch as it provides a means of transition from the mathematical expressions of physical quantities connected with a system at rest to the corresponding quantities for a similar system in uniform motion. The transformation of Lorentz only enables us to pass from a system at rest to one in *uniform* motion, but it has been postulated that there is a more general transformation which can be applied to systems moving in a more general manner. The case of a system moving with a constant acceleration has, in fact, been discussed by Einstein.*

The object of the present paper is to find all the transformations for which the electrodynamical equations are invariant. In the case of the simpler equations of the theory of electrons, it is proved that the transformations belong to a certain group which is isomorphic with the group of conformal transformations of a space of four dimensions. It is assumed, however, that the transformation is such that the total charge on a system of particles is unaltered.

I have great pleasure in thanking Mr. E. Cunningham for the stimulus which he gave to this research by the discovery of the formulæ of transformation in the case of an inversion in the four-dimensional space. These formulæ suggested the more general formulæ in terms of Jacobians, and the transition to the integral forms led at once to the present analysis. The two integral forms of the second order had been introduced previously by Mr. Hargreaves in another connection ; they enable us to give a very concise expression of the electrodynamical equations which promises to be of considerable importance in future developments. The integrals that occur in the equations differ from the usual surface and volume integrals by the fact that the quantities in the integrand are calculated at different points of space at different times, these times being specified by an arbitrary known law.

The invariance of the two integral forms of the second order leads at once to the formulæ of transformation of the electric and magnetic force,

* *Jahrbuch der Radioaktivität* (1907).

and to a number of identical relations which are shown to imply that the differential equation

$$dx^2+dy^2+dz^2-dt^2=0$$

is invariant. This is the differential equation of the characteristics of the electromagnetic equations and gives the form of an elementary wave front. Since this is a sphere, we have called the group of transformation the group of spherical wave transformations. The differential equation also expresses the condition that two neighbouring particles should be in a position to act on one another. The transformations which leave the electrodynamical equations for ponderable bodies unaltered in form must depend upon the form of the constitutive relations connecting the magnetic induction and electric force with the magnetic force and electric displacement when the bodies are in motion. Minkowski, Einstein, and Laub have proceeded in the opposite way and constructed a set of constitutive relations for bodies in *uniform* motion, by assuming that the transformations are Lorentzian transformations and transforming the known constitutive relations for bodies at rest.

This method, however, does not give the constitutive relations for the case of a dielectric whose motion is not uniform. The general spherical wave transformation can be applied to obtain a certain type of accelerated motion, but the dimensions and shapes of bodies are continually altering in the transformed system.

An attempt has been made to discover whether there are any types of constitutive relations which are invariant for the general space time transformation. It is shown in § 7 that it is possible to construct such relations on the assumption that a certain quadratic form is invariant for the transformation. These relations may be supposed to correspond to a special type of configuration and state of motion which preserves its character after any space time transformation, which satisfies certain limitations. The general theory of space time transformations is discussed with the help of two theorems on the transformation of integral forms. It is shown that when a transformation of variables is performed, and the forms equivalent to two given integral forms are known, then the integral forms may be multiplied together by the rules of Grassmann's calculus of extension, and the resulting integral form is equal to the one obtained by multiplying the equivalent integral forms in the same way. Secondly, it is shown that when a quadratic form and its equivalent are known, a pair of equivalent integral forms may be obtained from a given pair by a process of analysis analogous to reciprocation, the new integral forms being of order $n-m$, where n denotes the number of variables and m the order of the original integral forms. These theorems are very

useful for finding invariants.* Various invariants are calculated with a view of obtaining the general form of the equations of motion. Abraham's expression of the reaction of radiation on a moving electron is shown to be an invariant for the group of spherical wave transformations, provided it is multiplied by a certain factor.

2. *The Integral Equations of the Theory of Electrons.*

Let the electromagnetic units be those adopted by Lorentz, and let the units of time and length be so chosen that the velocity of radiation is unity.

Let curl H denote the vector whose components are

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \ \ldots,$$

when the axes of x, y, z are right-handed and the vector whose components are

$$\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y}, \ \ldots,$$

when the axes are left-handed. Then the fundamental equations† of the theory of electrons may be written

$$\left.\begin{aligned} \text{curl } H &= \frac{\partial E}{\partial t} + \rho w \\ \text{curl } E &= -\frac{\partial H}{\partial t} \\ \text{div } E &= \rho \\ \text{div } H &= 0 \end{aligned}\right\}, \qquad \text{(I)}$$

where E is the electric force, H the magnetic force, ρ the volume density of electricity, and ρw the convection current.

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* They appear to be closely connected with two theorems used by Ricci and Levi Civita, "Méthodes de calcul différential absolu et leurs applications," *Math. Ann.*, Bd. LIV (1901).

† H. A. Lorentz, *Ency. d. math. Wiss.*, Vol. 13, No. 5; and Vol. 14, No. 2. *The Theory of Electrons*, Leipzig (1909), p. 12.

These equations may be replaced by the two integral equations*

$$\iint (H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt) = 0, \quad \text{(II)}$$

$$\iint (E_x dy\,dz + E_y dz\,dx + E_y dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt)$$
$$= -\iiint (\rho w_x dy\,dz\,dt + \rho w_y dz\,dx\,dt + \rho w_z dx\,dy\,dt - \rho\, dx\,dy\,dz), \quad \text{(III)}$$

provided the integrals receive suitable interpretations. The interpretation that first suggests itself is obtained by regarding (x, y, z, t) as the coordinates of a point in a space of four dimensions. Let any closed two-dimensional manifold S_2 in this space be assigned by equating x, y, z, t to one-valued differentiable functions of two parameters α, β, and let S_2 be the boundary of a three-dimensional manifold S_3 in which the coordinates are like functions of three parameters α, β, γ, of which $\gamma = 0$ on S_2, and $\gamma < 0$ on S_3. Then any term such as $\iint H_x dy\,dz$ may be interpreted to mean $\iint H_x \frac{\partial (y, z)}{\partial (\alpha, \beta)} d\alpha\, d\beta$ taken over S_2, and any term such as $\iiint \rho\, dx\,dy\,dz$ may be interpreted to mean $\iiint \rho \frac{\partial (x, y, z)}{\partial (\alpha, \beta, \gamma)} d\alpha\, d\beta\, d\gamma$ taken over S_3.

The relations (II) and (III) may now be obtained with the aid of (I) by applying the generalized Green-Stokes theorem as given by Baker,† Poincaré,‡ and others.

In order that equations (II) and (III) may be equivalent to (I), the axes must form a right-handed system. If we wish to use left-handed axes we must change the sign of H in (II) and (III).

We shall now endeavour to give a simpler interpretation to the integrals occurring in equations (II) and (III).

Let S be an arbitrary closed surface in the (x, y, z) space, and let t be expressed in terms of x, y, z by an arbitrary law $t = t(x, y, z)$, which must be chosen, however, in such a way that t is a single-valued function which is finite together with its derivatives with regard to x, y, z at all points within S and on S itself. Let the coordinates of points on S be expressed in terms of two parameters α, β.

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* The integral forms occurring in these equations have been studied by Hargreaves, *Camb. Phil. Trans.*, Vol. 21, p. 107 (1908).

† *Camb. Phil. Trans.*, Vol. 18 (1900), p. 408.

‡ *Acta Math.*, t. 9 (1887), p. 321.

A term such as $\iint E_x dx\,dt$ may now be understood to mean*

$$\iint E_x \left[\frac{\partial(x,t)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(\alpha,\beta)} + \frac{\partial(x,t)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(\alpha,\beta)} + \frac{\partial(x,t)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(\alpha,\beta)}\right] d\alpha\,d\beta,$$

and a term such as $\iiint \rho w_x dy\,dz\,dt$ to mean

$$\iiint \rho w_x \frac{\partial(y,z,t)}{\partial(x,y,z)}\,dx\,dy\,dz.$$

The equation (II) may now be written

$$\iint \left[\left(E_x - H_y\frac{\partial t}{\partial z} + H_z\frac{\partial t}{\partial y}\right) dy\,dz + \left(E_y - H_z\frac{\partial t}{\partial x} + H_x\frac{\partial t}{\partial z}\right) dz\,dx + \left(E_z - H_x\frac{\partial t}{\partial y} + H_y\frac{\partial t}{\partial x}\right) dx\,dy\right]$$
$$= -\iiint \left(\rho w_x\frac{\partial t}{\partial x} + \rho w_y\frac{\partial t}{\partial y} + \rho w_z\frac{\partial t}{\partial z} - \rho\right) dx\,dy\,dz.$$

Transforming the left-hand side by means of Green's theorem, we get

$$\iiint \left[\frac{d}{dx}\left(E_x - H_y\frac{\partial t}{\partial z} + H_z\frac{\partial t}{\partial y}\right) + \frac{d}{dy}\left(E_y - H_z\frac{\partial t}{\partial x} + H_x\frac{\partial t}{\partial z}\right) + \frac{d}{dz}\left(E_z - H_x\frac{\partial t}{\partial y} + H_y\frac{\partial t}{\partial x}\right)\right] dx\,dy\,dz.$$

Now

$$\frac{dE_x}{dx} = \frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial t}\frac{\partial t}{\partial x},$$

$$\frac{dH_y}{dx} = \frac{\partial H_y}{\partial x} + \frac{\partial H_y}{\partial t}\frac{\partial t}{\partial x};$$

hence the above integral may be written

$$\iiint \left[\left(\frac{\partial E_z}{\partial t} - \frac{\partial H_z}{\partial y} + \frac{\partial H_y}{\partial z}\right)\frac{\partial t}{\partial x} + \left(\frac{\partial E_y}{\partial t} - \frac{\partial H_x}{\partial z} + \frac{\partial H_z}{\partial x}\right)\frac{\partial t}{\partial y} + \left(\frac{\partial E_z}{\partial t} - \frac{\partial H_y}{\partial x} + \frac{\partial H_x}{\partial y}\right)\frac{\partial t}{\partial z} + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\right] dx\,dy\,dz,$$

* The relation $\frac{\partial(x,t)}{\partial(\alpha,\beta)} = \frac{\partial(x,t)}{\partial(y,z)}\frac{\partial(y,z)}{\partial(\alpha,\beta)} + \frac{\partial(x,t)}{\partial(z,x)}\frac{\partial(z,x)}{\partial(\alpha,\beta)} + \frac{\partial(x,t)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(\alpha,\beta)}$ is a particular case of a general relation given in Scott's *Determinants*, p. 134.

and this is equivalent to

$$-\iiint\left(\rho w_x\frac{\partial t}{\partial x}+\rho w_y\frac{\partial t}{\partial y}+\rho w_z\frac{\partial t}{\partial z}-\rho\right)dx\,dy\,dz,$$

if equations (I) are satisfied and the axes form a right-handed system. In the same way, it can be shown that the integral on the left-hand side of (II) is zero, if equations (I) are satisfied.

Conversely, if equations (II) and (III) are satisfied for every closed surface S and for every law $t = t(x, y, z)$, such that $\partial t/\partial x$, $\partial t/\partial y$, $\partial t/\partial z$ are finite and continuous within S and on its surface, then equations (I) are also satisfied, provided, of course, that the quantities $E_x, \ldots, H_x, \ldots$ possess derivatives which behave in such a way that an application of Green's theorem is permissible.

It seems natural to regard equations (II) and (III) as the fundamental equations of the theory of electrons, because they do not require the assumptions that the medium is continuous and that vectors E, H can be associated with *every* point in space;* all that is required is that the quantities $E_x, \ldots, \rho w_x, \ldots$ shall be *integrable.*

It is clear that in the general case the quantities occurring in these equations are evaluated at different points of space at *different times.* The integrals are thus more general than the usual surface and volume integrals, and seem to be better adapted for purposes of measurement, the difficulty of measuring quantities at different points of space at the same time being avoided.

Equation (III) may be regarded as the definition of the electric charge associated with a system of particles. The triple integral represents the total charge on the particles.† Each particle is supposed to be within

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* In the case of a single pulse travelling across a medium in which the vectors E and H are initially zero, these vectors may exist at a given point at one time and not at another.

† This may be proved by showing that the triple integral remains invariant during the motion of a system of electrons.

Let an electron which was at the point (x, y, z) at time t be in a new position (x', y', z') at time t', where

$$x' = x + \epsilon w_x, \qquad z' = z + \epsilon w_z,$$

$$y' = y + \epsilon w_y, \qquad t' = t + \epsilon,$$

and ϵ is a small quantity which is a function of x, y, z, t. The integral form is transformed into

$$\rho' w'_x\,dy'\,dz'\,dt' + \rho' w'_y\,dz'\,dx'\,dt' + \rho' w'_z\,dx'\,dy'\,dt' - \rho'\,dx'\,dy'\,dz',$$

where

$$\rho w_x = \rho' w'_x\frac{\partial(y', z', t')}{\partial(y, z, t)} + \rho' w'_y\frac{\partial(z', x', t')}{\partial(y, z, t)} + \rho' w'_z\frac{\partial(x', y', t)}{\partial(y, z, t)} - \rho'\frac{\partial(x', y', z')}{\partial(y, z, t)},$$

... ...

the closed surface S at the moment when its charge is evaluated, but since the charges on the particles are evaluated at different times, the particles need not all be within the closed surface at a given time. This explains why the total charge is not

$$\iiint \rho \, dx \, dy \, dz.$$

The double integral expresses the total charge as a surface integral, but here again the particles over which the integration extends are not all on the surface at the same time, but at different times.

In the same way equation (II) may be regarded as a general way of expressing the fact that there is no free magnetism.

If Φ is the scalar electromagnetic potential and (A_x, A_y, A_z) the components of the electromagnetic vector potential, we have

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t}, \qquad H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}.$$

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or
$$\rho' w_x' \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} = \rho w_x \frac{\partial x'}{\partial x} + \rho w_y \frac{\partial x'}{\partial y} + \rho w_z \frac{\partial x'}{\partial z} + \rho \frac{\partial x'}{\partial t}.$$

... ...

This gives
$$\rho' w_x' \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} = \rho w_x \left[1 + \epsilon \frac{\partial w_x}{\partial x} + w_x \frac{\partial \epsilon}{\partial x}\right] + \rho w_y \left[\epsilon \frac{\partial w_x}{\partial y} + w_x \frac{\partial \epsilon}{\partial y}\right]$$
$$+ \rho w_z \left[\epsilon \frac{\partial w_x}{\partial z} + w_x \frac{\partial \epsilon}{\partial z}\right] + \rho \left[\epsilon \frac{\partial w_x}{\partial t} + w_x \frac{\partial \epsilon}{\partial t}\right],$$

... ...

$$\rho' \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} = \rho w_x \frac{\partial \epsilon}{\partial x} + \rho w_y \frac{\partial \epsilon}{\partial y} + \rho w_z \frac{\partial \epsilon}{\partial z} + \rho \left[1 + \frac{\partial \epsilon}{\partial t}\right].$$

Multiplying the last equation by w_x, and subtracting from the first, we get

$$\rho' \frac{\partial (x', y', z', t')}{\partial (x, y, z, t)} (w_x' - w_x) = \rho \epsilon \left[\frac{\partial w_x}{\partial t} + w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} + w_z \frac{\partial w_x}{\partial z}\right].$$

Putting
$$\frac{d}{dt} = \frac{\partial}{\partial t} + w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + w_z \frac{\partial}{\partial z},$$

we see that the equation is satisfied if

$$\left(1 + \frac{d\epsilon}{dt}\right)(w_x' - w_x) = \epsilon \frac{dw_x}{dt},$$

and if $\frac{d\epsilon}{dt}$ is a small quantity of the first order

$$w_x' - w_x = \epsilon \frac{dw_x}{dt},$$

in other words (w_x', w_y', w_z') are the component velocities at time $t + \epsilon$.

Since the integral form is an invariant, we may calculate its value by considering the swarm of electrons at a given time t. The temporal terms then disappear, and the integral reduces simply to

$$-\rho \, dx \, dy \, dz,$$

i.e., the total negative charge on the system of electrons.

These equations are equivalent to the single integral equation

$$\int (A_x dx + A_y dy + A_z dz - \Phi dt)$$
$$= \iint (H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt), \quad \text{(IV)}$$

provided a suitable interpretation is given to the integrals.

Take any closed curve C and a surface Σ bounded by this curve. Let t be expressed in terms of (x, y, z) by an arbitrary known law; then the line integral may be understood to mean

$$\int \left[\left(A_x - \Phi \frac{\partial t}{\partial x} \right) dx + \left(A_y - \Phi \frac{\partial t}{\partial y} \right) dy + \left(A_z - \Phi \frac{\partial t}{\partial z} \right) dz \right],$$

and can be transformed into the surface integral by means of Stokes's theorem. Since

$$\frac{d}{dy}\left(A_z - \Phi \frac{\partial t}{\partial z} \right) - \frac{d}{dz}\left(A_y - \Phi \frac{\partial t}{\partial y} \right)$$
$$= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} + \left(\frac{\partial A_z}{\partial t} + \frac{\partial \Phi}{\partial z} \right) \frac{\partial t}{\partial y} - \left(\frac{\partial A_y}{\partial t} + \frac{\partial \Phi}{\partial y} \right) \frac{\partial t}{\partial z}$$
$$= H_x - E_z \frac{\partial t}{\partial y} + E_y \frac{\partial t}{\partial z},$$

the surface integral which is obtained is

$$\iint \left[\left(H_x - E_z \frac{\partial t}{\partial y} + E_y \frac{\partial t}{\partial z} \right) dy\,dz + \left(H_y - E_x \frac{\partial t}{\partial z} + E_z \frac{\partial t}{\partial x} \right) dz\,dx \right.$$
$$\left. + \left(H_z - E_y \frac{\partial t}{\partial x} + E_x \frac{\partial t}{\partial y} \right) dx\,dy \right],$$

or $$\iint (H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt).$$

Hence equation (IV) is established.

3. *The Group of Point Transformations for which the Integral Equations of the Theory of Electrons are Invariant.*

Let us consider a transformation of coordinates from (x, y, z, t) to (x', y', z', t') which is biuniform within a certain domain of values of (x, y, z, t). We shall suppose that the choice of a transformation is

limited by the condition that the total charge on a system of particles is invariant. This is expressed analytically by the equation

$$\rho w_x\,dy\,dz\,dt+\rho w_y\,dz\,dx\,dt+\rho w_z\,dx\,dy\,dt-\rho\,dx\,dy\,dz$$
$$=\rho' w_x'\,dy'\,dz'\,dt'+\rho' w_y'\,dz'\,dx'\,dt'+\rho' w_z'\,dx'\,dy'\,dt'-\rho' dx'\,dy'\,dz', \quad (1)$$

provided the axes form a right-handed system in each case.

If the transformation is such that the integral equations of the theory of electrons are invariant, we must have

$$E_x dy\,dz+E_y dz\,dx+E_z dx\,dy-H_x dx\,dt-H_y dy\,dt-H_z dz\,dt$$
$$=E_x' dy'\,dz'+E_y' dz'\,dx'+E_z' dx'\,dy'-H_x' dx'\,dt'-H_y' dy'\,dt'-H_z' dz'\,dt',$$

$$H_x dy\,dz+H_y dz\,dx+H_z dx\,dy+E_x dx\,dt+E_y dy\,dt+E_z dz\,dt$$
$$=\theta[H_x' dy'\,dz'+H_y' dz'\,dx'+H_z' dx'\,dy'+E_x' dx'\,dt'+E_y' dy'\,dt'+E_z' dz'\,dt'],$$

where θ is a constant. (2)

These relations give two sets of equations connecting the quantities $E_x, \ldots, H_x, \ldots$ with $E_x', \ldots, H_x', \ldots$, viz.,

$$E_x=E_x'\frac{\partial(y',z')}{\partial(y,z)}+E_y'\frac{\partial(z',x')}{\partial(y,z)}+E_z'\frac{\partial(x',y')}{\partial(y,z)}-H_x'\frac{\partial(x',t')}{\partial(y,z)}$$
$$-H_y'\frac{\partial(y',t')}{\partial(y,z)}-H_z'\frac{\partial(z',t')}{\partial(y,z)},$$

$$-H_x=E_x'\frac{\partial(y',z')}{\partial(x,t)}+E_y'\frac{\partial(z',x')}{\partial(x,t)}+E_z'\frac{\partial(x',y')}{\partial(x,t)}-H_x'\frac{\partial(x',t')}{\partial(x,t)}$$
$$-H_y'\frac{\partial(y',t')}{\partial(x,t)}-H_z'\frac{\partial(z',t')}{\partial(x,t)}, \quad (3)$$

and

$$H_x=\theta\Big[H_x'\frac{\partial(y',z')}{\partial(y,z)}+H_y'\frac{\partial(z',x')}{\partial(y,z)}+H_z'\frac{\partial(x',y')}{\partial(y,z)}$$
$$+E_x'\frac{\partial(x',t')}{\partial(y,z)}+E_y'\frac{\partial(y',t')}{\partial(y,z)}+E_z'\frac{\partial(z',t')}{\partial(y,z)}\Big],$$

$$E_x=\theta\Big[H_x'\frac{\partial(y',z')}{\partial(x,t)}+H_y'\frac{\partial(z',x')}{\partial(x,t)}+H_z'\frac{\partial(x',y')}{\partial(x,t)}$$
$$+E_x'\frac{\partial(x',t')}{\partial(x,t)}+E_y'\frac{\partial(y',t')}{\partial(x,t)}+E_z'\frac{\partial(z',t')}{\partial(x,t)}\Big].$$

In order that these equations may be equivalent to one another* we must

* It is assumed here that the equations of transformation are independent of E_x, E_y, E_z, H_x, H_y, H_z.

have $\theta^2 = 1$ and eighteen relations of the types

$$\frac{\partial(y', z')}{\partial(x, y)} = \theta \frac{\partial(x', t')}{\partial(z, t)},$$

$$\frac{\partial(y', z')}{\partial(x, t)} = -\theta \frac{\partial(x', t')}{\partial(y, z)}.$$

There are clearly nine relations of the first type and nine relations of the second type. We shall now show that when $\theta^2 = 1$ these relations imply that there is a relation of the form

$$dx'^2+dy'^2+dz'^2-dt'^2 = \lambda^2(dx^2+dy^2+dz^2-dt^2);$$

for this purpose we shall require the following lemma.

LEMMA.—Let the sixteen quantities

$$(\alpha_1\alpha_2\alpha_3\alpha_4), \quad (\beta_1\beta_2\beta_3\beta_4), \quad (\gamma_1\gamma_2\gamma_3\gamma_4), \quad (\delta_1\delta_2\delta_3\delta_4)$$

be connected by the eighteen relations of type

$$\beta_2\gamma_3-\beta_3\gamma_2 = \alpha_1\delta_4-\alpha_4\delta_1, \qquad \gamma_2\alpha_3-\gamma_3\alpha_2 = \beta_1\delta_4-\beta_4\delta_1,$$

$$\alpha_2\beta_3-\alpha_3\beta_2 = \gamma_1\delta_4-\gamma_4\delta_1,$$

which imply that conjugate minors of the determinant $[\alpha_1\beta_2\gamma_3\delta_4]$ are equal. The identity

$$\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2)+\alpha_2(\alpha_3\beta_1-\alpha_1\beta_3)+\alpha_3(\alpha_1\beta_2-\alpha_2\beta_1) = 0$$

then gives $\alpha_1(\gamma_1\delta_4-\gamma_4\delta_1)+\alpha_2(\gamma_2\delta_4-\gamma_4\delta_2)+\alpha_3(\gamma_3\delta_4-\gamma_4\delta_3) = 0$

or $$\delta_4(\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3+\alpha_4\gamma_4) = \gamma_4(\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4\delta_4).$$

Introducing the notation

$$(\alpha\gamma) \equiv \alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3+\alpha_4\gamma_4,$$

we may obtain in the above way the equations

$$\delta_1(\alpha\gamma) = \gamma_1(\alpha\delta), \qquad \delta_3(\alpha\gamma) = \gamma_3(\alpha\delta),$$

$$\delta_2(\alpha\gamma) = \gamma_2(\alpha\delta), \qquad \delta_4(\alpha\gamma) = \gamma_4(\alpha\delta);$$

$$\delta_1(\beta\gamma) = \gamma_1(\beta\delta), \qquad \delta_3(\beta\gamma) = \gamma_3(\beta\delta),$$

$$\delta_2(\beta\gamma) = \gamma_2(\beta\delta), \qquad \delta_4(\beta\gamma) = \gamma_4(\beta\delta).$$

Hence, either $$(\alpha\gamma) = (\alpha\delta) = (\beta\gamma) = (\beta\delta) = 0$$

or all the quantities of type $\gamma_1\delta_2-\gamma_2\delta_1$ are zero. In the same way we can prove that either $$(\alpha\beta) = (\alpha\delta) = (\gamma\beta) = (\gamma\delta) = 0$$

or all the quantities of type $\beta_1\delta_2-\beta_2\delta_1$ are zero, and that either

$$(\beta\alpha) = (\beta\delta) = (\gamma\alpha) = (\gamma\delta) = 0$$

or all the quantities of type $\alpha_1\delta_2-\alpha_2\delta_1$ are zero. It follows from this that either (1) the six quantities $(\alpha\beta)$ are zero or (2) that the thirty-six quantities $(\alpha_1\delta_2-\alpha_2\delta_1)$, $(\beta_1\delta_2-\beta_2\delta_1)$, $(\gamma_1\delta_2-\gamma_2\delta_1)$, ... are all zero or (3) that there is a set of relations

$$(\alpha\gamma) = (\alpha\delta) = (\beta\gamma) = (\beta\delta) = 0,$$

$$\frac{\beta_1}{\delta_1} = \frac{\beta_2}{\delta_2} = \frac{\beta_3}{\delta_3} = \frac{\beta_4}{\delta_4}, \qquad \frac{\alpha_1}{\delta_1} = \frac{\alpha_2}{\delta_2} = \frac{\alpha_3}{\delta_3} = \frac{\alpha_4}{\delta_4}.$$

It is easy to see, however, that in the latter case we also have

$$(\gamma\delta) = (\alpha\beta) = 0.$$

Hence, in all cases,

$$(\alpha\gamma) = (\alpha\beta) = (\alpha\delta) = (\beta\gamma) = (\beta\delta) = (\gamma\delta) = 0.$$

Again, we have

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \alpha_1(\alpha_1\delta_4-\alpha_4\delta_1)+\alpha_2(\alpha_2\delta_4-\alpha_4\delta_2)+\alpha_3(\alpha_3\delta_4-\alpha_4\delta_3)$$
$$= \delta_4(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)-\alpha_4(\alpha_1\delta_1+\alpha_2\delta_2+\alpha_3\delta_3+\alpha_4\delta_4);$$

hence, since the last term is zero,

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \delta_4(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2).$$

This gives

$$(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2)(\delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2) = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = \lambda^4,$$

say, and there are similar equations in α, β, $\alpha\gamma$, $\beta\gamma$, $\beta\delta$, $\gamma\delta$. It follows that

$$\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2 = \beta_1^2+\beta_2^2+\beta_3^2+\beta_4^2$$
$$= \gamma_1^2+\gamma_2^2+\gamma_3^2+\gamma_4^2 = \delta_1^2+\delta_2^2+\delta_3^2+\delta_4^2 = \pm\lambda^2.$$

These conditions, combined with the previous set, imply that the sixteen quantities α_r, β_s, ... are the elements of an orthogonal matrix.

If the signs of the α's are all changed, the relations take the form

$$\beta_2\gamma_3-\beta_3\gamma_2 = -(\alpha_1\delta_4-\alpha_4\delta_1),$$

but the final result is unaltered; hence a set of relations of this type also imply that the quantities α are the elements of an orthogonal matrix.

Taking the quantities $\alpha, \beta, \ldots$ to be the elements of the matrix

$$\begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} & \frac{\partial x'}{i\partial t} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} & \frac{\partial y'}{i\partial t} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} & \frac{\partial z'}{i\partial t} \\ i\frac{\partial t'}{\partial x} & i\frac{\partial t'}{\partial y} & i\frac{\partial t'}{\partial z} & \frac{\partial t'}{\partial t} \end{vmatrix},$$

we have the result that, if eighteen relations of the type

$$\frac{\partial(y', z')}{\partial(x, y)} = \frac{\partial(x', t')}{\partial(z, t)}, \qquad \frac{\partial(y', z')}{\partial(x, t)} = -\frac{\partial(x', t')}{\partial(y, z)}$$

or of the type

$$\frac{\partial(y', z')}{\partial(x, y)} = -\frac{\partial(x', t')}{\partial(z, t)}, \qquad \frac{\partial(y', z')}{\partial(x, t)} = \frac{\partial(x', t')}{\partial(y, z)}$$

are satisfied, then the matrix is an orthogonal one, and this implies that

$$dx'^2+dy'^2+dz'^2-dt'^2 = \lambda^2[dx^2+dy^2+dz^2-dt^2];$$

in other words, that the differential equation

$$(dx)^2+(dy)^2+(dz)^2-(dt)^2 = 0$$

is an invariant. The converse is also true, and may be easily verified.

This differential equation expresses the condition that two neighbouring particles should be in a position to act on one another; it also indicates that an elementary wave starting from a given point will have the form of a sphere. For this reason we shall call a transformation which leaves the differential equation invariant, a *spherical wave transformation*. Our analysis shows that a transformation which leaves the integral equations of the theory of electrons invariant is necessarily a spherical wave transformation. We must next inquire whether all spherical wave transformations are relevant for our purpose. If we call t' the local time, we must exclude transformations which make the local time run backwards as t increases. Hence, in order that a transformation may be relevant, the condition

$$\frac{\partial t'}{\partial t} > 0$$

must be satisfied.

Secondly, we must exclude transformations in which the axes are changed from a right-handed system to a left-handed system, for then equation (1) would imply a change from positive electricity to negative electricity. Such a transformation, however, becomes relevant when equation (1) is replaced by

$$\rho w_x dy\,dz\,dt+\rho w_y dz\,dx\,dt+\rho w_z dx\,dy\,dt-\rho\,dx\,dy\,dz$$
$$=-\rho' w'_x dy'dz'dt'-\rho' w'_y dz'dx'dt'-\rho' w'_z dx'dy'dt'+\rho' dx'dy'dz', \quad (1)'$$

equation (2) by

$$E_x dy\,dz+E_y dz\,dx+E_z dx\,dy-H_x dx\,dt-H_y dy\,dt-H_z dz\,dt$$
$$=-E'_x dy'dz'-E'_y dz'dx'-E'_z dx'dy'-H'_x dx'dt'-H'_y dy'dt'-H'_z dz'dt', \quad (2)'$$

and equation (3) by

$$H_x dy\,dz+H_y dz\,dx+H_z dx\,dy+E_x dx\,dt+E_y dy\,dt+E_z dz\,dt$$
$$=\theta[-H'_x dy'dz'-H'_y dz'dx'-H'_z dx'dy'+E'_x dx'dt'+E'_y dy'dt'+E'_z dz'dt']. \quad (3)'$$

These relations also imply that the transformation is a spherical wave transformation, and so a transformation which changes a right-handed system of axes into a left-handed system is relevant, but the formulæ of transformation of the components of the electric and magnetic force are not the same as before. The sign of θ will be determined later by the condition that a right-handed set of axes is transformed into a left-handed set. If

$$\frac{\partial(x', y', z')}{\partial(x, y, z)} > 0,$$

corresponding sets of axes are both right-handed or both left-handed. If, on the other hand,

$$\frac{\partial(x', y', z')}{\partial(x, y, z)} < 0,$$

the axes are right-handed in one system and left-handed in the other.

It is easy to establish a relation between the two quantities

$$\frac{\partial t'}{\partial t} \quad \text{and} \quad \frac{\partial(x', y', z')}{\partial(x, y, z)}.$$

Since

$$\frac{\partial(y', z')}{\partial(y, z)}=\theta\frac{\partial(x', t')}{\partial(x, t)}, \qquad \frac{\partial(z', x')}{\partial(y, z)}=\theta\frac{\partial(y', t')}{\partial(x, t)},$$
$$\frac{\partial(x', y')}{\partial(y, z)}=\theta\frac{\partial(z', t')}{\partial(x, t)},$$

we have

$$\frac{\partial(x', y', z')}{\partial(x, y, z)} = \theta\Big[\frac{\partial t'}{\partial t}\Big\{\Big(\frac{\partial x'}{\partial x}\Big)^2+\Big(\frac{\partial y'}{\partial x}\Big)^2+\Big(\frac{\partial z'}{\partial x}\Big)^2-\Big(\frac{\partial t'}{\partial x}\Big)^2\Big\}$$

$$-\frac{\partial x'}{\partial x}\Big\{\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial t}+\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial t}+\frac{\partial z'}{\partial x}\frac{\partial z'}{\partial t}-\frac{\partial t'}{\partial x}\frac{\partial t'}{\partial t}\Big\}\Big]$$

or
$$\frac{\partial(x', y', z')}{\partial(x, y, z)} = \theta\frac{\partial t'}{\partial t}\Big[\Big(\frac{\partial x'}{\partial x}\Big)^2+\Big(\frac{\partial y'}{\partial x}\Big)^2+\Big(\frac{\partial z'}{\partial x}\Big)^2-\Big(\frac{\partial t'}{\partial x}\Big)^2\Big]. \qquad (A)$$

We also have

$$\frac{\partial(x', y', z', t')}{\partial(x, y, z, t)} = \frac{\partial(y', z')}{\partial(y, z)}\frac{\partial(x', t')}{\partial(x, t)}+\frac{\partial(z', x')}{\partial(y, z)}\frac{\partial(y', t')}{\partial(x, t)}+\ldots$$

$$= \theta\Big[\Big\{\frac{\partial(x', t')}{\partial(x, t)}\Big\}^2+\Big\{\frac{\partial(y', t')}{\partial(x, t)}\Big\}^2+\ldots-\Big\{\frac{\partial(y', z')}{\partial(x, t)}\Big\}^2-\ldots\Big]$$

$$= -\theta\Big[\Big\{\Big(\frac{\partial x'}{\partial x}\Big)^2+\Big(\frac{\partial y'}{\partial x}\Big)^2+\Big(\frac{\partial z'}{\partial x}\Big)^2-\Big(\frac{\partial t'}{\partial x}\Big)^2\Big\}$$

$$\times\Big\{\Big(\frac{\partial x'}{\partial t}\Big)^2+\Big(\frac{\partial y'}{\partial t}\Big)^2+\Big(\frac{\partial z'}{\partial t}\Big)^2-\Big(\frac{\partial t'}{\partial t}\Big)^2\Big\}$$

$$-\Big\{\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial t}+\frac{\partial y'}{\partial x}\frac{\partial y'}{\partial t}+\frac{\partial z'}{\partial x}\frac{\partial z'}{\partial t}-\frac{\partial t'}{\partial x}\frac{\partial t'}{\partial t}\Big\}^2\Big].$$

Therefore

$$\frac{\partial(x', y', z', t')}{\partial(x, y, z, t)} = -\theta\Big[\Big(\frac{\partial x'}{\partial x}\Big)^2+\Big(\frac{\partial y'}{\partial x}\Big)^2+\Big(\frac{\partial z'}{\partial x}\Big)^2-\Big(\frac{\partial t'}{\partial x}\Big)^2\Big]$$

$$\times\Big[\Big(\frac{\partial x'}{\partial t}\Big)^2+\Big(\frac{\partial y'}{\partial t}\Big)^2+\Big(\frac{\partial z'}{\partial t}\Big)^2-\Big(\frac{\partial t'}{\partial t}\Big)^2\Big].$$

This gives us the relation*

$$-\frac{\partial(x', y', z')}{\partial(x, y, z)}\Big[\Big(\frac{\partial x'}{\partial t}\Big)^2+\Big(\frac{\partial y'}{\partial t}\Big)^2+\Big(\frac{\partial z'}{\partial t}\Big)^2-\Big(\frac{\partial t'}{\partial t}\Big)^2\Big] = \frac{\partial t'}{\partial t}\frac{\partial(x', y', z', t')}{\partial(x, y, z, t)}, \qquad (B)$$

which holds for any spherical wave transformation.

We shall now introduce the further restriction that the inequality

$$(dx')^2+(dy')^2+(dz')^2 < (dt')^2$$

is a consequence of $\quad (dx)^2+(dy)^2+(dz)^2 < (dt)^2.$

This means that, if a particle is moving with a velocity less than that of light in one system of coordinates, it is also moving with a velocity less than that of light in the transformed system.

* I am indebted to a referee for calling my attention to this relation and the necessity of distinguishing between the two types of transformation.

Putting

$$(dx')^2+(dy')^2+(dz')^2-(dt')^2 = \lambda^2[(dx)^2+(dy)^2+(dz)^2-(dt)^2], \qquad (C)$$

we see that λ^2 is positive, and therefore

$$\left(\frac{\partial x'}{\partial t}\right)^2+\left(\frac{\partial y'}{\partial t}\right)^2+\left(\frac{\partial z'}{\partial t}\right)^2-\left(\frac{\partial t'}{\partial t}\right)^2$$

is negative. This shows that $\dfrac{\partial(x', y', z')}{\partial(x, y, z)}$ and $\dfrac{\partial t'}{\partial t}\dfrac{\partial(x', y', z', t')}{\partial(x, y, z, t)}$ must have the same sign. Hence, if $\dfrac{\partial t'}{\partial t}$ is positive, $\dfrac{\partial(x', y', z')}{\partial(x, y, z)}$ must have the same sign as the Jacobian. Accordingly, a transformation which changes a right-handed system of axes into a right-handed system must have a positive Jacobian; a transformation which changes a right-handed system of axes into a left-handed system must have a negative Jacobian.

The sign of θ may now be determined from equation (A). Since

$$\left(\frac{\partial x'}{\partial x}\right)^2+\left(\frac{\partial y'}{\partial x}\right)^2+\left(\frac{\partial z'}{\partial x}\right)^2-\left(\frac{\partial t'}{\partial x}\right)^2 = \lambda^2,$$

it is necessarily positive. Consequently θ must have the same sign as $\dfrac{\partial(x', y', z')}{\partial(x, y, z)}$, and therefore the same sign as the Jacobian.

We can now obtain the formulæ of transformation in the two possible cases.

(i) When the Jacobian is positive, $\theta = +1$, and the formulæ of transformation are

$$E_x = E_x'\frac{\partial(y', z')}{\partial(y, z)}+E_y'\frac{\partial(z, x')}{\partial(y, z)}+E_z'\frac{\partial(x', y')}{\partial(y, z)}$$
$$-H_x'\frac{\partial(x', t')}{\partial(y, z)}-H_y'\frac{\partial(y', t')}{\partial(y, z)}-H_z'\frac{\partial(z', t')}{\partial(y, z)},$$

$$-H_x = E_x'\frac{\partial(y', z')}{\partial(x, t)}+E_y'\frac{\partial(z', x')}{\partial(x, t)}+E_z'\frac{\partial(x', y')}{\partial(x, t)}$$
$$-H_x'\frac{\partial(x', t')}{\partial(x, t)}-H_y'\frac{\partial(y', t')}{\partial(x, t)}-H_z'\frac{\partial(z', t')}{\partial(x, t)},$$

$$\rho w_x = \rho' w_x'\frac{\partial(y', z', t')}{\partial(y, z, t)}+\rho' w_y'\frac{\partial(z', x', t')}{\partial(y, z, t)}+\rho' w_z'\frac{\partial(x', y', t')}{\partial(y, z, t)}-\rho'\frac{\partial(x', y', z')}{\partial(y, z, t)},$$

$$-\rho = \rho' w_x'\frac{\partial(y', z', t')}{\partial(x, y, z)}+\rho' w_y'\frac{\partial(z', x', t')}{\partial(x, y, z)}+\rho' w_z'\frac{\partial(x', y', t')}{\partial(x, y, z)}-\rho'\frac{\partial(x', y, z')}{\partial(x, y, z)}.$$

These equations may be simplified by using the relations of type

$$\frac{\partial(y, z, t')}{\partial(x, y, z)} = \theta \frac{\partial x'}{\partial t} \lambda^2,$$

which are proved in the same way as (A), the quantity λ^2 being defined by equation (C).

The new equations are

$$\rho w_x = \lambda^2 \left[\rho' w'_x \frac{\partial x'}{\partial x} + \rho' w'_y \frac{\partial y}{\partial x} + \rho' w'_z \frac{\partial z'}{\partial x} - \rho' \frac{\partial t'}{\partial x} \right],$$

$$-\rho = \lambda^2 \left[\rho' w'_x \frac{\partial x'}{\partial t} + \rho' w'_y \frac{\partial y'}{\partial t} + \rho' w'_z \frac{\partial z'}{\partial t} - \rho' \frac{\partial t'}{\partial t} \right],$$

and these imply that

$$\rho w_x dx + \rho w_y dy + \rho w_z dz - \rho\, dt = \lambda^2 [\rho' w'_x dx' + \rho' w'_y dy' + \rho' w'_z dz' - \rho' dt'].$$

The formulæ connecting the electromagnetic potentials are obtained by putting

$$A_x dx + A_y dy + A_z dz - \Phi dt = \theta [A'_x dx' + A'_y dy' + A'_z dz' - \Phi' dt'].$$

Since $\theta = +1$, they are

$$A_x = A'_x \frac{\partial x'}{\partial x} + A'_y \frac{\partial y'}{\partial x} + A'_z \frac{\partial z}{\partial x} - \Phi' \frac{\partial t}{\partial x},$$

$$-\Phi = A'_x \frac{\partial x'}{\partial t} + A'_y \frac{\partial y'}{\partial t} + A'_z \frac{\partial z'}{\partial t} - \Phi' \frac{\partial t'}{\partial t}.$$

(ii) When the Jacobian is negative and $\partial t'/\partial t > 0$, the axes are right-handed in the original system and left-handed in the system specified by the dashed letters. To obtain the correct formulæ of transformation we must change the sign of H' in equations (2) and (3) and put $\theta = -1$. There is also a doubt about the sign in the equation

$$\rho w\, dy dz dt + \rho w_y dz dx dt + \rho w_z dx dy dt - \rho\, dx dy dz$$

$$= \pm (\rho' w_x dy' dz' dt' + \rho w_y dz' dx' dt' + \rho' w_z dx' dy' dt' - \rho' dx' dy' dz'),$$

the sign depending upon whether positive electricity is transformed into negative electricity or positive electricity. If the negative sign be taken there must be a corresponding alteration in equation (3).

4. *Spherical Wave Transformations and the Group of Conformal Transformations of a Space of Four Dimensions.*

The group of spherical wave transformations may be reduced to a known group by putting $t = is$. The quadratic form which remains invariant is then of the type

$$\lambda^2(dx^2+dy^2+dz^2+ds^2),$$

and so the transformation is a conformal one.

The group of conformal transformations in a space of four dimensions has been studied by Sophus Lie,* who has shown that it is composed of reflexions, translations, rotations, magnifications, and inversions.† The transformations which are of importance in the present case are the imaginary ones, and it should be noticed that by a combination of two imaginary inversions we can obtain a transformation of the type

$$x' = \frac{kx}{z+is}, \quad y' = \frac{ky}{z+is}, \quad z' = k\,\frac{x^2+y^2+z^2+s^2-a^2}{2a(z+is)},$$

$$s' = k\,\frac{x^2+y^2+z^2+s^2+a^2}{2ia(z+is)},$$

which is quite different from an inversion or simple displacement. This corresponds to the real spherical wave transformation‡

$$x' = \frac{kx}{z-t}, \quad y' = \frac{ky}{z-t}, \quad z' = k\,\frac{x^2+y^2+z^2-a^2-t^2}{2a(z-t)},$$

$$-t' = k\,\frac{x^2+y^2+z^2+a^2-t^2}{2a(z-t)}.$$

An imaginary rotation in the four-dimensional space may be specified, in a particular case, by the equations

$$x' = x\cos iw + s\sin iw, \qquad y' = y,$$

$$s' = -x\sin iw + s\cos iw, \qquad z' = z.$$

Putting

$$\tanh w = v,$$

* *Göttinger Nachrichten* (1871), *Transformationgruppen*, Bd. 3, p. 351.

† Every transformation belonging to the group is a birational transformation.

‡ See a paper by the author "On the Conformal Transformations of a Space of Four Dimensions and their Applications to Geometrical Optics," *Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 70 (1909).

we obtain Lorentz's transformation

$$x' = \frac{x+vt}{\sqrt{(1-v^2)}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t+vx}{\sqrt{(1-v^2)}}.$$

The formulæ of transformation of the electric and magnetic vectors are obtained at once from the general formulæ; they are

$$\begin{aligned}
A_x &= \beta(A'_x - v\Phi'), & -\Phi &= \beta(vA'_x - \Phi'),\\
A_y &= A'_y, & A_z &= A'_z,\\
E_x &= E'_x, & H_x &= H'_x,\\
E_y &= \beta(E'_y - vH'_z), & H_y &= \beta(H'_y + vE'_z),\\
E_z &= \beta(E'_z + vH'_y), & H_z &= \beta(H'_z - vE'_y),\\
\rho w_x &= \beta(\rho' w'_x - v\rho'), & \rho w_y &= \rho' w'_y, \quad \rho w_z = \rho' w'_z,\\
& & -\rho &= \beta(v\rho' w'_x - \rho'),
\end{aligned}$$

where

$$\beta = \frac{1}{\sqrt{(1-v^2)}}.$$

These agree with the formulæ of Einstein, Lorentz's formulæ for the convection currents being slightly different.

In the case of an inversion with regard to a hypersphere whose centre is at the origin, and whose radius is a real quantity k, the formulæ of transformation are

$$x' = \frac{k^2 x}{x^2+y^2+z^2+s^2}, \quad y' = \frac{k^2 y}{x^2+y^2+z^2+s^2}, \quad z' = \frac{k^2 z}{x^2+y^2+z^2+s^2},$$
$$s' = \frac{k^2 s}{x^2+y^2+z^2+s^2},$$

Putting $s = it$, $x^2+y^2+z^2 = r^2$, we get the real spherical wave transformation

$$x' = \frac{k^2 x}{r^2-t^2}, \quad y' = \frac{k^2 y}{r^2-t^2}, \quad z' = \frac{k^2 z}{r^2-t^2}, \quad t' = \frac{k^2 t}{r^2-t^2}.$$

This transformation has a negative Jacobian; the formulæ for transforming the components of the electric and magnetic force have been obtained by E. Cunningham.*

It should be noticed that

$$\frac{dt'}{dt} = \frac{r^2+t^2}{(r^2-t^2)^2},$$

* *Proc. London Math. Soc.*, Ser, 2, Vol. 8, p. 77.

and is positive; hence t' increases as t increases, if (x, y, z) are kept constant.

The transformation which corresponds to a reflexion in the four-dimensional space is also of considerable interest. In the particular case when the reflecting space passes through the plane $x = 0$, $s = 0$, the reflexion may be replaced by a rotation round the plane $x = 0$, $s = 0$, and a reflexion in the space $x = 0$. The corresponding transformation is thus made up out of a transformation of Lorentz

$$x' = \frac{x-vt}{\sqrt{(1-v^2)}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-vx}{\sqrt{(1-v^2)}},$$

and a change in the sign of x'. Putting

$$\frac{v}{\sqrt{(1-v^2)}} = \frac{2u}{1-u^2}, \qquad \frac{1}{\sqrt{(1-v^2)}} = \frac{1+u^2}{1-u^2},$$

and changing the sign of x', we get

$$x' = -\frac{1+u^2}{1-u^2}x + \frac{2u}{1-u^2}t,$$

$$y' = y,$$

$$z' = z,$$

$$t' = \frac{1+u^2}{1-u^2}t - \frac{2u}{1-u^2}x.$$

The quantity u is introduced because the angle of rotation in the four-dimensional space is twice the angle between the reflecting space and the space $x = 0$. Its geometrical meaning in the case of the spherical wave transformation is indicated by the equation

$$x'-ut' = -(x-ut),$$

which implies that a plane moving with the constant velocity u is transformed into itself. Further, when $x = ut$, we have

$$x' = x, \quad y' = y, \quad z' = z, \quad t' = t;$$

hence every point of the plane is transformed into itself.*

The formulæ of transformation of the electromagnetic vectors are found from the first set of equations for a transformation with negative Jacobian.

* Geometrically the transformation is equivalent to a reflexion in a moving plane mirror.

The general infinitesimal spherical wave transformation is

$$x' = x+\epsilon[p(x^2-y^2-z^2+t^2)+2qxy+2rxz+2sxt+\mu x+hy+gz+lt+a],$$

$$y' = y+\epsilon[q(y^2-z^2-x^2+t^2)+2pxy+2ryz+2syt-hx+\mu y+fz+mt+b],$$

$$z' = z+\epsilon[r(z^2-x^2-y^2+t^2)+2pxz+2qyz+2szt-gx-fy+\mu z+nt+c],$$

$$t' = t+\epsilon[s(x^2+y^2+z^2+t^2)+2pxt+2qyt+2rzt+lx+my+nz+\mu t+d],$$

where the coefficients are all constants, and ϵ is a quantity whose square may be neglected. Since there are fifteen arbitrary constants the group is a fifteen parameter group.

We have

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y} = \frac{\partial z'}{\partial z} = \frac{\partial t'}{\partial t} = 1+\epsilon[\mu+2px+2qy+2rz+2st] = \lambda,$$

$$dt'^2-dx'^2-dy'^2-dz'^2 = \lambda^2[dt^2-dx^2-dy^2-dz^2].$$

If (x, y, z) is kept constant as t varies, the corresponding point (x', y', z') moves along a parabola, but in one type of transformation it moves along a straight line with constant acceleration. This is the case, for example, when

$$x' = x+\epsilon(y^2+z^2-x^2-t^2), \qquad y' = y(1-2\epsilon x),$$

$$z' = z(1-2\epsilon x), \qquad t' = t(1-2\epsilon x),$$

for, since quantities of order ϵ^2 may be neglected, the first equation may be written

$$x' = x+\epsilon(y^2+z^2-x^2-t'^2).$$

Hence, if (x, y, z) are kept constant, the point (x', y', z') moves with constant acceleration γ given by

$$\gamma = -2\epsilon.$$

Substituting for γ, we get

$$x' = x-\tfrac{1}{2}\gamma(y^2+z^2-x^2-t^2), \qquad y' = y(1+\gamma x),$$

$$z' = z(1+\gamma x), \qquad t' = t(1+\gamma x).$$

The last equation agrees with the one obtained by Einstein.*

In the case of the general infinitesimal transformation, the expression

$$(x'-x)^2+(y'-y)^2+(z'-z)^2-(t'-t)^2,$$

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* *Jahrbuch der Radioaktivität*, Band IV. (1907), p. 457.

is of order ϵ^2, but there is one type of transformation in which it is of order ϵ^3 at least. The formulæ of transformation are then

$$x' = x+\epsilon[2x(lx+my+nz-t)-l\ (x^2+y^2+z^2-t^2)],$$
$$y' = y+\epsilon[2y(lx+my+nz-t)-m(x^2+y^2+z^2-t^2)],$$
$$z' = z+\epsilon[2z\,(lx+my+nz-t)-n\ (x^2+y^2+z^2-t^2)],$$
$$t' = t+\epsilon[2t\,(lx+my+nz-t)-\ \ (x^2+y^2+z^2-t^2)],$$

where $$l^2+m^2+n^2 = 1,$$

and signify that (x', y', z'), (x, y, z) are successive positions of a particle which is moving with the velocity of light.

Real spherical wave transformations may be obtained geometrically in the following way. Let the space time point (x, y, z, t) be represented by a sphere of radius t having its centre at the point (x, y, z). Then, if we apply a real conformal transformation of space to these representative spheres, the new set of spheres may be taken as the representative spheres of a new set of space time points (x', y', z', t'). These are connected with the original set by a system of equations which define a spherical wave transformation. This theorem has already been established in the case of an inversion,* and since any real conformal transformation of space can be built up from inversions, it follows that the general transformation obtained in the above way is a spherical wave transformation.

It should be noticed that the general spherical wave transformation cannot be obtained in this way, because $t' = 0$ always corresponds to $t = 0$. If, however, we combine these transformations with the real spherical wave transformation obtained by increasing or decreasing the radii of all the representative spheres by the same amount, it is possible to obtain any spherical wave transformation by a suitable combination. The proof of this will be left to the reader.

When we use the representative spheres the differential equation

$$(dx)^2+(dy)^2+(dz)^2-(dt)^2 = 0,$$

admits of a very simple interpretation, as it implies that the two consecutive representative spheres specified by (x, y, z, t) and $(x+dx, y+dy, z+dz, t+dt)$ touch one another internally.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 70.

5. *The Transformation of Integral Forms.*

The general theory of the transformation of physical problems by means of a change of coordinates can be developed in a convenient manner by studying a number of integral forms, examining the way in which they are related to one another, and obtaining the formulæ by means of which they can be transformed.*

We shall commence by studying the simple case of a transformation in two variables from (x, y) to (x', y'). Suppose that

$$a\,dx+b\,dy = a'dx'+b'dy', \tag{1}$$

$$A\,dx+B\,dy = A'dx'+B'dy', \tag{2}$$

then
$$a = a'\frac{\partial x'}{\partial x}+b'\frac{\partial y'}{\partial x}, \qquad b = a'\frac{\partial x'}{\partial y}+b'\frac{\partial y'}{\partial y},$$

$$A = A'\frac{\partial x'}{\partial x}+B'\frac{\partial y'}{\partial x}, \qquad B = A'\frac{\partial x'}{\partial y}+B'\frac{\partial y'}{\partial y},$$

and it is easy to see that

$$aB-Ab = (a'B'-A'b')\frac{\partial(x', y')}{\partial(x, y)}.$$

This implies that $(aB-Ab)\,dx\,dy = (a'B'-A'b')\,dx'dy'.$ (3)

Now this relation can be obtained from the previous pair by the process of multiplication used in Grassmann's calculus of extension.† In this calculus, the sign of a product depends on the order of the terms; thus

$$dx\,dy = -dy\,dx \quad \text{and} \quad dx\,dx = 0.$$

This rule can also be applied to the case in which the quantities a and b are differential operators; thus, if

$$a = \frac{\partial}{\partial x}, \quad b = \frac{\partial}{\partial y}, \quad a' = \frac{\partial}{\partial x'}, \quad b' = \frac{\partial}{\partial y'},$$

$$\left(\frac{\partial B}{\partial x}-\frac{\partial A}{\partial y}\right)dx\,dy = \left(\frac{\partial B'}{\partial x'}-\frac{\partial A'}{\partial y'}\right)dx'dy', \tag{4}$$

a relation which may be obtained directly by means of Green's theorem.

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* The theory of integral invariants has been developed by Poincaré, *Mécanique céleste*, t. 3; *Méthodes nouvelles de la Mécanique céleste*, t. 3, p. 33; Goursat, *Liouville's Journal* (6), t. 4, p. 331; Koenigs, *Comptes rendus*, t. 122 (1906), pp. 25-27.

† Scott's *Determinants*, p. 16.

Next, suppose that the formula of transformation of a quadratic differential form is known, *e.g.*,

$$E\,dx^2+2F\,dx\,dy+G\,dy^2 = E'dx'^2+2F'dx'dy'+G'dy'^2\,; \tag{5}$$

then taking two independent differentials dx, dy, δx, δy, and writing $dx+\lambda\delta x$, $dy+\lambda\delta y$, ... in place of dx, dy, we get the formula of transformation of the bilinear form

$$E\,dx\,\delta x+F(dx\,\delta y+dy\,\delta x)+G\,dy\,\delta y = E'dx'\delta x'+F'(dx'\delta y'+dy'\delta x')+G'dy'\delta y'. \tag{6}$$

We may multiply this equation by itself, multiplying both sets of differentials according to Grassmann's rule. The resulting equation is

$$(EG-F^2)\,dx\,dy\,\delta x\,\delta y = (E'G'-F'^2)\,dx'dy'\delta x'\delta y'.$$

This gives
$$\sqrt{(EG-F^2)}\,dx\,dy = \sqrt{(E'G'-F'^2)}\,dx'dy', \tag{7}$$

or
$$\Delta\,dx\,dy = \Delta'dx'dy'.$$

Multiplying (6) by (1) according to Grassmann's rule, we obtain

$$[(Eb-Fa)\delta x-(Ga-Fb)\delta y]\,dx\,dy = [(E'b'-F'a')\delta x'-(G'a'-F'b')\delta y']\,dx'dy'.$$

This gives the formula of transformation of a linear form

$$\frac{Eb-Fa}{\Delta}\,\delta x-\frac{Ga-Fb}{\Delta}\,\delta y = \frac{E'b'-F'a'}{\Delta}\,\delta x'-\frac{G'a'-F'b'}{\Delta'}\,\delta y', \tag{8}$$

which may be called the reciprocal of the first.

Multiplying this equation by

$$a\,dx+b\,dy = a'dx'+b'dy',$$

$$dx\,\frac{\partial}{\partial x}+dy\,\frac{\partial}{\partial y} = dx'\,\frac{\partial}{\partial x'}+dy'\,\frac{\partial}{\partial y'},$$

respectively, we get

$$\frac{Eb^2-2Fab+Ga^2}{\Delta}\,dx\,dy = \frac{E'b'^2-2F'a'b'+G'a'^2}{\Delta'}\,dx'dy',$$

$$\left[\frac{\partial}{\partial x}\left(\frac{Ga-Fb}{\Delta}\right)+\frac{\partial}{\partial y}\left(\frac{Eb-Fa}{\Delta}\right)\right]dx\,dy = \left[\frac{\partial}{\partial x'}\left(\frac{G'a'-F'b'}{\Delta'}\right)+\frac{\partial}{\partial y'}\left(\frac{E'b'-F'a'}{\Delta'}\right)\right]dx'dy'.$$

These formulæ are well known in the differential geometry of a surface.*

We shall now show that the above theory may be extended to transformations in any number of variables. In the first place we must show that the law of multiplication still holds. Let there be n variables $x_1, \ldots, x_n$, and suppose that

$$\Sigma a_r dx_r = \Sigma a'_r dx'_r,$$
$$\Sigma b_r dx_r = \Sigma b'_r dx'_r;$$

then by multiplication we may obtain an integral form of the second order†

$$\Sigma (a_r b_s - a_s b_r)\, dx_r dx_s = \Sigma (a'_r b'_s - a'_s b'_r)\, dx'_r dx'_s,$$

the multiplication being performed by Grassmann's rule. To verify this we have only to replace $dx_r dx_s$ by $\dfrac{\partial (x_r, x_s)}{\partial (\alpha, \beta)} d\alpha\, d\beta$, and notice that the last equation may be written

$$\begin{vmatrix} \Sigma a_r \dfrac{\partial x_r}{\partial \alpha}, & \Sigma b_r \dfrac{\partial x_r}{\partial \alpha} \\ \Sigma a_r \dfrac{\partial x_r}{\partial \beta}, & \Sigma b_r \dfrac{\partial x_r}{\partial \beta} \end{vmatrix} d\alpha\, d\beta = \begin{vmatrix} \Sigma a'_r \dfrac{\partial x'_r}{\partial \alpha}, & \Sigma b'_r \dfrac{\partial x'_r}{\partial \alpha} \\ \Sigma a'_r \dfrac{\partial x'_r}{\partial \beta}, & \Sigma b'_r \dfrac{\partial x'_r}{\partial \beta} \end{vmatrix} d\alpha\, d\beta.$$

Similarly, if we take three integral forms,

$$\Sigma a_r dx_r = \Sigma a'_r dx'_r, \qquad (\alpha)$$
$$\Sigma b_r dx_r = \Sigma b'_r dx'_r, \qquad (\beta)$$
$$\Sigma c_r dx_r = \Sigma c'_r dx'_r, \qquad (\gamma)$$

and multiply them together by Grassmann's rule, we obtain the integral

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* Darboux, *Théorie générale des Surfaces*, t. 3, p. 193.

In particular, if $dx^2 + dy^2 = \lambda [dx'^2 + dy'^2]$,

we may deduce from the identity

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = \frac{\partial V}{\partial x'} dx' + \frac{\partial V}{\partial y'} dy',$$

that

$$\frac{\partial V}{\partial y} dx - \frac{\partial V}{\partial x} dy = \frac{\partial V}{\partial y'} dx' - \frac{\partial V}{\partial x'} dy',$$

and on multiplying by (9), we get the well known relation

$$\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) dx\, dy = \left(\frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2} \right) dx' dy'.$$

† An integral form of the second order must be carefully distinguished from a quadratic differential form.

form of the third order

$$\sum_{r,s,t}\begin{vmatrix} a_r & a_s & a_t \\ b_r & b_s & b_t \\ c_r & c_s & c_t \end{vmatrix} dx_r dx_s dx_t = \sum_{r,s,t}\begin{vmatrix} a'_r & a'_s & a'_t \\ b'_r & b'_s & b'_t \\ c'_r & c'_s & c'_t \end{vmatrix} dx'_r dx'_s dx'_t.$$

It will be seen also that this equation may be obtained by multiplying the integral form of the second order derived from (α) and (β), by (γ). The question then arises whether integral forms of any order may be multiplied together by Grassmann's rule. To show that this is the case we shall consider the two integral forms

$$\sum_{s,t,\ldots} A_{s,t,\ldots} dx_s dx_t \ldots = \sum_{s,t,\ldots} A'_{s,t,\ldots} dx'_s dx'_t, \ldots,$$

$$\sum_{p,q,r,\ldots} B_{p,q,r,\ldots} dx_p dx_q dx_r = \sum_{p,q,r,\ldots} B'_{p,q,r,\ldots} dx'_p dx'_q dx'_r \ldots,$$

of orders k and m respectively. Choosing $m+k$ variables α, β, ..., we have $\frac{(m+k)!}{m!\,k!}$ relations of the type

$$\sum_{s,t,\ldots} A_{s,t,\ldots} \frac{\partial(x_s, x_t, \ldots)}{\partial(\alpha, \beta, \ldots)} = \sum_{s,t,\ldots} A'_{s,t,\ldots} \frac{\partial(x'_s, x'_t, \ldots)}{\partial(\alpha, \beta, \ldots)},$$

and the same number of relations of the type

$$\sum_{p,q,r,\ldots} B_{p,q,r,\ldots} \frac{\partial(x_p, x_q, x_r, \ldots)}{\partial(\gamma, \delta, \epsilon, \ldots)} = \sum_{p,q,r,\ldots} B'_{p,q,r,\ldots} \frac{\partial(x'_p, x'_q, x'_r, \ldots)}{\partial(\gamma, \delta, \epsilon, \ldots)}.$$

These may be arranged in conjugate pairs in such a way that the whole set of variables α, β, γ, δ, ... occur in each pair. Multiplying conjugate pairs together, attributing proper signs to each product, and adding, we find from the properties of the minors of a determinant that the coefficient of a term such as $A_{s,t,\ldots}B_{p,q,r,\ldots}$ is zero, unless the quantities $s, t, p, q, r, \ldots$ are all different. When these quantities are different the coefficient is simply the determinant $\frac{\partial(x_s, x_t, \ldots, x_p, x_q, x_r, \ldots)}{\partial(\alpha, \beta, \gamma, \ldots)}$ with a proper sign. When multiplied by $d\alpha\, d\beta\, d\gamma \ldots$ this yields $dx_s dx_t \ldots dx_p dx_q dx_r \ldots$, and it is clear from this that the multiplication of the integral forms by Grassmann's rule is justified.

Taking the case of a transformation from four variables (x, y, z, t) to (x', y', z', t'). If

$$H_x dy\, dz + H_y dz\, dx + H_z dx\, dy + E_x dx\, dt + E_y dy\, dt + E_z dz\, dt$$
$$= H'_x dy' dz' + H'_y dz' dx' + H'_z dx' dy' + E'_x dx' dt' + E'_y dy' dt' + E'_z dz' dt',$$

and

$$E_x dy\,dz + E_y dz\,dx + E_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt$$
$$= E'_x dy'dz' + E'_y dz'dx' + E'_z dx'dy' - H_x dx'dt' - H'_y dy'dt' - H'_z dz'dt',$$

we obtain, on multiplication,

$$(E_x^2 + E_y^2 + E_z^2 - H_x^2 - H_y^2 - H_z^2)\, dx\,dy\,dz\,dt$$
$$= (E_x'^2 + E_y'^2 + E_z'^2 - H_x'^2 - H_y'^2 - H_z'^2)\, dx'dy'dz'dt',$$

while, if either form be multiplied by itself, we obtain

$$(E_x H_x + E_y H_y + E_z H_z)\, dx\,dy\,dz\,dt = (E'_x H'_x + E'_y H'_y + E'_z H'_z) dx'dy'dz'dt'.$$

These equations indicate the invariance of the property that at a surface of discontinuity, or in a spherical wave, the electric force is equal in magnitude to the magnetic force and perpendicular to it.

A space time transformation from the variables (x, y, z, t) to (x', y', z', t') can be used to transform the whole motion in one dynamical system into a corresponding motion in another, as far as the kinematics is concerned, provided the velocities (w_x, w_y, w_z), (w'_x, w'_y, w'_z) of corresponding points are such that the equations

$$dx' = w'_x dt', \quad dy' = w'_y dt', \quad dz' = w'_z dt',$$

are a consequence of the relations

$$dx = w_x dt, \quad dy = w_y dt, \quad dz = w_z dt.$$

This condition is satisfied, if

$$dx' - w'_x dt' = \nu_{11}(dx - w_x dt) + \nu_{12}(dy - w_y dt) + \nu_{13}(dz - w_z dt),$$
$$dy' - w'_y dt' = \nu_{21}(dx - w_x dt) + \nu_{22}(dy - w_y dt) + \nu_{23}(dz - w_z dt),$$
$$dz' - w'_z dt' = \nu_{31}(dx - w_x dt) + \nu_{32}(dy - w_y dt) + \nu_{33}(dz - w_z dt).$$

Multiplying these equations together by Grassmann's rule, we get

$$dx'dy'dz' - w'_x dy'dz'dt' - w'_y dz'dx'dt - w'_z dx'dy'dt'$$
$$= \begin{vmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{vmatrix} (dx\,dy\,dz - w_x dy\,dz\,dt - w_y dz\,dx\,dt - w_z dx\,dy\,dt).$$

This shows that there is an integral invariant of the form

$$\theta(w_x dy\,dz\,dt + w_y dz\,dx\,dt + w_z dx\,dy\,dt - dx\,dy\,dz).$$

This fact has already been used in § 3 and will be required again in § 7.

We shall now obtain some further properties of a general space time transformation.

When an expression has the same form in the dashed letters as in the original ones it will be convenient to say that it is an invariant.

Let us suppose that the bilinear form

$$\begin{aligned} &A\,dx\,\delta x+B\,dy\,\delta y+C\,dz\,\delta z+D\,dt\,\delta t+F\,(dy\,\delta z+dz\,\delta y)\\ &\quad+G\,(dz\,\delta x+dx\,\delta z)+H\,(dx\,\delta y+\delta x\,dy)+U(dx\,\delta t+\delta x\,dt)\\ &\quad+V(dy\,\delta t+dt\,\delta y)+W(dz\,\delta t+dt\,\delta z) \end{aligned} \tag{1}$$

is an invariant. If we multiply it by itself four times according to Grassmann's rule, we obtain the invariant

$$\begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{vmatrix} dx\,dy\,dz\,dt\,\delta x\,\delta y\,\delta z\,\delta t.$$

Putting $dx = \delta x$, ..., and denoting the determinant by Δ, we have the invariant

$$\sqrt{\Delta}\,dx\,dy\,dz\,dt. \tag{2}$$

Now, let $$u_x\,dy\,dz\,dt+u_y\,dz\,dx\,dt+u_z\,dx\,dy\,dt+u_t\,dx\,dy\,dz \tag{3}$$

be an integral invariant of the third order.

Multiplying by (1) and rejecting the factor $\sqrt{\Delta}\,dx\,dy\,dz\,dt$, we obtain an invariant

$$v_x\,dx+v_y\,dy+v_z\,dz+v_t\,dt, \tag{4}$$

where

$$\left.\begin{aligned} \sqrt{\Delta}\,v_x &= Au_x+Hu_y+Gu_z+Uu_t\\ \sqrt{\Delta}\,v_y &= Hu_x+Bu_y+Fu_z+Vu_t\\ \sqrt{\Delta}\,v_z &= Gu_x+Fu_y+Cu_z+Wu_t\\ \sqrt{\Delta}\,v_z &= Uu_x+Vu_y+Wu_z+Du_t \end{aligned}\right\}. \tag{5}$$

Multiplying (3) and (4), we obtain the invariant

$$\begin{aligned} &[Au_x^2+Bu_y^2+Cu_z^2+Du_t^2+2Fu_yu_z+2Gu_zu_x+2Hu_xu_y\\ &\qquad+2Uu_xu_t+2Vu_yu_t+2Wu_zu_t]\,\frac{dx\,dy\,dz\,dt}{\sqrt{\Delta}}; \end{aligned} \tag{6}$$

hence the quantity in the square brackets divided by Δ is an invariant.

Conversely, if we are given an integral form of the first order (4), we may obtain a *reciprocal* integral form of the third order (3), by multiply-

ing (4) by (1) three times and rejecting the factor $dx\,dy\,dz\,dt\,\sqrt{\Delta}$. The relations expressing the quantities u_x in terms of the quantities v_x are exactly the same as those obtained by solving equations (5).

Next, suppose we are given an integral form of the second order

$$B_x dy\,dz + B_y dz\,dx + B_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt, \qquad (7)$$

which is an invariant. Then multiplying it twice by (1), and rejecting the factor $\sqrt{\Delta}\,dx\,dy\,dz\,dt$, we obtain a reciprocal integral invariant

$$D_x dy\,dz + D_y dz\,dx + D_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt, \qquad (8)$$

where

$$\left.\begin{aligned}
\sqrt{\Delta}\,D_x = {} & (BC-F^2)E_x + (FG-CH)E_y + (HF-BG)E_z \\
& + (HW-VG)B_x + (BW-VF)B_y + (FW-CV)B_z \\
& \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
-\sqrt{\Delta}\,H_x = {} & (HW-VG)E_x + (GU-AW)E_y + (AV-HU)E_z \\
& + (AD-U^2)B_x + (HD-UV)B_y + (GD-UW)B_z \\
& \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}\right\}. \qquad (9)$$

The relation between these two invariants is evidently a mutual one.

Multiplying them together, we obtain the integral invariant

$$(E_x D_x + E_y D_y + E_z D_z - H_x B_x - H_y B_y - H_z B_z)\,dx\,dy\,dz\,dt \qquad (10)$$

and the absolute invariant

$$\frac{1}{\Delta}\,[(BC-F^2)E_x^2 + (CA-G^2)E_y^2 + \dots]. \qquad (11)$$

The theory of reciprocal invariants can evidently be extended to the case in which there are n variables, and a bilinear integral form is known to be invariant.

If we suppose that a biquadratic integral form

$$\begin{aligned}
& \kappa_{11} dy\,dz\,\delta y\,\delta z + \kappa_{22} dz\,dx\,\delta z\,\delta x + \kappa_{33} dx\,dy\,\delta x\,\delta y \\
& + \kappa_{44} dx\,dt\,\delta x\,\delta t + \kappa_{55} dy\,dt\,\delta y\,\delta t + \kappa_{66} dz\,dt\,\delta z\,\delta t \\
& + \kappa_{23}(dz\,dx\,\delta x\,\delta y + dx\,dy\,\delta z\,\delta x) + \dots
\end{aligned} \qquad (12)$$

is invariant for a transformation, we may multiply it by itself and obtain the invariant

$$(\kappa_{11}\kappa_{44} + \dots + 2\kappa_{23}\kappa_{14} + \dots)\,dx\,dy\,dz\,dt\,\delta x\,\delta y\,\delta z\,\delta t.$$

Putting $dx = \delta x$, and extracting the square root, we obtain an invariant

$$\sqrt{\Theta}\, dx\, dy\, dz\, dt. \tag{13}$$

Now, let

$$B_x dy\, dz + B_y dz\, dx + B_z dx\, dy + E_x dx\, dt + E_y dy\, dt + E_z dz\, dt$$

be an invariant of the second order. Multiplying it by (12) and rejecting a factor $\sqrt{\Theta}\, dx\, dy\, dz\, dt$, we obtain an invariant

$$D_x \delta y\, \delta z + D_y \delta z\, \delta x + D_z \delta x\, \delta y - H_x \delta x\, \delta t - H_y \delta y\, \delta t - H_z \delta z\, \delta t,$$

where

$$\sqrt{\Theta}\, D_x = \kappa_{11} E_x + \kappa_{12} E_y + \kappa_{13} E_z + \kappa_{14} B_x + \kappa_{15} B_y + \kappa_{16} B_z,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$-\sqrt{\Theta}\, H_x = \kappa_{41} E_x + \kappa_{42} E_y + \kappa_{43} E_z + \kappa_{44} B_x + \kappa_{45} B_y + \kappa_{46} B_z.$$

The relation between the two invariants will be a mutual one if the coefficients κ_{11}, ... are the elements of an orthogonal matrix.

The Invariants of a Spherical Wave Transformation.

Starting from the fundamental invariants

$$d\tau^2 = \lambda^2 [dt^2 - dx^2 - dy^2 - dz^2], \tag{1}$$

$$\lambda^4\, dx\, dy\, dz\, dt, \tag{2}$$

$$A_x dx + A_y dy + A_z dz - \Phi\, dt, \tag{3}$$

$$H_x dy\, dz + H_y dz\, dx + H_z dx\, dy + E_x dx\, dt + E_y dy\, dt + E_z dz\, dt, \tag{4}$$

$$E_x dy\, dz + E_y dz\, dx + E_z dx\, dy - H_x dx\, dt - H_y dy\, dt - H_z dz\, dt, \tag{5}$$

$$\rho w_x dy\, dz\, dt + \rho w_y dz\, dx\, dt + \rho w_z dx\, dy\, dt - \rho\, dx\, dy\, dz, \tag{6}$$

we may obtain a number of others by the methods of multiplication and reciprocation. It will be sufficient to enumerate these if we mention the equations from which they are derived,

(1 and 6) $$\frac{1}{\lambda^2} [\rho w_x dx + \rho w_y dy + \rho w_z dz - \rho\, dt], \tag{7}$$

(5 and 4) $$(E_x^2 + E_y^2 + E_z^2 - H_x^2 - H_y^2 - H_z^2)\, dx\, dy\, dz\, dt, \tag{8}$$

(4 and 4) $$(E_x H_x + E_y H_y + E_z H_z)\, dx\, dy\, dz\, dt, \tag{9}$$

(3 and 6) $$\rho [A_x w_x + A_y w_y + A_z w_z - \Phi]\, dx\, dy\, dz\, dt, \tag{10}$$

(6 and 7) $$\frac{\rho^2}{\lambda^2} (1 - w^2)\, dx\, dy\, dz\, dt, \tag{11}$$

(5 and 7) $$\frac{\rho}{\lambda^2}\left[(E_x-w_zH_y+w_yH_z)\,dy\,dz\,dt+(E_y-w_xH_z+w_zH_x)\,dz\,dx\,dt\right.$$
$$\left.+(E_z-w_yH_x+w_xH_y)\,dx\,dy\,dt-(w_xE_x+w_yE_y+w_zE_z)\,dx\,dy\,dz\right], \quad (12)$$

(12) $$\frac{\rho}{\lambda^4}\left[(E_x+w_yH_z-w_zH_y)\,\delta x+(E_y+w_zH_x-w_xH_z)\,\delta y\right.$$
$$\left.+(E_z+w_xH_y-w_yH_x)\,\delta z-(w_xE_x+w_yE_y+w_zE_z)\,\delta t\right], \quad (13)$$

(13 and 2) $$\rho\,dx\,dy\,dz\,dt\left[(E_x+w_yH_z-w_zH_y)\,\delta x+(E_y+w_zH_x-w_xH_z)\,\delta y\right.$$
$$\left.+(E_z+w_xH_y-w_yH_x)\,\delta z-(w_xE_x+w_yE_y+w_zE_z)\,\delta t\right]. \quad (14)$$

These invariants represent quantities which are of considerable importance in the theory of electrons.* Another invariant which is of some importance is obtained in the following way.

Let†

$$w_1=\frac{w_x}{\sqrt{(1-w^2)}},\quad w_2=\frac{w_y}{\sqrt{(1-w^2)}},\quad w_3=\frac{w_z}{\sqrt{(1-w^2)}},\quad w_4=\frac{1}{\sqrt{(1-w^2)}},$$

$$ds^2=dt^2-dx^2-dy^2-dz^2,$$

so that $$w_1=\frac{dx}{ds},\quad w_2=\frac{dy}{ds},\quad w_3=\frac{dz}{ds},\quad w_4=\frac{dt}{ds}.$$

Then $$\frac{dw_1}{ds}=\frac{\dot{w}_x}{1-w^2}+\frac{w_x(w\dot{w})}{(1-w^2)^2},$$

$$\frac{d^2w_1}{ds^2}=\frac{\ddot{w}_x}{(1-w^2)^{\frac{1}{2}}}+\frac{3\dot{w}_x(w\dot{w})}{(1-w^2)^{\frac{2}{2}}}+\frac{w_x}{(1-w^2)^{\frac{3}{2}}}\left\{w\ddot{w}+\frac{3(w\dot{w})^2}{1-w^2}+\dot{w}^2+\frac{(w\dot{w})^2}{1-w^2}\right\}.$$

Hence

$$\left(\frac{dw_1}{ds}\right)^2+\left(\frac{dw_2}{ds}\right)^2+\left(\frac{dw_3}{ds}\right)^2-\left(\frac{dw_4}{ds}\right)^2$$
$$=\frac{\dot{w}^2}{(1-w^2)^2}+\frac{2(w\dot{w})^2}{(1-w^2)^3}+\frac{w^2(w\dot{w})^2}{(1-w^2)^4}-\frac{(w\dot{w})^2}{(1-w^2)^4}$$
$$=\frac{\dot{w}^2}{(1-w^2)^2}+\frac{(w\dot{w})^2}{(1-w^2)^3}.$$

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* Some of these are already known in the case of Lorentz's transformations. Cf. Planck, *Ann. der Physik*, Bd. 26, p. 1 (1908). Minkowski, *Göttinger Nachrichten* (1908), *Born. Ann. d. Physik*, Bd. 28 (1909).

† Cf. Minkowski, *Gött. Nachr.* (1908).

Thus

$$\frac{d^2w_1}{ds^2}-w_1\left\{\left(\frac{dw_1}{ds}\right)^2+\left(\frac{dw_2}{ds}\right)^2+\left(\frac{dw_3}{ds}\right)^2-\left(\frac{dw_4}{ds}\right)^2\right\}$$

$$=\frac{\ddot{w}_x}{(1-w^2)^{\frac{3}{2}}}+\frac{3\dot{w}_x(w\dot{w})}{(1-w^2)^{\frac{5}{2}}}+\frac{w_x}{(1-w^2)^{\frac{5}{2}}}\left\{(w\ddot{w})+\frac{3(w\dot{w})^2}{1-w^2}\right\}.$$

Now Abraham* has given a formula for the reaction of radiation upon a moving electron, the x component of the reaction being

$$K_x=\frac{2e^2}{3(1-w^2)}\left[\ddot{w}_x+\dot{w}_x\frac{3(w\dot{w})}{1-w^2}+\frac{w_x}{1-w^2}\left\{(w\ddot{w})+\frac{3(w\dot{w})^2}{1-w^2}\right\}\right];$$

accordingly,

$$K_x dt=\frac{2e^2}{3c^2}\left[\frac{d^2w_1}{ds^2}-w_1\left\{\left(\frac{dw_1}{ds}\right)^2+\left(\frac{dw_2}{ds}\right)^2+\left(\frac{dw_3}{ds}\right)^2-\left(\frac{dw_4}{ds}\right)^2\right\}\right]ds$$

$$=K_1 ds.$$

To complete the symmetry of the result, we define quantities K_t and K_4 by the equations

$$K_t dt=\frac{2e^2}{3c^2}\left[\frac{d^2w_4}{ds^2}-w_4\left\{\left(\frac{dw_1}{ds}\right)^2+\left(\frac{dw_2}{ds}\right)^2+\left(\frac{dw_3}{ds}\right)^2-\left(\frac{dw_4}{ds}\right)^2\right\}\right]ds$$

$$=K_4 ds;$$

we then have

$$w_xK_x+w_yK_y+w_zK_z=K_t,$$

$$w_1K_1+w_2K_2+w_3K_3+w_4K_4=0.$$

Laue† has shown recently that Abraham's formula may be derived by means of the principle of relativity. We shall complete this result by showing that

$$(K_x\delta x+K_y\delta y+K_z\delta z-K\,\delta t)\,dt$$

is an integral invariant for the whole group of spherical wave transformations. It will be sufficient to prove this for the case of the transformation

$$x'=\frac{x}{r^2-t^2},\quad y'=\frac{y}{r^2-t^2},\quad z'=\frac{z}{r^2-t^2},\quad t'=\frac{t}{r^2-t^2}.$$

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* *Theorie der Elektricität*, Vol. II, p. 123.

† *Ann. d. Phys.*, Bd. 28, p. 436 (1908).

Now, if $$ds' = \frac{ds}{r^2}, \qquad \rho^2 = r^2 - t^2,$$

$$\frac{dx'}{ds'} = \left(1 - \frac{2x^2}{\rho^2}\right)\frac{dx}{ds} - \frac{2xy}{\rho^2}\frac{dy}{ds} - \frac{2xz}{\rho^2}\frac{dz}{ds} + \frac{2xt}{\rho^2}\frac{dt}{ds},$$

$$\frac{d^2x'}{ds'^2} = (\rho^2 - 2x^2)\frac{d^2x}{ds^2} - 2xy\frac{d^2y}{ds^2} - 2xz\frac{d^2z}{ds^2} + 2xt\frac{d^2t}{ds^2}$$

$$+ 2x - 2\frac{dx}{ds}\rho\frac{d\rho}{ds} + 4x\left(\frac{d\rho}{ds}\right)^2,$$

$$\frac{d^3x'}{ds'^3}ds' = ds'\left[(\rho^2 - 2x^2)\frac{d^3x}{ds^3} - 2xy\frac{d^3y}{ds^3} - 2xz\frac{d^3z}{ds^3} + 2xt\frac{d^3t}{ds^3}\right.$$

$$\left. - 4r\frac{d^2r}{ds^2}\frac{dx}{ds} + 8x\frac{dr}{ds}\frac{d^2r}{ds^2}\right].$$

Hence

$$\left[\frac{d^3x'}{ds'^3} - \frac{dx'}{ds'}\left\{\left(\frac{d^2x'}{ds'^2}\right)^2 + \left(\frac{d^2y'}{ds'^2}\right)^2 + \left(\frac{d^2z'}{ds'^2}\right)^2 - \left(\frac{d^2t'}{ds'^2}\right)^2\right\}\right]ds$$

$$= ds\left[(\rho^2 - 2x^2)\left\{\frac{d^3x}{ds^3} - \frac{dx}{ds}\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 - \left(\frac{d^2t}{ds^2}\right)^2\right]\right\}\right.$$

$$- 2xy\left\{\frac{d^3y}{ds^3} - \frac{dy}{ds}\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 - \left(\frac{d^2t}{ds^2}\right)^2\right]\right\}$$

$$- 2xz\left\{\frac{d^3z}{ds^3} - \frac{dz}{ds}\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 - \left(\frac{d^2t}{ds^2}\right)^2\right]\right\}$$

$$\left. + 2xt\left\{\frac{d^3t}{ds^3} - \frac{dt}{ds}\left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 - \left(\frac{d^2t}{ds^2}\right)^2\right]\right\}\right].$$

Now $$\frac{\partial x}{\partial x'} = \rho^2 - 2x^2, \qquad \frac{\partial y}{\partial x'} = -2xy, \qquad \ldots;$$

therefore

$$(K'_x\delta x' + K'_y\delta y' + K'_z\delta z - K'_t\delta t')\,dt' = (K_x\delta x + K_y\delta y + K_z\delta z - K_t\delta t)\,dt.$$

The Electrodynamical Equations for Ponderable Bodies.

We shall consider a system of electrodynamical equations for moving ponderable bodies of the type used by Minkowski,* Einstein and Laub.†

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* *Göttinger Nachrichten* (1908).

† *Annalen der Phys.*, Bd. 26 (1908).

These equations are of the form

$$\left.\begin{aligned} \text{curl } H &= \frac{\partial D}{\partial t}+s, \qquad \text{div } D = \rho \\ \text{curl } E &= -\frac{\partial B}{\partial t}, \qquad \text{div } B = 0 \end{aligned}\right\} \tag{I}$$

where the vectors E, H, D, B denote the electric force, magnetic force, electric displacement, and magnetic induction respectively, s denotes the current and ρ the volume density of electricity. The equations differ from those used by Lorentz by the fact that the vector $H-[Pw]$ occurring in Lorentz's equations is replaced here by the vector H.* It should be remarked that Frank† has obtained Minkowski's equations by a process of averaging in the case of non-magnetic bodies.

We shall suppose that the electric polarisation P and the magnetic polarisation Q are connected with D, E, B, and H by the formulæ

$$P = D-E, \qquad Q = B-H.$$

The electrodynamical equations (I) can be replaced by the two integral equations

$$\iint (B_x\,dy\,dz+B_y\,dz\,dx+B_z\,dx\,dy+E_x\,dx\,dt+E_y\,dy\,dt+E_z\,dz\,dt) = 0, \tag{II}$$

$$\iint (D_x\,dy\,dz+D_y\,dz\,dx+D_z\,dx\,dy-H_x\,dx\,dt-H_y\,dy\,dt-H_z\,dz\,dt)$$
$$= -\iiint (s_x\,dy\,dz\,dt+s_y\,dz\,dx\,dt+s_z\,dx\,dy\,dt-\rho\,dx\,dy\,dz). \tag{III}$$

These equations are unaltered in form by a transformation from (x, y, z, t) to (x', y', z', t') if

$$B_x\,dy\,dz+B_y\,dz\,dx+B_z\,dx\,dy+E_x\,dx\,dt+E_y\,dy\,dt+E_z\,dz\,dt$$
$$= \theta[B'_x\,dy'\,dz'+B'_y\,dz'\,dx'+B'_z\,dx'\,dy'+E'_x\,dx'\,dt'+E'_y\,dy'\,dt'+E_z\,dz'\,dt'], \tag{IV}$$

$$D_x\,dy\,dz+D_y\,dz\,dx+D_z\,dx\,dy-H_x\,dx\,dt-H_y\,dy\,dt-H_z\,dz\,dt$$
$$= \phi[D'_x\,dy'\,dz'+D'_y\,dz'\,dx'+D'_z\,dx'\,dy'-H'_x\,dx'\,dt'-H'_y\,dy'\,dt'-H'_z\,dz'\,dt'], \tag{V}$$

$$s_x\,dy\,dz\,dt+s_y\,dz\,dx\,dt+s_z\,dx\,dy\,dt-\rho\,dx\,dy\,dz$$
$$= \phi[s'_x\,dy'\,dz'\,dt'+s'_y\,dz'\,dx'\,dt'+s'_z\,dx'\,dy'\,dt'-\rho'\,dx'\,dy'\,dz'], \tag{VI}$$

where θ and ϕ are constants.

* This simply means that a different definition is adopted for H, the object being to retain the symmetry of the equations.

† *Ann. d. Phys.*, Bd. 27, p. 1059 (1908).

The formulæ of transformation of the type used by Einsten and Laub in the case of Lorentz's transformation are obtained by putting $\theta = \phi = 1$. If we use a spherical wave transformation with positive Jacobian, we can deduce the relation

$$E_x dy\,dz + E_y dz\,dx + E_z dx\,dy - B_x dx\,dt - B_y dy\,dt - B_z dz\,dt$$
$$= E'_x dy'dz' + E'_y dz'dx' + E'_z dx'dy' - B'_x dx'dt' - B'_y dy'dt' - B'_z dz'dt', \quad \text{(VII)}$$

and this, combined with (V), gives

$$P_x dy\,dz + P_y dz\,dx + P_z dx\,dy - Q_x dx\,dt - Q_y dy\,dt - Q_z dz\,dt$$
$$= P'_x dy'dz' + P'_y dz'dx' + P'_z dx'dy' - Q'_x dx'dt' - Q'_y dy'dt' - Q'_z dz'dt', \quad \text{(VIII)}$$

which enables us to obtain the formulæ of transformation of the polarisation vectors.

The constitutive relations are obtained by Einstein and Laub by assuming that all the bodies in the dashed system of coordinates are at rest, and that in this system

$$D' = \epsilon G', \qquad B' = \mu H', \qquad s' = \sigma E'.$$

This gives a set of constitutive relations for the case in which a system of bodies are moving with constant velocity w.

The constitutive relations that are obtained in this way may be written in the form *

$$\left.\begin{aligned} D + [wH] &= \epsilon \{E + [wB]\} \\ B - [wE] &= \mu \{H - [wD]\} \\ \frac{s_w - |w|\rho}{\sqrt{(1-w^2)}} &= \sigma \{E + [wB]\}_w \\ s_{\bar{w}} &= \frac{\sigma \{E + [wB]\}_{\bar{w}}}{\sqrt{(1-w^2)}} \end{aligned}\right\}, \quad \text{(IX)}$$

where the suffix w denotes that the component in the direction of w, $\bar{w}$, a component in a direction perpendicular to w is to be taken.

The first two equations are seen to be invariant for the group of spherical wave tranformations when we obtain them in the following way. The expression

$$\frac{\lambda [dt - w_x dx - w_y dy - w_z dz]}{\sqrt{(1-w^2)}} \quad \text{(X)}$$

* Minkowski, *Göttinger Nachrichten* (1908); Einstein and Laub, *Ann. d. Phys.* (1908).

is seen from (1), § 6, to be an invariant. If we multiply this by the invariants

$$B_x dy\,dz + B_y dz\,dx + B_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt,$$

$$D_x dy\,dz + D_y dz\,dx + D_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt,$$

$$H_x dy\,dz + H_y dz\,dx + H_z dx\,dy + D_x dx\,dt + D_y dy\,dt + D_z dz\,dt,$$

$$E_x dy\,dz + E_y dz\,dx + E_z dx\,dy - B_x dx\,dt - B_y dy\,dt - B_z dz\,dt,$$

we obtain the invariants

$$\frac{\lambda}{\sqrt{(1-w^2)}}\,[(B_x - w_y E_z + w_z E_y)\,dy\,dz\,dt + (B_y - w_z E_x + w_x E_z)\,dz\,dx\,dt + (B_z - w_x E_y + w_y E_x)\,dx\,dy\,dt - (w_x B_x + w_y B_y + w_z B_z)\,dx\,dy\,dz], \quad (1)$$

$$\frac{\lambda}{\sqrt{(1-w^2)}}\,[(D_x + w_y H_z - w_z H_y)\,dy\,dz\,dt + (D_y + w_z H_x - w_x H_z)\,dz\,dx\,dt + (D_z + w_x H_y - w_y H_x)\,dx\,dy\,dt - (w_x D_x + w_y D_y + w_z D_z)\,dx\,dy\,dz], \quad (2)$$

$$\frac{\lambda}{\sqrt{(1-w^2)}}\,[(H_x - w_y D_z + w_z D_y)\,dy\,dz\,dt + (H_y - w_z D_x + w_x D_z)\,dz\,dx\,dt + (H_z - w_x D_y + w_y D_x)\,dx\,dy\,dt - (w_x H_x + w_y H_y + w_z H_z)\,dx\,dy\,dz], \quad (3)$$

$$\frac{\lambda}{\sqrt{(1-w^2)}}\,[(E_x + w_y B_z - w_z B_y)\,dy\,dz\,dt + (E_y + w_z B_x - w_x B_z)\,dz\,dx\,dt + (E_z + w_x B_y - w_y B_x)\,dx\,dy\,dt - (w_x E_x + w_y E_y + w_z E_z)\,dx\,dy\,dz], \quad (4)$$

and the constitutive relations are obtained by making the first of these equal to μ times the third, and the second equal to ϵ times the fourth, where ϵ and μ are invariants. These, however, are not the only constitutive relations which remain invariant,* for we may obtain the two integral invariants

$$\left(\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}\right) dy\,dz\,dt + \left(\frac{\partial E_y}{\partial t} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x}\right) dz\,dx\,dt + \left(\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}\right) dx\,dy\,dt + \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx\,dy\,dz\,; \quad (5)$$

$$\left(\frac{\partial H_x}{\partial t} + \frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z}\right) dy\,dz\,dt + \left(\frac{\partial H_y}{\partial t} + \frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x}\right) dz\,dx\,dt + \left(\frac{\partial H_z}{\partial t} + \frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y}\right) dx\,dy\,dt - \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right) dx\,dy\,dz. \quad (6)$$

A set of constitutive relations given by two linear relations between

* This was pointed out to me by Mr. Hassé,

these six integral forms of the third order is invariant for the group of spherical wave transformations.

To obtain the other constitutive relations, we construct an integral form of the third order reciprocal to (X) and write it in the form

$$\frac{\rho_0}{1-w^2}\,[w_x\,dy\,dz\,dt+w_y\,dz\,dx\,dt+w_z\,dx\,dy\,dt-dx\,dy\,dz], \tag{7}$$

where ρ_0 is a quantity which will be determined presently.

Subtracting this from the invariant

$$s_x\,dy\,dz\,dt+s_y\,dz\,dx\,dt+s_z\,dx\,dy\,dt-\rho\,dx\,dy\,dz, \tag{8}$$

we may obtain a set of constitutive relations by identifying the resulting invariant with

$$\frac{\sigma\lambda}{\sqrt{(1-w^2)}}\,[(E_x+w_yB_z-w_zB_y)\,dy\,dz\,dt+(E_y+w_zB_x-w_xB_z)\,dz\,dx\,dt$$
$$+(E_z+w_xB_y-w_yB_x)\,dx\,dy\,dt-(E_xw_x+E_yw_y-E_zw_z)\,dx\,dy\,dz].$$

If we multiply the first coefficient in this invariant by w_x, the second by w_y, the third by w_z, the fourth by 1, and add, the result is zero. The same must hold in the case of the invariant to which it is equated; therefore

$$s_xw_x+s_yw_y+s_zw_z-\rho-\frac{\rho_0}{1-w^2}\,[w_x^2+w_y^2+w_z^2-1]=0$$

or

$$\rho=\rho_0+(sw).$$

Hence we have the constitutive relations

$$s-\frac{\rho-(sw)}{1-w^2}\,w=\frac{\sigma\lambda}{\sqrt{(1-w^2)}}\,\{E+[wB]\},$$

which agree with those obtained by Minkowski, Einstein and Laub, in the case when $\lambda=1$.

We shall now show that it is possible to construct a set of constitutive relations which are invariant for a much wider class of transformations.

Let the transformation be biuniform within a certain domain of values of (x, y, z, t), and such that the bilinear form

$$A\,dx\,\delta x+B\,dy\,\delta y+C\,dz\,\delta z+D\,dt\,\delta t+F(dy\,\delta z+dz\,\delta y)$$
$$+G(dz\,\delta x+\delta z\,dx)+H(dx\,\delta y+dy\,\delta x)+U(dx\,\delta t+\delta x\,dt)$$
$$+V(dy\,\delta t+\delta y\,dt)+W(dz\,\delta t+\delta z\,dt), \tag{1}$$

is an invariant. When we put $dx = \delta x, \ldots$ this implies that a certain quadratic form is an invariant.

Let us suppose that the constitutive relations connecting B_x, B_y, B_z, E_x, E_y, E_z with H_x, H_y, H_z, D_x, D_y, D_z, are given by the circumstance that the invariant reciprocal to

$$D_x dy\,dz + D_y dz\,dx + D_z dx\,dy - H_x dx\,dt - H_y dy\,dt - H_z dz\,dt$$

is an invariant multiple of

$$B_x dy\,dz + B_y dz\,dx + B_z dx\,dy + E_x dx\,dt + E_y dy\,dt + E_z dz\,dt.$$

This assumption preserves the analogy with the electron equations where the two fundamental integral invariants of the second order are reciprocals with regard to the quadratic form

$$\lambda^2[dx^2 + dy^2 + dz^2 - dt^2].$$

In the present case the relations between the two sets of vectors are of the type

$$\left.\begin{aligned}
\kappa\sqrt{\Delta}\,B_x &= -(BC-F^2)H_x - (FG-CH)H_y - (HF-BG)H_z \\
&\quad + (HW-VG)D_x + (BW-VF)D_y + (FW-CV)D_z \\
\kappa\sqrt{\Delta}\,E_x &= -(HW-VG)H_x - (GU-AW)H_y - (AV-HU)H_z \\
&\quad + (AD-U^2)D_x + (HD-UV)D_y + (GD-UW)D_z \\
\frac{1}{\kappa}\sqrt{\Delta}\,D_x &= (BC-F^2)E_x + (FG-CH)E_y + (HF-BG)E_z \\
&\quad + (HW-VG)B_x + (BW-VF)B_y + (FW-CV)B_z \\
-\frac{1}{\kappa}\sqrt{\Delta}\,H_x &= (HW-VG)E_x + (GU-AW)E_y + (AV-HU)E_z \\
&\quad + (AD-U^2)B_x + (HD-UV)B_y + (GD-UW)B_z
\end{aligned}\right\}, \qquad (2)$$

where Δ denotes the determinant

$$\begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{vmatrix}.$$

To obtain the other constitutive relations we start with the assumption that there is an integral invariant of the type*

$$\theta[w_x dy\,dz\,dt + w_y dz\,dx\,dt + w_z dx\,dy\,dt - dx\,dy\,dz].$$

* This assumption is justified by the remark made on p. 249.

From this we may obtain the reciprocal invariant

$$v_x dx+v_y dy+v_z dz+v_t dt,$$

where

$$\left.\begin{aligned}\sqrt{\Delta}\, v_x &= \theta(Aw_x+Hw_y+Gw_z+U)\\ \sqrt{\Delta}\, v_y &= \theta(Hw_x+Bw_y+Fw_z+V)\\ \sqrt{\Delta}\, v_z &= \theta(Gw_x+Fw_y+Cw_z+W)\\ \sqrt{\Delta}\, v_t &= \theta(Uw_x+Vw_y+Ww_z+D)\end{aligned}\right\}. \tag{3}$$

Multiplying these and rejecting the invariant factor $\sqrt{\Delta}\, dx\, dy\, dz\, dt$, we obtain the invariant

$$\frac{\theta^2}{\Delta}\,[Aw_x^2+Bw_y^2+Cw_z^2+D+2Fw_yw_z+2Gw_zw_x+2Hw_xw_y \\ +2Uw_x+2Vw_y+2Ww_z] = \Theta. \tag{4}$$

Multiplying

$$D_x dy\, dz+D_y dz\, dx+D_z dx\, dy+H_x dx\, dt-H_y dy\, dt-H_z dz\, dt$$

by

$$v_x dx+v_y dy+v_z dz+v_t dt,$$

we obtain the invariant

$$(v_tD_x-v_yH_z+v_zH_y)dy\, dz\, dt+(v_tD_y-v_zH_x+v_xH_z)dx\, dy\, dt \\ +(v_tD_z-v_xH_y+v_yH_x)dx\, dy\, dt+(v_xD_x+v_yD_y+v_zD_z)dx\, dy\, dz. \tag{5}$$

We now assume that

$$s_x dy\, dz\, dt+s_y dz\, dx\, dt+s_z dx\, dy\, dz-\rho\, dx\, dy\, dz \\ -\theta[w_x dy\, dz\, dt+w_y dz\, dx\, dt+w_z dx\, dy\, dt-dx\, dy\, dz]$$

is an invariant multiple of this invariant. This gives the relation

$$s_xv_x+s_yv_y+s_zv_z+\rho v_t-\theta[v_xw_x+v_yw_y+v_zw_z+v_t] = 0,$$

or

$$s_xv_x+s_yv_y+s_zv_z+\rho v_t = \Theta\sqrt{\Delta}. \tag{6}$$

The constitutive relations are of the type

$$\left.\begin{aligned}s_x-\theta w_x &= \sigma(v_tD_x-v_yH_z+v_zH_y)\\ s_y-\theta w_y &= \sigma(v_tD_y-v_zH_x+v_xH_z)\\ s_z-\theta w_z &= \sigma(v_tD_z-v_xH_y+v_yH_x)\end{aligned}\right\}, \tag{7}$$

and can be expressed in terms of E and B by means of the relations connecting these quantities with D and H.

With these constitutive relations the equations of electrodynamics are invariant for a very large class of space time transformations. The relations simplify considerably when the quadratic form is of a simple type.

The vanishing of the quadratic form may be supposed to represent the condition that two neighbouring particles are in a position to act on one another.

We shall now verify that in a particular case the constitutive relations obtained in this way agree with the ones obtained by Minkowski, Einstein, and Laub.

Starting with a system at rest, we assume that the invariant quadratic form is

$$dx^2+dy^2+dz^2-\epsilon\mu\, dt^2.$$

This gives

$$\Delta = -\epsilon\mu,$$

and if we take

$$\kappa = \sqrt{\left(-\frac{\epsilon}{\mu}\right)},$$

the relations (2) take the form

$$B = \mu H, \qquad E = \epsilon^{-1} D.$$

Now make the Lorentzian transformation

$$x = \frac{x'-ut'}{\sqrt{(1-u^2)}}, \quad y = y', \quad z = z', \quad t = \frac{t'-ux'}{\sqrt{(1-u^2)}};$$

the quadratic form then becomes

$$dx'^2\left[\frac{1-\epsilon\mu\,.\,u^2}{1-u^2}\right]+dy'^2+dz'^2+dt'^2\frac{u^2-\epsilon\mu}{1-u^2}-2u\frac{1-\epsilon\mu}{1-u^2}\,dx'dt'.$$

We again have $\Delta = -\epsilon\mu$, and the relations (2) take the form

$$B_x = \mu H_x,$$

$$B_y = \frac{\epsilon\mu-u^2}{\epsilon(1-u^2)}\,H_y+u\,\frac{\epsilon\mu-1}{\epsilon(1-u^2)}\,D_z,$$

$$B_z = \frac{\epsilon\mu-u^2}{\epsilon(1-u^2)}\,H_z-u\,\frac{\epsilon\mu-1}{\epsilon(1-u^2)}\,D_y,$$

$$E_x = \frac{1}{\epsilon}\,D_x,$$

$$E_y = \frac{1-u^2\epsilon\mu}{\epsilon(1-u^2)}\,D_y+u\,\frac{\epsilon\mu-1}{\epsilon(1-u^2)}\,H_z,$$

$$E_z = \frac{1-u^2\epsilon\mu}{\epsilon(1-u^2)}\,D_z-u\,\frac{\epsilon\mu-1}{\epsilon(1-u^2)}\,H_y.$$

These are equivalent to the relations

$$B_x = \mu H_x,$$
$$B_y + uE_z = \mu[H_y + uD_z],$$
$$B_z - uE_y = \mu[H_z - uD_y],$$
$$E_x = \epsilon^{-1} D_x,$$
$$E_y - uB_z = \epsilon^{-1}[D_y - uH_z],$$
$$E_z + uB_y = \epsilon^{-1}[D_y + uH_y],$$

obtained by Minkowski, Einstein, and Laub.

It is also easy to verify that the constitutive relations (7) take the form

$$\frac{s_x - u\rho}{\sqrt{(1-u^2)}} = \sigma E_x, \qquad s_y = \sigma \frac{E_y - uB_z}{\sqrt{(1-u^2)}}, \qquad s_z = \sigma \frac{E_z - uB_y}{\sqrt{(1-u^2)}},$$

given by Minkowski, Einstein, and Laub.

We have thus shown that the scheme of constitutive relations indicated by the invariance of a quadratic form agrees with the known scheme of relations in particular cases, and is invariant for any relevant space time transformation which is biuniform in a certain domain.

These relations are not the most general possible, and so the configuration and state of motion under consideration is of a special type, but the relations are sufficiently general for most ordinary purposes.

[*Note added October 8th*, 1909.]

Spherical wave transformations are not the only ones which can be used to transform problems occurring in the theory of electrons, for there are large classes of transformations which can be applied to particular problems, but cannot be applied to an arbitrary problem. The equations of transformation in this case involve the magnitudes of the electric and magnetic forces occurring in the particular problem.

Let us suppose that the electrodynamical field in a particular problem is of such a nature that the components (E_x, E_y, E_z), (H_x, H_y, H_z) of the electric and magnetic forces are connected by the relations

$$E_x H_x + E_y H_y + E_z H_z = 0,$$
$$E_x^2 + E_y^2 + E_z^2 = H_x^2 + H_y^2 + H_z^2,$$

and let (s_x, s_y, s_z) be the components of Poynting's vector.

Then, if a_x, a_y, a_z, a, s are five functions of x, y, z, t satisfying the relations

$$a_x s_x + a_y s_y + a_z s_z = as,$$

$$s_x^2 + s_y^2 + s_z^2 = s^2,$$

a transformation of coordinates for which

$$dx^2 + dy^2 + dz^2 - dt^2 + (a_x dx + a_y dy + a_z dz - a\,dt)(s_x dx + s_y dy + s_z dz - s\,dt)$$
$$= \lambda(dx'^2 + dy'^2 + dz'^2 - dt'^2),$$

λ being a function of x', y', z', t', is in general suitable for the transformation of the problem in question. The above equation implies that, if the velocity of radiation is represented by unity in the first system of coordinates, it is also represented by unity in the second system.

[*March 15th*, 1910.—

The relation $a_x s_x + a_y s_y + a_z s_z = as$ can be omitted.]

SYMBOLICAL EXPRESSIONS FOR THE ELIMINANT OF TWO BINARY QUANTICS

By A. L. DIXON.

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THE object of this communication is to establish the following propositions.

I (*a*). Two quantics, each of the n-th degree, are represented, in the Aronhold-Clebsch symbolism, by

$$(ax+a'x')^n, \qquad (bx+b'x')^n,$$

the products of the a's and a''s denoting coefficients of one quantic, and of the b's and b''s of the other, and equivalent symbols being distinguished by suffixes.

Two sets of n symbols a_r, a'_r and b_r, b'_r are taken, and A_r, B_r, Θ_r are defined by

$$A_r \equiv b_r \frac{\partial}{\partial a_r} + b'_r \frac{\partial}{\partial a'_r}, \qquad B_r \equiv a_r \frac{\partial}{\partial b_r} + a'_r \frac{\partial}{\partial b'_r},$$

$$\Theta_r \equiv 1 + \frac{A_r B_r}{(n-1)^2} + \frac{A_r^2 B_r^2}{(n-1)^2 (n-2)^2} + \ldots + \frac{A_r^{n-1} B_r^{n-1}}{\{(n-1)!\}^2}.$$

(i) Then the eliminant of the two quantics is given by

$$\prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r)\, \Theta_1 \Theta_2 \ldots \Theta_n \prod (a_r a'_s - a'_r a_s) \prod (b_r b'_s - b'_r b_s) \qquad (r \leqslant n,\ s \leqslant n,\ r \neq s),$$

in which any term

$$A_\alpha^l B_\alpha^l A_\beta^m B_\beta^m \ldots \prod (a_r a'_s - a'_r a_s) \prod (b_r b'_s - b'_r b_s)$$

which occurs is to be understood as denoting

$$[A_\alpha^l A_\beta^m \ldots \prod (a_r a'_s - a'_r a_s)][B_\alpha^l B_\beta^m \ldots \prod (b_r b'_s - b'_r b_s)].$$

(ii) The condition that the quantics should have $t+1$ common roots is given by the vanishing of the covariant

$$\prod_{r=1}^{r=n-t} (a_r b'_r - a'_r b_r)\, \Theta_1 \Theta_2 \ldots \Theta_{n-t} \prod (a_r a'_s - a'_r a_s) \prod (b_r b'_s - b'_r b_s) \times \prod_{r=1}^{r=n-t} (a_r x + a'_r x')^t \prod_{r=1}^{r=n-t} (b_r x + b'_r x')^t \qquad (r \leqslant n-t,\ s \leqslant n-t,\ r \neq s).$$

I (*b*). (i) The discriminant of a quantic of the $(n+1)$-th degree

$$(ax+a'x')^{n+1} \equiv (bx+b'x')^{n+1}$$

is given by

$$\prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r)^2 \Theta_1 \Theta_2 \dots \Theta_n \prod (a_r a'_s - a'_r a_s) \prod (b_r b'_s - b'_r b_s);$$

(ii) and, similarly, the condition that $a(ax+a'x')^n$ and $a'(ax+a'x')^n$ should have $t+1$ common roots is got from I (*a*) (ii) by squaring the first factor.

Results I are obtained by transformation of the formula for the eliminant given by me in a paper entitled "On a Form of the Eliminant of Two Quantics" (*Proceedings*, Ser. 2, Vol. 6, p. 469), this formula being itself a transformation of Bézout's determinant (*loc. cit.*, p. 475, § 7).

II (*a*). (i) For two quantics of the same or different degrees

$$(ax+a'x')^m \quad \text{and} \quad (bx+b'x')^n,$$

the eliminant is given by*

$$\prod \left(\frac{\partial}{\partial a_p}\frac{\partial}{\partial a'_q} - \frac{\partial}{\partial a'_p}\frac{\partial}{\partial a_q}\right) \prod \left(\frac{\partial}{\partial b_r}\frac{\partial}{\partial b'_s} - \frac{\partial}{\partial b'_r}\frac{\partial}{\partial b_s}\right)$$

$$\times \prod (a_p a'_q - a'_p a_q) \prod (b_r b'_s - b'_r b_s) \prod (a_p b'_r - a'_p b_r)$$

$$(p \leqslant n,\ q \leqslant n,\ p \neq q;\ r \leqslant m,\ s \leqslant m,\ r \neq s).$$

(ii) The covariant whose vanishing is the condition that the quantics should have $t+1$ common roots is got from this by taking

$$p \leqslant n-t,\ q \leqslant n-t,\ p \neq q;\quad r \leqslant m-t,\ s \leqslant m-t,\ r \neq s$$

and completing the operand by the addition of the factor

$$\prod (a_p x + a'_p x')^t \prod (b_r x + b'_r x')^t.$$

II (*b*). (i) The discriminant of a quantic of the $(n+1)$-th degree

$$(ax+a'x')^{n+1} \equiv (bx+b'x')^{n+1}$$

is got from II (*a*) (i) by taking $m = n$ and prefixing the factor

$$\prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r);$$

and (ii) the condition that $a(ax+a'x')^n$ and $a'(ax+a'x')^n$ should have $t+1$ common roots is got from II (*a*) (ii) by prefixing the factor

$$\prod_{r=1}^{r=n-t} (a_r b'_r - a'_r b_r).$$

* For an equivalent form, see later, § 11, p. 276.

III. (i) The condition that $\phi(x, x')$ of degree m, and $\psi(x, x')$ of degree n, should have $t+1$ common roots is the vanishing of the covariant

$$\Pi\phi\left(\frac{d}{dx_p}, \frac{d}{dx'_p}\right)\Pi\psi\left(\frac{d}{dy_r}, \frac{d}{dy'_r}\right)$$

$$\times\Pi(x_p x'_q - x'_p x_q)\,\Pi(y_r y'_s - y'_r y_s)\,\Pi(x_p y'_r - x'_p y_r)\,\Pi(x x'_p - x' x_p)^t\,\Pi(x y'_r - x' y_r)^t$$

$$(p, q = 1, 2, 3, \ldots, n-t,\ p \neq q\ ;\ r, s = 1, 2, 3, \ldots, m-t,\ r \neq s),$$

where x_p, x'_p, y_r, y'_r are arbitrary quantities.

(ii) The eliminant is got from this by putting $t = 0$.

In both cases there is an extraneous factor

$$\Pi(x_p x'_q - x'_p x_q)\,\Pi(y_r y'_s - y'_r y_s).$$

Results II and III are obtained by transformation of the Euler-Sylvester determinant.

1. In order to make the scope of the method clearer, I will begin by finding a symbolical form for the eliminant of two quadratics. Let

$$\phi(x) = a_x^2 = (ax + a'x')^2, \qquad \psi(x) = b_x^2 = (bx + b'x')^2,$$

so that a^2, $2aa'$, a'^2 denote coefficients of $\phi(x)$ and b^2, $2bb'$, b'^2 coefficients of $\psi(x)$.

Then the eliminant E* is given by the equation

$$XYE = \begin{vmatrix} \dfrac{\phi(x_1)\psi(y_1) - \phi(y_1)\psi(x_1)}{x_1 y'_1 - x'_1 y_1}, & \dfrac{\phi(x_2)\psi(y_1) - \phi(y_1)\psi(x_2)}{x_2 y'_1 - x'_2 y_1} \\ \dfrac{\phi(x_1)\psi(y_2) - \phi(y_2)\psi(x_1)}{x_1 y'_2 - x'_1 y_2}, & \dfrac{\phi(x_2)\psi(y_2) - \phi(y_2)\psi(x_2)}{x_2 y'_2 - x'_2 y_2} \end{vmatrix},$$

where X is $x_1 x'_2 - x_2 x'_1$, Y is $y_1 y'_2 - y_2 y'_1$.

Substitute for $\phi(x)$ and $\psi(x)$ their symbolical forms, giving both the a's and b's the same suffix as the x's.

Then, since

$$(ax + a'x')^2 (by + b'y')^2 - (ay + a'y')^2 (bx + b'x')^2$$

$$= (ab' - a'b)(xy' - x'y)\{(ax + a'x')(by + b'y') + (ay + a'y')(bx + b'x')\},$$

we may take out $(a_1 b'_1 - a'_1 b_1)(a_2 b'_2 - a'_2 b_2)$ as a factor and obtain the

* *Proceedings*, Ser. 2, Vol. 6, pp. 469, 475.

result that $XYE/(a_1b_1'-a_1'b_1)(a_2b_2'-a_2'b_2)$ is equal to

$$\begin{vmatrix} (a_1x_1+a_1'x_1')(b_1y_1+b_1'y_1')+(a_1y_1+a_1'y_1')(b_1x_1+b_1'x_1'), & \\ \qquad (a_2x_2+a_2'x_2')(b_2y_1+b_2'y_1')+(a_2y_1+a_2'y_1')(b_2x_2+b_2'x_2') & \\ (a_1x_1+a_1'x_1')(b_1y_2+b_1'y_2')+(a_1y_2+a_1'y_2')(b_1x_1+b_1'x_1'), & \\ \qquad (a_2x_2+a_2'x_2')(b_2y_2+b_2'y_2')+(a_2y_2+a_2'y_2')(b_2x_2+b_2'x_2') & \end{vmatrix}.$$

Now split this determinant up into four others by taking corresponding elements of columns. A specimen one of the four will be

$$(a_1x_1+a_1'x_1')(a_2x_2+a_2'x_2')\begin{vmatrix} b_1y_1+b_1'y_1', & b_2y_1+b_2'y_1' \\ b_1y_2+b_1'y_2', & b_2y_2+b_2'y_2' \end{vmatrix}.$$

In the relation thus obtained interchange x_1, x_1' with x_2, x_2', which will change the sign of X, and subtract the second result from the first. The left-hand side will be doubled, and on the right-hand side we have four terms, the term corresponding to the one given above being

$$\begin{vmatrix} a_1x_1+a_1'x_1', & a_2x_1+a_2'x_1' \\ a_1x_2+a_1'x_2', & a_2x_2+a_2'x_2' \end{vmatrix}\begin{vmatrix} b_1y_1+b_1'y_1', & b_2y_1+b_2'y_1' \\ b_1y_2+b_1'y_2', & b_2y_2+b_2'y_2' \end{vmatrix},$$

which is

$$\begin{vmatrix} a_1, & a_1' \\ a_2, & a_2' \end{vmatrix}\begin{vmatrix} x_1, & x_1' \\ x_2, & x_2' \end{vmatrix}\begin{vmatrix} b_1, & b_1' \\ b_2, & b_2' \end{vmatrix}\begin{vmatrix} y_1, & y_1' \\ y_2, & y_2' \end{vmatrix}.$$

Thus, dividing out X and Y, we have the result

$$2E = \begin{vmatrix} a_1, & a_1' \\ b_1, & b_1' \end{vmatrix}\begin{vmatrix} a_2, & a_2' \\ b_2, & b_2' \end{vmatrix}$$

$$\times\left\{\begin{vmatrix} a_1, & a_1' \\ a_2, & a_2' \end{vmatrix}\begin{vmatrix} b_1, & b_1' \\ b_2, & b_2' \end{vmatrix}+\begin{vmatrix} b_1, & b_1' \\ a_2, & a_2' \end{vmatrix}\begin{vmatrix} a_1, & a_1' \\ b_2, & b_2' \end{vmatrix}+\begin{vmatrix} a_1, & a_1' \\ b_2, & b_2' \end{vmatrix}\begin{vmatrix} b_1, & b_1' \\ a_2, & a_2' \end{vmatrix}+\begin{vmatrix} b_1, & b_1' \\ b_2, & b_2' \end{vmatrix}\begin{vmatrix} a_1, & a_1' \\ a_2, & a_2' \end{vmatrix}\right\}$$

or

$$E = (a_1, b_1')(a_2, b_2')\{(a_1, a_2')(b_1, b_2')-(a_1, b_2')(a_2, b_1')\}.$$

2. For the general case of two quantics of degree n put

$$\phi(x) \equiv (ax+a'x')^n, \qquad \psi(x) \equiv (bx+b'x')^n.$$

The eliminant E^* is given by the equation

$$XYE = \Delta,$$

where X is the determinant

$$|\, x_r^{n-1},\ x_r^{n-2}x_r',\ x_r^{n-3}x_r'^2,\ \ldots,\ x_r'^{n-1}\,|,$$

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* *Proceedings*, Ser. 2, Vol. 6, pp. 469, 475.

Y is the determinant

$$| y_r^{n-1},\ y_r^{n-2}y_r,\ y_r^{n-3}y_r'^2,\ \ldots,\ y_r'^{n-1} |,$$

and Δ is the determinant whose elements are

$$\frac{\phi(x_r)\,\psi(y_s)-\phi(y_s)\,\psi(x_r)}{x_r y_s'-x_r' y_s},$$

x_r being the same throughout the r-th column and y_s throughout the s-th row. But, putting in the symbolical expressions for ϕ and ψ, we have

$$\begin{aligned}\phi(x)\,\psi(y)-\phi(y)\,\psi(x) &= (ax+a'x')^n\,(by+b'y')^n-(ay+a'y')^n\,(bx+b'x')^n\\ &= (ab'-a'b)\,(xy'-x'y)\\ &\quad\times\left\{\begin{array}{l}(ax+a'x')^{n-1}\,(by+b'y')^{n-1}\\ +(ax+a'x')^{n-2}\,(by+b'y')^{n-2}\,(ay+a'y')\,(bx+b'x')\\ +(ax+a'x')^{n-3}\,(by+b'y')^{n-3}\,(ay+a'y')^2\,(bx+b'x')^2\\ \quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \qquad\qquad\qquad\qquad\qquad +(ay+a'y')^{n-1}\,(bx+b'x')^{n-1}\end{array}\right\}.\end{aligned}$$

Thus the coefficient of $(ab'-a'b)$ in $[\phi(x)\,\psi(y)-\phi(y)\,\psi(x)]/(xy'-x'y)$ may be written

$$\left\{1+\frac{AB}{(n-1)^2}+\frac{A^2B^2}{(n-1)^2\,(n-2)^2}+\ldots+\frac{A^{n-1}B^{n-1}}{[(n-1)!]^2}\right\}(ax+a'x')^{n-1}\,(by+b'y')^{n-1},$$

where $$A\equiv b\frac{\partial}{\partial a}+b'\frac{\partial}{\partial a'},\qquad B\equiv a\frac{\partial}{\partial b}+a'\frac{\partial}{\partial b'},$$

and A^s operates only on $(ax+a'x')^{n-1}$, and B^s only on $(bx+b'x')^{n-1}$.

Giving the same suffixes to the a's and b's as to the x's, so that a_r, a_r', b_r, b_r' occur only in the r-th column of Δ, we see that we may write

$$\begin{aligned}XYE &= \prod_{r=1}^{r=n}(a_r b_r'-a_r' b_r)\\ &\times\prod_{r=1}^{r=n}\left\{1+\frac{A_rB_r}{(n-1)^2}+\frac{A_r^2B_r^2}{(n-1)^2\,(n-2)^2}+\ldots+\frac{A_r^{n-1}B_r^{n-1}}{[(n-1)!]^2}\right\}\\ &\times(a_1x_1+a_1'x_1')^{n-1}\,(a_2x_2+a_2'x_2')^{n-1}\ \ldots\ (a_nx_n+a_n'x_n')^{n-1}\\ &\times\begin{vmatrix}(b_1y_1+b_1'y_1')^{n-1}, & (b_2y_1+b_2'y_1')^{n-1}, & (b_3y_1+b_3'y_1')^{n-1}, & \ldots, & (b_ny_1+b_n'y_1')^{n-1}\\ (b_1y_2+b_1'y_2')^{n-1}, & (b_2y_2+b_2'y_2')^{n-1}, & (b_3y_2+b_3'y_2')^{n-1}, & \ldots, & (b_ny_2+b_n'y_2')^{n-1}\\ (b_1y_3+b_1'y_3')^{n-1}, & (b_2y_3+b_2'y_3')^{n-1}, & (b_3y_3+b_3'y_3')^{n-1}, & \ldots, & (b_ny_3+b_n'y_3')^{n-1}\\ \quad\ldots & \quad\ldots & \quad\ldots & \quad\ldots & \quad\ldots\\ (b_1y_n+b_1'y_n')^{n-1}, & (b_2y_n+b_2'y_n')^{n-1}, & (b_3y_n+b_3'y_n')^{n-1}, & \ldots, & (b_ny_n+b_n'y_n')^{n-1}\end{vmatrix}.\end{aligned}$$

If we write P for the product of the coefficients in the expansion of $(1+x)^{n-1}$, the determinant just written is equal to

$$P\,Y\,\Pi(b_r b'_s - b_s b'_r).$$

If we interchange (x_1, x'_1) and (x_2, x'_2), we change the sign of X, so that, by adding and subtracting the results obtained by making all possible interchanges, we may replace the single product

$$(a_1 x_1 + a'_1 x'_1)^{n-1}(a_2 x_2 + a'_2 x'_2)^{n-1} \ldots (a_n x_n + a'_n x'_n)^{n-1}$$

by the determinant whose r-th row is

$$|\,(a_1 x_r + a'_1 x'_r)^{n-1},\ (a_2 x_r + a'_2 x'_r)^{n-1},\ \ldots,\ (a_n x_r + a'_n x'_r)^{n-1}\,|,$$

that is to say by $PX\Pi(a_r a'_s - a_s a'_r)$, whilst on the left-hand side we shall get $(n!)XYE$.

Therefore, if, for brevity, we denote the operator

$$1 + \frac{A_r B_r}{(n-1)^2} + \frac{A_r^2 B_r^2}{(n-1)^2 (n-2)^2} + \ldots + \frac{A_r^{n-1} B_r^{n-1}}{[(n-1)!]^2}$$

by Θ_r, we have the result

$$(n!)E = P^2 \prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r)\,\Theta_1\Theta_2\Theta_3 \ldots \Theta_n\,\Pi(a_r a'_s - a'_r a_s)\,\Pi(b_r b'_s - b'_r b_s)$$

in which it is to be understood that any term

$$A_\alpha^l B_\alpha^l A_\beta^m B_\beta^m \ldots \Pi(a_r a'_s - a'_r a_s)\,\Pi(b_r b'_s - b'_r b_s)$$

which occurs is to be taken as denoting

$$[A_\alpha^l A_\beta^m \ldots \Pi(a_r a'_s - a'_r a_s)]\,[B_\alpha^l B_\beta^m \ldots \Pi(b_r b'_s - b'_r b_s)].$$

3. This expression as it stands is a sum of n^n terms, but since all the terms in the expansion of $\Pi\Theta$ which have the same composition and differ only by interchange of suffixes are equivalent, the number of distinct terms is $(2n-1)!/n!\,(n-1)!$, and these are equal in pairs owing to the symmetry of the original expression in the a's and b's.

4. The condition that the quantics should have two common roots is that not only Δ should vanish, but also all its first minors, and similarly the condition for $t+1$ common roots is that the t-th minors of Δ should all vanish. But in this case Δ will vanish identically if we regard a_s, a'_s and b_s, b'_s, for all values of s greater than $n-t$ as arbitrary parameters, and further, it is easy to see that if we drop out the factors $(a_s b'_s - a'_s b_s)$ for all such values of s, and also the operator Θ_s and the factors $(a_s b'_p - a'_s b_p)$, for $s > n-t$, $p > n-t$, $p \neq s$, the result will still vanish identically.

But now, as a_s, a'_s, b_s, b'_s $(s > n-t)$ are arbitrary quantities, it is a mere question of notation to write $a_s = x'$ and $a'_s = -x$, so that if we replace all of the arbitrary quantities by the same x, x', we get the result I*a* (ii) given at the beginning of the paper.

Discriminants.

5. Clebsch* pointed out how the symbolical expression for discriminants can be deduced from that for the eliminant of two quantics.

If the given quantic, of degree $n+1$, be written

$$(ax+a'x')^{n+1} \equiv (bx+b'x')^{n+1},$$

the discriminant is the eliminant of

$$\phi(x) = a\,(ax+a'x')^n = b\,(bx+b'x')^n,$$

$$\psi(x) = a'(ax+a'x')^n = b'(bx+b'x')^n.$$

But $\phi(x)\,\psi(y) - \phi(y)\,\psi(x)$

$$= ab'\,[(ax+a'x')^n\,(by+b'y')^n - (ay+a'y')^n\,(bx+b'x')^n]$$

$$= \tfrac{1}{2}(ab'-a'b)\,[(ax+a'x')^n\,(by+b'y')^n - (ay+a'y')^n\,(bx+b'x')^n],$$

owing to the equivalence of a, a' and b, b', so that the formula for the discriminant is at once obtained from that already given, by replacing the factor

$$\prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r)$$

by its square

$$\prod_{r=1}^{r=n} (a_r b'_r - a'_r b_r)^2,$$

all the other factors remaining the same.

6. The other expressions II and III which I have obtained are derived from the Euler-Sylvester determinant by a transformation similar to the one I have just used for Bézout's determinant. Sylvester's method is to take $\phi(x, x')$ of degree m, and $\psi(x, x')$ of degree n, and to form a deter-

* "Ueber symbolische Darstellung algebraischer Formen" (*Crelle*, Bd. 59, 1861, p. 25). In this paper Clebsch who, with Salmon, was the first to investigate the symbolical expression of eliminants, gave a general—very complicated—result for the eliminant of two quantics of the same degree.

minant of the coefficients of

$$x^r x'^{n-r-1}\phi(x, x') \quad \text{and} \quad x^s x'^{m-s-1}\psi(x, x'),$$

for $r = 0, 1, 2, \ldots, n-1$; $s = 0, 1, 2, \ldots, m-1$.

Calling this determinant, of order $m+n$, E, it is obvious that on multiplying it by the determinant X, also of order $m+n$, whose r-th row is

$$|\, x_r^{m+n-1},\ x_r^{m+n-2}x_r',\ x_r^{m+n-3}x_r'^2,\ \ldots,\ x_r'^{m+n-1}\,|,$$

we have a determinant* which has n rows of the form

$$|\, x_1^r x_1'^{n-r-1}\phi(x_1, x_1'),\ x_2^r x_2'^{n-r-1}\phi(x_2, x_2'),\ \ldots,\ x_{m+n}^r x_{m+n}'^{n-r-1}\phi(x_{m+n}, x_{m+n}')\,| \qquad (r = 0, 1, 2, \ldots, n-1),$$

and m rows of the form

$$|\, x_1^s x_1'^{m-s-1}\psi(x_1, x_1'),\ x_2^s x_2'^{m-s-1}\psi(x_2, x_2'),\ \ldots,\ x_{m+n}^s x_{m+n}'^{m-s-1}\psi(x_{m+n}, x_{m+n}')\,| \qquad (s = 0, 1, 2, \ldots, m-1).$$

If, further, we multiply this by certain numerical multiples of two determinants Λ and M, of orders n and m, whose r-th rows are

$$|\, \lambda_r^{n-1},\ \lambda_r^{n-2}\lambda_r',\ \lambda_r^{n-3}\lambda_r'^2,\ \ldots,\ \lambda_r'^{n-1}\,|$$

and

$$|\, \mu_r^{m-1},\ \mu_r^{m-2}\mu_r',\ \mu_r^{m-3}\mu_r'^2,\ \ldots,\ \mu_r'^{m-1}\,|,$$

we may write

$$\Lambda \mathrm{M} X E = \Delta,$$

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* This determinant, when expanded in terms of minors of the n-th and m-th orders, gives for the eliminant

$$\Sigma \prod_p \phi(x_p, x_p') \prod_r \psi(x_r, x_r') / \prod (x_p x_r' - x_p' x_r),$$

where x_p, x_p' is any one of a set of n arbitrary quantities taken out of the $m+n$, and x_r, x_r' any one of the remaining m, and the summation is for all possible divisions into two such sets. Again, putting all the x' equal to unity, we may write

$$\phi(x) = e^{x\Omega} a_m, \quad \psi(x) = e^{x\Omega'} b_n,$$

where

$$\Omega \equiv a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \ldots + m a_{m-1} \frac{d}{da_m},$$

$$\Omega' \equiv b_0 \frac{d}{db_1} + 2b_1 \frac{d}{db_2} + \ldots + n b_{n-1} \frac{d}{db_n},$$

so that the eliminant is given by

$$\Sigma \left\{ \left(\Sigma_p x_p \Omega_p + \Sigma_r x_r \Omega_r' \right)^{mn} / \prod (x_p - x_r) \right\},$$

operating on the product of a_m repeated n times, and b_n repeated m times, the $m+n$ factors of the operand being considered distinct till after the operations are performed.

where Δ is the determinant which has n rows of the form

$$|(\lambda_r x_1+\lambda'_r x'_1)^{n-1}\phi(x_1, x'_1),\ (\lambda_r x_2+\lambda'_r x'_2)^{n-1}\phi(x_2, x'_2),\ \ldots,$$
$$(\lambda_r x_{m+n}+\lambda'_r x'_{m+n})^{n-1}\phi(x_{m+n}, x'_{m+n})|,$$

and m rows of the form

$$|(\mu_r x_1+\mu'_r x'_1)^{m-1}\psi(x_1, x'_1),\ (\mu_r x_2+\mu'_r x'_2)^{m-1}\psi(x_2, x'_2),\ \ldots,$$
$$(\mu_r x_{m+n}+\mu'_r x'_{m+n})^{m-1}\psi(x_{m+n}, x'_{m+n})|.$$

7. Now substitute the symbolical expressions

$$\phi(x, x') \equiv (ax+a'x')^m, \quad \psi(x, x') \equiv (bx+b'x')^n,$$

giving the a's and a''s the same suffix as the λ's and λ''s, and the b's and b''s the same suffix as the μ's and μ''s. Then the elements of Δ are of the two forms

$$(\lambda_r x_s+\lambda'_r x'_s)^{n-1}(a_r x_s+a'_r x'_s)^m \quad \text{and} \quad (\mu_r x_s+\mu'_r x'_s)^{m-1}(b_r x_s+b'_r x'_s)^n.$$

But, disregarding certain numerical coefficients, $(\lambda x+\lambda' x')^{n-1}(ax+a'x')^m$ may be written either as

$$\left(a\frac{\partial}{\partial\lambda}+a'\frac{\partial}{\partial\lambda'}\right)^m(\lambda x+\lambda' x')^{m+n-1}, \tag{A}$$

or as

$$\left(\lambda\frac{\partial}{\partial a}+\lambda'\frac{\partial}{\partial a'}\right)^{n-1}(ax+a'x')^{m+n-1}, \tag{B}$$

with similar substitutions for

$$(\mu x+\mu' x')^{m-1}(bx+b'x')^n.$$

Taking the first form (A) we may take out the factors

$$\prod_{r=1}^{r=n}\left(a_r\frac{\partial}{\partial\lambda_r}+a'_r\frac{\partial}{\partial\lambda'_r}\right)^m \prod_{r=1}^{r=m}\left(b_r\frac{\partial}{\partial\mu_r}+b'_r\frac{\partial}{\partial\mu'_r}\right)^n$$

$\left[\text{which may be written } \prod_{r=1}^{r=n}\phi\left(\frac{\partial}{\partial\lambda_r}, \frac{\partial}{\partial\lambda'_r}\right)\prod_{r=1}^{r=m}\psi\left(\frac{\partial}{\partial\mu_r}, \frac{\partial}{\partial\mu'_r}\right)\right]$, and the determinant which is left is

$$\text{AMXII}(\lambda_r\mu'_s-\lambda'_r\mu_s).$$

Changing the notation to the more usual one of x, x', y, y', we have the result III (ii), viz.,

$$\Pi(x_p x'_q-x'_p x_q)\,\Pi(y_r y'_s-y'_r y_s)\,\text{E}$$
$$=\Pi\phi\left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x'_p}\right)\Pi\psi\left(\frac{\partial}{\partial y_r}, \frac{\partial}{\partial y'_r}\right)\Pi(x_p x'_q-x'_p x_q)\,\Pi(y_r y'_s-y'_r y_s)$$
$$\times\Pi(x_p y'_r-x'_p y_r)$$

$$(p, q = 1, 2, 3, \ldots, n,\ p\neq q;\ r, s = 1, 2, 3, \ldots, m,\ r\neq s).$$

8. Taking the second form (B) we get, on treating it in a similar manner, and dividing out by X,

$$\text{AME} = \Pi\left(\lambda_r \frac{\partial}{\partial a_r} + \lambda'_r \frac{\partial}{\partial a'_r}\right)^{n-1} \Pi\left(\mu_s \frac{\partial}{\partial b_s} + \mu'_s \frac{\partial}{\partial b'_s}\right)^{m-1} D, \qquad \text{(C)}$$

where D is the product of all the differences

$$a_p a'_q - a'_p a_q, \quad b_r b'_s - b'_r b_s, \quad a_p b'_r - a'_p b_r.$$

But in this if we interchange any two λ, λ''s, or any two μ, μ''s, we change the sign of Λ or M, so that adding and subtracting the results so obtained, we may replace

$$\Pi\left(\lambda_r \frac{\partial}{\partial a_r} + \lambda'_r \frac{\partial}{\partial a'_r}\right)^{n-1}$$

by the determinant, whose elements are

$$\left(\lambda_r \frac{\partial}{\partial a_s} + \lambda'_r \frac{\partial}{\partial a'_s}\right)^{n-1},$$

that is to say, by $\quad \Lambda\Pi\left(\dfrac{\partial}{\partial a_p}\dfrac{\partial}{\partial a'_q} - \dfrac{\partial}{\partial a'_p}\dfrac{\partial}{\partial a_q}\right)$

and $\quad \Pi\left(\mu_s \dfrac{\partial}{\partial b_s} + \mu'_s \dfrac{\partial}{\partial b'_s}\right)^{m-1}$ by $\text{M}\Pi\left(\dfrac{\partial}{\partial b_r}\dfrac{\partial}{\partial b'_s} - \dfrac{\partial}{\partial b'_r}\dfrac{\partial}{\partial b_s}\right)$,

merely numerical factors being disregarded. Thus we have finally the result IIa (i), p. 266.

9. To find the further conditions that the quantics should have $t+1$ roots in common, consider Euler's method of finding the eliminant, and its extension to finding these further conditions, as given in Salmon, *Higher Algebra*, pp. 78, 79, §§ 81, 82. It is not difficult to see that these are the conditions that the determinant, of order $m+n-2t$, whose elements are of the two forms

$$(\lambda_r x_s + \lambda'_r x'_s)^{n-t-1}(a_r x_s + a'_r x'_s)^m \quad \text{and} \quad (\mu_r x_s + \mu'_r x'_s)^{m-t-1}(b_r x_s + b'_r x'_s)^n,$$

should vanish identically, that is, for all values of x_s, x'_s.

But this again is equivalent to the vanishing of a determinant of order $m+n-t$, which has t rows arbitrary, the others being of the form just given.

If, then, we take the t arbitrary rows to be of the form

$$|\,(x_1 y'_s + x'_1 y_s)^{m+n-t-1},\ (x_2 y'_s + x'_2 y_s)^{m+n-t-1},\ \ldots,\ (x_{m+n-t} y'_s + x'_{m+n-t} y_s)^{m+n-t-1}\,|,$$

we have, in analogy with (C), § 8,

$$\Pi\left(\lambda_r\frac{\partial}{\partial a_r}+\lambda'_r\frac{\partial}{\partial a'_r}\right)^{n-t-1}\Pi\left(\mu_s\frac{\partial}{\partial b_s}+\mu'_s\frac{\partial}{\partial b'_s}\right)^{m-t-1}D,$$

where D is now the product of all the differences

$$a_p a'_q - a'_p a_q, \quad b_r b'_s - b'_r b_s, \quad y_u y'_v - y'_u y_v,$$
$$a_p b'_r - a'_p b_r, \quad a_p y'_u - a'_p y_u, \quad b_r y'_u - b'_r y_u.$$

Taking out the irrelevant factors $y_u y'_v - y'_u y_v$, we may replace each y, y' by $-x$, x', and obtain the result IIa (ii), p. 266.

10. I will now show that the discriminant of a quantic

$$\phi(x, x') \equiv (ax+a'x')^{n+1} \equiv (bx+b'x')^{n+1}$$

may be derived from this form also, by prefixing the product

$$\Pi(a_r b'_r - a'_r b_r).$$

The discriminant is the eliminant of $\partial\phi/\partial x$ and $\partial\phi/\partial x'$, so that if we take two sets of n pairs of arbitrary quantities y_r, y'_r and z_r, z'_r, and form an Euler-Sylvester determinant out of the coefficients of

$$x^r x'^{n-r}\left(y_r\frac{\partial\phi}{\partial x}+y'_r\frac{\partial\phi}{\partial x'}\right) \quad \text{and} \quad x^r x'^{n-r}\left(z_r\frac{\partial\phi}{\partial x}+z'_r\frac{\partial\phi}{\partial x'}\right),$$

i.e., of $$(ay_r+a'y'_r)\,x^r x'^{n-r}(ax+a'x')^n,$$

and $$(az_r+a'z'_r)\,x^r x'^{n-r}(ax+a'x')^n,$$

the result must be the discriminant multiplied by

$$\Pi_r(y_r z'_r - y'_r z_r),$$

since it will vanish if any factor $(y_r z'_r - y'_r z_r)$ of this product is zero.

But now treating this determinant just as I have treated the one which is the eliminant of two separate quantics, we shall get results identical with those already obtained, except that a factor

$$\Pi_r(a_r y_r + a'_r y'_r)\,\Pi_r(b_r z_r + b'_r z'_r)$$

is prefixed all through.

We may, if we please, suppose y_r, y'_r to be the same throughout, and also z_r, z'_r, and so get the curious result that the covariant got by prefixing the factor

$$\Pi_r(a_r y + a'_r y')\,\Pi_r(b_r z + b'_r z')$$

to IIa (i) is the product of the discriminant by $(yz'-y'z)^n$.

But, otherwise, keeping the (y, y')'s and (z, z')'s distinct, we see that to interchange y_r, y'_r and z_r, z'_r, is to change the sign of the whole, so that we may put

$$(a_r y_r + a'_r y'_r)(b_r z_r + b'_r z'_r) - (a_r z_r + a'_r z'_r)(b_r y_r + b'_r y'_r),$$

in the place of $$(a_r y_r + a'_r y'_r)(b_r z_r + b'_r z'_r),$$

and still have the same result save as to a numerical factor 2.

That is to say, we may put

$$\prod_r (a_r b'_r - a'_r b_r)(z_r y'_r - z'_r y_r)$$

in the place of $$\prod_r (a_r y_r + a'_r y'_r) \prod_r (b_r z_r + b'_r z'_r),$$

and so the result II*b* (i) is established.

11. [*April*, 1910.—Some other expressions for the eliminant which can be deduced immediately from the relation (C) of § 8, seem worthy of notice.

The result of operating on D, the product of all the differences

$$a_p a'_q - a'_p a_q, \quad b_r b'_s - b'_r b_s, \quad a_p b'_r - a'_p b_r,$$

with the operator $$\Pi \alpha_p \left(\frac{\partial}{\partial a_p}, \frac{\partial}{\partial a'_p}\right) \Pi \beta_r \left(\frac{\partial}{\partial b_r}, \frac{\partial}{\partial b'_r}\right),$$

where α_p, β_r are any homogeneous functions of two variables of degree $n-1$, $m-1$ respectively, is the product of E into the two determinants, of orders n, m, whose elements are the coefficients of $\alpha_1, \alpha_2, \ldots, \alpha_n$, and of $\beta_1, \beta_2, \ldots, \beta_n$ respectively.

Or, again, the result of operating on D with any operator, which is homogeneous and of degree $n-1$ in $\frac{\partial}{\partial a_p}$, $\frac{\partial}{\partial a'_p}$, for all values of p, and of degree $m-1$ in $\frac{\partial}{\partial b_r}$, $\frac{\partial}{\partial b'_r}$, for all values of r, is a multiple of E, unless it vanish.]

THE APPLICATION TO DIRICHLET'S SERIES OF BOREL'S EXPONENTIAL METHOD OF SUMMATION

By G. H. HARDY.

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1. The series

$$\Sigma a_n e^{-\lambda_n s}, \tag{1}$$

where

$$\lambda_{n+1} > \lambda_n, \quad \lambda_n \to \infty,$$

is called a *generalised Dirichlet's series.** If $\lambda_n = n$, it is a power series in $x = e^{-s}$: if $\lambda_n = \log n$, it is an *ordinary Dirichlet's series*

$$\Sigma a_n n^{-s}. \tag{2}$$

The series (1) possesses two lines of convergence

$$R(s) = s_0, \quad R(s) = \bar{s}, \tag{3}$$

where $s_0 \leqslant \bar{s}$: the series is convergent to the right of the first and absolutely convergent to the right of the second. Thus the series

$$\Sigma(-1)^{n-1} n^{-s}, \tag{4}$$

which represents the function

$$(1-2^{1-s})\,\zeta(s) \tag{5}$$

is convergent for $R(s) > 0$, and absolutely convergent for $R(s) > 1$. On the other hand, for the series Σn^{-s}, we have $s_0 = \bar{s} = 1$. It is evident that in the case of any ordinary Dirichlet's series $0 \leqslant \bar{s} - s_0 \leqslant 1$. We may, of course, have $s_0 = \bar{s} = -\infty$, the series then converging abso-

* The literature which concerns Dirichlet's series in general, considered as functions of a complex variable s, is rather scattered. The theory was first attacked seriously by Cahen: "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues," *Annales Sc. de l'École Normale Supérieure* (sér. 3), t. XI, p. 75. See the dissertation of W. Schnee (Berlin, 1908), and his article in the *Rendiconti del Circolo Mat. di Palermo*, t. XXVII, where numerous references are given to papers by Landau and others who have contributed to the theory; a later dissertation by H. Bohr (Copenhagen, 1910); and Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*. The general theory has been rather overwhelmed in the mass of its applications to the problem of the distribution of primes and to the general analytical theory of numbers.

lutely all over the plane, as in the case of the series $\Sigma a^n n^{-s}$ ($|a|<1$); or, we may have $s_0=\bar{s}=\infty$, the series being never convergent (as in the last example when $|a|>1$). In the general case the inequality $\bar{s}-s_0\leqslant 1$ need not hold: we may even have $s_0=-\infty$, $\bar{s}=\infty$, as in the case of the series

$$\Sigma(-1)^{n-1}n^{-a}(\log n)^{-s} \quad (0<a<1),$$

which converges conditionally for all values of s.

When $\lambda_n=n$, s_0 and $\bar{s}$ are necessarily equal, and the function represented by the series has necessarily at least one singular point on its line of convergence. No such result holds in the general case.*

The series Σn^{-s} has the one singular point $s=1$, which lies on the line of convergence: and Landau† has shown that, when $a_n>0$, the real point on the line of convergence is always a singular point of the function represented by the series. On the other hand, the function (5) is an integral function of s. The line of convergence may also be a line of essential singularities, across which the function cannot be continued: Landau‡ has given as an example the series

$$\Sigma 2^{-s2^n}.$$

2. These circumstances make the problem of the summation of a Dirichlet's series, even in the narrower sense, more difficult and at the same time more interesting than the corresponding problem for power-series. In this connection some very interesting results have been stated in three recent notes in the *Comptes Rendus*, by MM. Bohr and Riesz.

M. Bohr§ considers the application to Dirichlet's series of Cesàro's method of summation by mean values.‖ He begins by proving the following general theorem :—

If Σa_n *is summable or finite* (Ck),¶ *and* $f_n\to 0$ *as* $n\to\infty$, *and the series*

$$\Sigma|\Delta f_n|, \quad \Sigma n|\Delta^2 f_n|, \quad \ldots, \quad \Sigma n^k|\Delta^{k+1}f_n|$$

---

* It is easy to see, however, that $s_0=\bar{s}$ in all cases in which the increase of λ_n is sufficiently rapid to ensure the convergence of $\Sigma e^{-\lambda_n s}$ for all positive values of s.

† *Math. Annalen*, Bd. LXI, p. 537.

‡ *Sitzungsberichte der Akademie zu München*, Bd. XXXVI, p. 191.

§ *Comptes Rendus*, Jan. 11, 1909.

‖ Bromwich, *Infinite Series*, pp. 310 *et seq.*

¶ *Summable* (Ck) means summable by Cesàro's k-th mean: see *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 257. By saying that Σa_n is *finite* (Ck), we mean that, in the notation of the paper just referred to, S_n^k/A_n^k oscillates finitely as $n\to\infty$.

are convergent, then the series $\Sigma a_n f_n$ *is summable* (*Ck*), *and its sum is equal to that of the series*

$$\Sigma S_n^k \Delta^{k+1} f_n,*$$

which is absolutely convergent.

This theorem is substantially the same as one which I proved in a paper† which I communicated to this Society in 1907, and which M. Bohr had not seen at the time of the publication of his note. It includes, as my theorem did not, the case in which Σa_n is only *finite* (*Ck*). On the other hand the conditions are less simple than they may be made, in that [as Dr. Bromwich (*Math. Annalen*, Bd. 65, p. 361) has proved] the convergence of $\Sigma n^k |\Delta^{k+1} f_n|$ involves that of all the other series mentioned in the final condition.

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* The notation is that of the paper referred to in the preceding footnote.

† Referred to in the two preceding footnotes.

[This paper unfortunately contains several inaccuracies, which have been pointed out to me by M. Bohr himself (see Bohr, *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen, 1910) and by Mr. J. E. Littlewood. The fact is that, finding the whole difficulty of the investigation to lie in the algebraical work (pp. 258–261), I was careless in writing pp. 261–3, which do not involve any point of very serious difficulty.

In the first place, it should have been explicitly stated that, throughout the proof of Theorem I, the condition $f_n \to 0$ is assumed to be satisfied: this is, of course, implied in calling f_n a "convergence factor" (p. 264). §§ 8, 9, 10 should be modified as follows. We consider first the terms in T_n^k for which $i = 0$. These give

$${}_0T_n^k = \sum_{j=0}^{n} \binom{n-j+k}{k} S_j^k \Delta^{k+1} f_j.$$

In this expression f_m is to be replaced by zero when $m > n$. But we may imagine this convention abandoned without affecting the limit of ${}_0T_n^k/A_n^k$. For this introduces a set of new terms in number depending only on k; for these terms we have $n-j < K$, and so

$$\binom{n-j+k}{k} < K,$$

and also

$$|S_j^k/A_n^k| < K.$$

And, as each of them involves a factor f_m which tends to zero as $n \to \infty$, their sum also tends to zero as $n \to \infty$.

Again,

$$\binom{n-j+k}{k} - A_n^k = \frac{1}{k!}\{(n-j+1)(n-j+2)\ldots(n-j+k) - (n+1)(n+2)\ldots(n+k)\}$$

is negative and numerically less than Kn^k $(0 \leqslant j \leqslant n)$. Also since

$$(n-j+1)\ldots(n-j+k) - (n+1)(n+2)\ldots(n+k) = -j\,\frac{d}{dx}\{(x+1)(x+2)\ldots(x+k)\},$$

From this theorem M. Bohr deduces the existence of a *line of summability* (Ck) for the series (2), viz.,

$$R(s) = s_k = \overline{\lim_{n\to\infty}} \{(S_n^k\, k!\, n^{-k})/\log n\}.$$

The series is summable (Ck) if $R(s) > s_k$, and not so summable if $R(s) < s_k$. Also

$$0 \leqslant s_k - s_{k+1} \leqslant 1,$$

and the function represented by the series is regular for $R(s) > s_k$. Thus for the series (4), $s_k = -k$, and the series is summable, by one or other of Cesàro's means, all over the plane.

For the general series (1) there are wider possibilities as to summability. Thus, if $\lambda_n = n$, $s_k = s_0 = \bar{s}$ for all values of k. On the other hand, the series

$$\Sigma(-1)^{n-1} n^a (\log n)^{-s} \quad (k-1 < a < k)$$

---

where $n-j < x < n$, it follows that $\binom{n-j+k}{k} - A_n^k$ is numerically less than

$$K\nu n^{k-1} \quad (0 \leqslant j \leqslant \nu < n).$$

Hence
$$\frac{{}_0T_n^k}{A_n^k} = \epsilon_n + \sum_0^n S_j^k \Delta^{k+1} f_j + R_0,$$

where
$$|R_0| < \frac{K\nu}{n} \sum_{j=0}^{\nu} j^k |\Delta^{k+1} f| + K \sum_{\nu+1}^{n} j^k |\Delta^{k+1} f_j| :$$

and by choosing first ν and then n sufficiently large we see that $R_0 \to 0$.

It follows that
$${}_0T_n^k/A_n^k \to \sum_0^\infty S_j^k \Delta^{k+1} f_j.$$

Next we consider

$${}_iT_n^k = \binom{k+1}{i} \sum_{j=0}^{n} \binom{n-j-i+k}{k-i} S_j^k \Delta^{k+1-i} f_{j+i}.$$

Since
$$\binom{n-j-i+k}{k-i} = (n-j+1)(n-j+2)\ldots(n-j+k-i)/(k-i)! < Kn^{k-i},$$

it follows that
$$\left|\frac{{}_iT_n^k}{A_n^k}\right| < \frac{K}{n^i} \sum_{j=0}^{n} |S_j^k \Delta^{k+1-i} f_{j+i}| < \frac{K}{n^i} \sum_{j=0}^{n} j^k |\Delta^{k+1-i} f_{j+i}|$$

$$< K \sum_{j=0}^{n} \left(\frac{j}{n}\right)^i j^{k-i} |\Delta^{k+1-i} f_{j+i}|.$$

By dividing the range of summation into the two parts $(0, \nu)$, $(\nu+1, n)$, and choosing first ν and then n sufficiently large, we see that

$${}_iT_n^k/A_n^k \to 0.$$

This completes the proof of Theorem A, the statement of which requires no correction. The theorem, as M. Bohr points out, remains true when $f_n \to L$ $(L \neq 0)$, provided that the expression for the sum of the series is modified by the addition of a term SL, where S is the sum of the series Σa_n. This follows at once from Theorem A on replacing f_n by $f_n - L$.

Theorem B requires modification only in the replacing of $\Sigma S_j^k \Delta^{k+1} f_j$ by $\Sigma S_j'^k \Delta^{k+1} f_j$, where

is summable (Ck) all over the plane, but never summable $(C, k-1)$, so that $s_k = -\infty$, $s_{k-1} = \infty$: the sum represents an integral function of s.

3. I propose now to consider the application to the ordinary Dirichlet's series of Borel's method of summation. It is easy to see, by the consideration of two simple examples, that it either may or may not be possible, by the use of Borel's method, to continue a function partially represented by a Dirichlet's series outside the region of convergence of the series.

Suppose first that the series is Σn^{-s}, which converges for $R(s) > 1$, and represents the function $\zeta(s)$, whose only singularity is a simple pole for $s = 1$. Then Borel's integral is

$$\int_0^\infty e^{-x} u(x)\, dx, \tag{6}$$

S'^k is defined as in the text. Then $|S'^k_j| < \epsilon A^k$ for $j \geqslant m_0$, and so

$$\left| \sum_m^{m'} S'^k_j \Delta^{k+1} f_j \right| < \epsilon \sum_m^{m'} A^k_j |\Delta^{k+1} f_j| < \epsilon K,$$

for $m' > m \geqslant m_0$; whence the truth of the theorem follows.

The corollaries remain valid: but the first of them requires a few words of proof. The sum of the series $a'_0 f_0 + a'_1 f_1 + \ldots$ is $\Sigma S'^k_j \Delta^{k+1} f_j$, and is therefore continuous; and the sum of $a_0 f_0 + a_1 f_1 + \ldots$ differs from this by Sf_0, and is therefore also continuous.

It is to be observed that the series

$$\Sigma S^k_j \Delta^{k+1} f_j$$

is, in general, neither uniformly convergent nor continuous. Suppose, *e.g.*, that

$$a_0 = 1, \quad a_1 = -1, \quad a_2 = 1, \quad \ldots,$$

so that $k = 1$, $S = \frac{1}{2}$, $a'_0 = 1 - \frac{1}{2} = \frac{1}{2}$; and that f_n is a function, such as x^n, that has the limit 0 for $0 \leqslant x < 1$, and the limit 1 for $x = 1$, the interval of values of x under consideration being $0 \leqslant x \leqslant 1$. Then for $x < 1$, we have

$$\Sigma a_n f_n = \Sigma S^1_j \Delta^2 f_j = \tfrac{1}{2}\Sigma (j+1)\, \Delta^2 f_j + \Sigma S'^1_j \Delta^2 f_j = \tfrac{1}{2} f_0 + \Sigma S'^1_j \Delta^2 f_j;$$

the last series being a uniformly convergent series whose limit as $x \to 1$ is plainly zero. Thus $\Sigma a_n f_n$ and $\Sigma S^1_j \Delta^2 f_j$ have the limit $\frac{1}{2}$. But the last series is discontinuous. For $x = 1$ its sum is zero, but then

$$\Sigma a_n f_n = LS + \Sigma S^1_j \Delta^2 f_j = \tfrac{1}{2}.$$

These facts, I may point out, are clearly recognised in my earlier paper in Vol. 2, Ser. 2, of these *Proceedings* (pp. 247 *et seq.*).

The genesis of the inaccuracies that I have explained is to be found in a momentary confusion, more natural than excusable, between the relations

$$S^k_n / A^k_n \to S, \quad S^k_n \to S.$$

It is remarkable that so careless a blunder should not have led to more serious error in my results.—*Added, February*, 1910.]

where* $$u(x) = \sum_1^\infty \frac{x^{n-1}}{n^s(n-1)!}.$$

Since† $$u(x) \sim x^{-s}e^x,$$

it follows that (6) is convergent (and then absolutely) if, and only if, $R(s) > 1$, when the original series is convergent.

Secondly, consider the series (4), which represents an integral function of s. Here‡

$$u(x) = \sum_1^\infty \frac{(-x)^{n-1}}{n^s(n-1)!} \sim -\frac{1}{\Gamma(s)} x^{-1}(\log x)^{s-1}.$$

Hence (6) is absolutely convergent for all values of s. It is easy to see that the same is true of

(7) $$\int_0^\infty e^{-x} u^{(\lambda)}(x)\,dx,$$

and so the series is absolutely summable all over the plane.§

4. I shall now prove that, *if the series* Σa_n *is summable, so is* $\Sigma a_n n^{-s}$, *where* $R(s) > 0$.

We are given that the integral (6) is convergent when

$$u(x) = \sum_1^\infty \frac{a_n x^{n-1}}{(n-1)!}.$$

It follows‖ that the integral

(8) $$\int_0^\infty e^{-x} u(xy)\,dx$$

is uniformly convergent for $0 \leqslant y \leqslant 1$. We have therefore, if $R(s) > 0$,

(9) $$\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} dy \int_0^\infty e^{-x} u(xy)\,dx = \int_0^\infty e^{-x}\,dx \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} u(xy)\,dy.$$

---

* We are applying Borel's method to $a_1 + a_2 + a_3 + \ldots$ and not to $0 + a_1 + a_2 + \ldots$. The equation would in the latter case be

$$u(x) = \sum_1^\infty \frac{x^n}{n^s n!}.$$

The summability of $a_1 + a_2 + \ldots$ implies that of $0 + a_1 + a_2 + \ldots$, whereas the converse is not true: see Bromwich, *Infinite Series*, p. 273, and a paper by the present writer in the *Quarterly Journal*, there referred to.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 402.

‡ *Ibid.* The formula fails if s is zero or a negative integer; then $u(x)$ tends exponentially to zero as $x \to \infty$.

§ Borel, *Leçons sur les séries divergentes*, p. 99.

‖ Bromwich, *Infinite Series*, pp. 433 *et seq.*

This follows from the following general theorem, which is in substance due to De la Vallée-Poussin.* He considers only the case in which the variables x, y are both real (which is all that is wanted here, the contour C being the line $0 \leqslant y \leqslant 1$). We shall require the more general form of the theorem in § 9.

If $f(x, y)$ is a continuous function of the real variable x and the real or complex variable y, and $\int_0^\infty f(x, y)\,dx$ is uniformly convergent for all values of y lying on a finite contour C, and $\int_C |\phi(y)|\,|dy|$ is convergent, then

$$\int_C \phi(y)\,dy \int_0^\infty f(x, y)\,dx = \int_0^\infty dx \int_C \phi(y) f(x, y)\,dy.$$

The following proof is an adaptation of the proof given by Dr. Bromwich (*Infinite Series*, p. 448) of an analogous theorem for series.

Since $\int_0^\infty f\,dx$ is a continuous function of y, the integral on the left hand is certainly convergent. Given ϵ, we can choose X, so that $\left|\int_{x_1}^\infty f\,dx\right| \leqslant \epsilon$ for $x_1 \geqslant X$. Also, in virtue of the continuity of f,

$$\int_C \phi\,dy \int_0^{x_1} f\,dx = \int_0^{x_1} dx \int_C f\phi\,dy.$$

Hence $$\left|\int_C \phi\,dy \int_0^\infty f\,dx - \int_0^{x_1} dx \int_C f\phi\,dy\right| = \left|\int_C \phi\,dy \int_{x_1}^\infty f\,dx\right| \leqslant \epsilon \int_C |\phi|\,|dy|,$$

for $x \geqslant X$, and so

$$\int_0^\infty dx \int_C f\phi\,dy = \lim_{x_1 \to \infty} \int_0^{x_1} dx \int_C f\phi\,dy = \int_C \phi\,dy \int_0^\infty f\,dx.$$

Now $$\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} u(xy)\,dy = \sum_1^\infty \frac{a_n x^{n-1}}{(n-1)!} \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} y^{n-1}\,dy$$

$$= \Gamma(s) \sum_1^\infty \frac{a_n x^{n-1}}{n^s (n-1)!}.\dagger$$

The last series represents the integral function associated with the series $\Sigma a_n n^{-s}$; and so this series is summable. Incidentally we have proved that its sum is

$$\frac{1}{\Gamma(s)} \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \psi(y)\,dy,$$

where $\psi(y)$ is the sum of the series $\Sigma a_n y^n$.

We notice further that *the sum of the series $\Sigma a_n n^{-s}$ represents a function of s regular for all values of s whose real part is positive.* This follows‡ at once from the fact that

$$\int_0^1 \frac{\partial}{\partial s} \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \psi(y)\,dy = \int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{s-1} \log\log\left(\frac{1}{y}\right) \psi(y)\,dy$$

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* *Annales de la Société Scientifique de Bruxelles*, t. XVI.

† The justification of the term-by-term integration presents no difficulty.

‡ Bromwich, *Infinite Series*, p. 438.

is uniformly convergent throughout any continuous region, in the plane of s, which lies entirely to the right of the imaginary axis.

We have thus established the existence of a line $R(s) = \sigma_0$, such that a Dirichlet's series is summable everywhere on its right and nowhere on its left. In other words, *the region of summability is a half plane:* it may, of course, include the whole plane ($\sigma_0 = -\infty$), or none of it ($\sigma_0 = \infty$).

5. I shall now prove further that, *if the series* Σa_n *is absolutely summable, so is* $\Sigma a_n n^{-s}$, *where* $R(s) > 0$: *i.e.*, that *the region of absolute summability is also a half plane.*

We are assuming now that

$$\int_0^\infty e^{-x} \, | u^{(\lambda)}(x) | \, dx \tag{10}$$

is convergent for $\lambda = 0, 1, 2, \ldots$, and we have to show that

$$\int_0^\infty e^{-x} \, | u_s^{(\lambda)}(x) | \, dx, \tag{11}$$

where

$$u_s(x) = \sum_1^\infty \frac{a_n x^{n-1}}{n^s (n-1)!},$$

is convergent. Now

$$u_s(x) = \frac{1}{\Gamma(s)} \int_0^1 \left\{ \log\left(\frac{1}{y}\right) \right\}^{s-1} u(xy) \, dy,$$

$$| u_s(x) | \leqslant \frac{1}{| \Gamma(s) |} \int_0^1 \left\{ \log\left(\frac{1}{y}\right) \right\}^{R(s)-1} | u(xy) | \, dy.$$

The integral

$$\int_0^\infty e^{-x} \, | u(xy) | \, dx$$

is uniformly convergent for $0 \leqslant y \leqslant 1$; and this enables us to show, by an argument precisely similar to that employed in the preceding section, that

$$\int_0^\infty e^{-x} \, dx \int_0^1 \left\{ \log\left(\frac{1}{y}\right) \right\}^{R(s)-1} | u(xy) | \, dy,$$

and, *a fortiori*,

$$\int_0^\infty e^{-x} \, | u_s(x) | \, dx,$$

is convergent. Again,

$$u_s^{(\lambda)}(x) = \frac{1}{\Gamma(s)} \int_0^1 \left\{ \log\left(\frac{1}{y}\right) \right\}^{s-1} u^{(\lambda)}(xy) \, y^\lambda \, dy,$$

and, using this equation in exactly the same way, we can show that the remainder of the integrals (11) are convergent.

We thus establish the existence of a *line of absolute summability*

$$R(s) = \bar{\sigma}:$$

and clearly $$\sigma_0 \leqslant \bar{\sigma}.$$

6. It is evident that the same reasoning might have been applied to the more general series

$$\Sigma a_n \{\phi(n)\}^{-s},$$

provided only that we could find an expression of $\{\phi(n)\}^{-s}$ in the form

$$\{\phi(n)\}^{-s} = \int_0^\infty e^{-nw} \psi_s(w)\, dw = \int_0^1 \psi_s \left\{\log\left(\frac{1}{y}\right)\right\} y^{n-1}\, dy,$$

where ψ_s is a function such that

$$\int_0^1 \left| \psi_s \left\{\log\left(\frac{1}{y}\right)\right\} \right| dy$$

is convergent.

Suppose, for example, that

$$\phi(n) = \theta(n+a) = (n+a)^{a_0} \{\log(n+a)\}^{a_1} \{\log_2(n+a)\}^{a_2} \ldots,$$

the number of factors being finite, $\log_2 n$, $\log_3 n$, ..., denoting $\log\log n$, $\log\log\log n$, ..., and a, a_0, a_1, ... being real, and a and a_0 positive.

Then it may be deduced from a formula given by Pincherle* that

$$\{\theta(k)\}^{-s} = \int_0^\infty e^{-kw} \psi_s(w)\, dw,$$

where $$\psi_s(w) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{-wt} \{\theta(-t)\}^{-s}\, dt;$$

the path of integration being a straight line, and $e_{p-1} < -\lambda < k$, where p is the number of logarithmic factors of $\theta(k)$, and e_0, e_1, e_2, e_3, ... denote the numbers 1, e, e^e, e^{e^e}, ...

Hence we deduce

$$\{\phi(n)\}^{-s} = \int_0^\infty e^{-(n+a)w} \psi_s(w)\, dw$$

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* *Mem. di Bologna* (4), t. 8, pp. 125 *et seq.*

for all values of n greater than a definite n_0. It may further be proved that as $w \to 0$,

$$\psi_s(w) \sim \frac{w^{a_0 s-1}}{\Gamma(a_0 s)} \left\{ \log\left(\frac{1}{w}\right) \right\}^{-a_1 s} \dots;$$

and, using these facts, we are able to extend the conclusions of §§ 4, 5 to the more general series

$$\Sigma \frac{a_n}{(n+a)^{a_0 s} \{\log(n+a)\}^{a_1 s} \dots}.$$

On the other hand, it is by no means true that the regions of summability or of absolute summability of the general series (1) are necessarily half planes. It is sufficient to consider the series

$$\sum_1^\infty (-1)^{n-1} e^{-ns} = \sum_1^\infty (-1)^{n-1} x^n = \frac{x}{1+x}.$$

This series is summable (absolutely) if $R(x) > -1$. If $s = \xi + i\eta$, this condition is

$$e^{-\xi} \cos \eta > -1;$$

and the region of summability is obtained by cutting out of the plane of s an infinite succession of curvilinear areas whose general shape is easily sketched.

7. The series (4) is a simple example of a series summable by Borel's method all over the plane. For it Borel's method is more effective than any of Cesàro's: later on I shall define a large class of series, all of which resemble (4) in this respect. But it must not be imagined that Borel's method, even as applied to ordinary Dirichlet's series, is *always* as effective as even the simplest of Cesàro's; or that, even when the function represented by the series is regular all over the plane, it can always be continued by exponential summation.

Consider, for example, the series

$$1^{-s} + 0 + 0 + \dots - 8^{-s} + 0 + \dots + 27^{-s} + 0 + \dots, \tag{12}$$

in which $a_n = (-1)^{k-1}$, when $n = k^3$, and $a_n = 0$ otherwise. I shall prove that *this series is summable when, and only when, it is convergent, absolutely summable when, and only when, it is absolutely convergent.* Thus the function

$$(1 - 2^{1-3s}) \zeta(3s)$$

represented by the series when $R(s) > 0$, although an integral function of s, cannot be continued by the use of Borel's method.

It is convenient to consider instead of (12) the series

$$0+1^{-s}+0+\ldots-8^{-s}+0+\ldots+27^{-s}+\ldots,$$

which is certainly summable (absolutely summable) if (12) is summable (absolutely summable). Then

$$u(x) = \sum_1^\infty \frac{(-1)^{n-1}}{n^{3s}} \frac{x^{n^3}}{(n^3)!}.$$

I have considered elsewhere* the asymptotic properties of functions of this type, but it will be necessary now to obtain rather more precise information.

We divide the range of integration into two sets of intervals i_ν, j_ν, i_ν being the interval $(\nu-\delta)^3 \leqslant x \leqslant (\nu+\delta)^3$, where δ is a small fixed positive number. We shall consider first whether the series

$$\sum_{(\nu)} \int_{(i_\nu)} e^{-x} u(x)\, dx \tag{13}$$

is convergent.

If $$v_n = n^{-3s} x^{n^3}/(n^3)!,$$

we may write $u(x)$ in the form

$$\begin{aligned} u(x) &= (-1)^{\nu-1} v_\nu + \left(\sum_1^{\nu-1} + \sum_{\nu+1}^\infty \right) (-1)^{n-1} v_n \\ &= (-1)^{\nu-1} v_\nu + S_1 + S_2, \end{aligned}$$

say. It is easy to see that, when x lies in i_ν, v_ν is, to a first approximation, of the order of e^x: a more precise approximation will be obtained later. A straightforward application of Stirling's theorem gives the formula

$$v_n/v_{n+1} = (1+\epsilon_n)\, x^{-3n^2-3n-1} \exp\left\{(9n^2+9n+3)\log n + \tfrac{9}{2}(n+1)\right\},$$

where $\epsilon_n \to 0$ as $n \to \infty$. If $n \geqslant \nu$, we have

$$\begin{aligned} x^{-3n^2-3n-1} &> (\nu+\delta)^{-9n^2-9n-3} \geqslant (n+\delta)^{-9n^2-9n-3} \\ &> n^{-9n^2-9n-3} \exp\left\{-\frac{\delta}{n}(9n^2+9n+3)\right\} \\ &> K n^{-9n^2-9n-3} e^{-9\delta n}, \end{aligned}$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 335 *et seq.*; *Messenger of Math.*, Vol. 39, pp. 28 *et seq.*

and, accordingly, $v_n/v_{n+1} > Ke^{(\frac{8}{3}-\delta)n} > e^{4n}$:

and it follows that

$$|S_2| < Ke^{-4\nu}|v_\nu|. \tag{14}$$

Again, $$S_1 = \left(\sum_1^{n_0} + \sum_{n_0+1}^{\nu-1}\right)(-1)^{n-1}v_n = S_1' + S_1'',$$

say. We choose n_0, once for all, so as to justify the application of Stirling's theorem to the terms for which $n > n_0$. Then, if $n_0 < n < \nu$, we have

$$\begin{aligned} x^{-3n^2-3n-1} &< (\nu-\delta)^{-9n^2-9n-3} \leqslant (n+1-\delta)^{-9n^2-9n-3} \\ &< n^{-9n^2-9n-3}\exp\left[-\left\{\frac{1-\delta}{n} - \frac{(1-\delta)^2}{2n^2}\right\}(9n^2+9n+3)\right] \\ &< Kn^{-9n^2-9n-3}e^{-9(1-\delta)n}, \end{aligned}$$

and so $$v_n/v_{n+1} < Ke^{-(\frac{8}{3}-\delta)n} < e^{-4n}.$$

It follows that

$$|S_1''| < Ke^{-4\nu}|v_\nu|. \tag{15}$$

Also S_1'' is a polynomial in n, of degree n_0^3, while v_ν is (roughly) of order e^x or e^{ν^3}. Hence, when ν is large

$$|S_1'| < Ke^{-4\nu}|v_\nu|. \tag{16}$$

From (14), (15), and (16) it follows that, in i_ν,

$$u(x) = (-1)^{\nu-1}v_\nu(1+\rho_\nu), \tag{17}$$

where $$|\rho_\nu| < Ke^{-4\nu}.$$

Again, $$v_\nu = \frac{\nu^{-3s}x^{\nu^3}}{(\nu^3)!} = \frac{\nu^{-3s}(\nu+t)^{3\nu^3}}{(\nu^3)!} \quad (|t|<\delta)$$

$$= \frac{\nu^{-3s}}{\sqrt{(2\pi)}}\exp\{3\nu^3\log(\nu+t) - 3(\nu^3+\tfrac{1}{2})\log\nu + \nu^3 + r_\nu\},$$

where $$|r_\nu| < K/\nu^3.$$

We easily deduce that

$$v_\nu = \frac{\nu^{-3s-\frac{3}{2}}}{\sqrt{(2\pi)}}e^{x-\frac{3}{2}\nu t^2}(1+\epsilon_\nu), \tag{18}$$

where $$|\epsilon_\nu| < K/\nu.$$

From this it follows that

$$|e^{-x}\{u(x) - (-1)^{\nu-1}v_\nu\}| < K\nu^{-3R(s)-\frac{3}{2}}e^{-4\nu},$$

which is obviously, whatever the value of s, even when multiplied by the

length of the interval i_ν, the general term of an absolutely convergent series. Hence the convergence of (13) depends entirely upon that of the series

$$\Sigma(-1)^{\nu-1}\int_{(i_\nu)} e^{-x}v_\nu\,dx, \tag{19}$$

or of

$$\Sigma(-1)^{\nu-1}\nu^{-3s-\frac{3}{2}}\int_{(\nu-\delta)^3}^{(\nu+\delta)^3} e^{-\frac{3}{2}\nu t^2}(1+\epsilon_\nu)\,dx. \tag{20}$$

Now
$$\begin{aligned}\int_{(\nu-\delta)^3}^{(\nu+\delta)^3} e^{-\frac{3}{2}\nu t^2}dx &= 3\int_{-\delta}^{\delta}(\nu+t)^2 e^{-\frac{3}{2}\nu t^2}\,dt\\ &= 6\int_0^{\delta}(\nu^2+t^2)e^{-\frac{3}{2}\nu t^2}\,dt\\ &= 6\nu^2(1+\eta_\nu)\int_0^{\infty} e^{-\frac{3}{2}\nu t^2}dt\\ &= \nu^{\frac{3}{2}}\sqrt{(2\pi)}(1+\eta_\nu),\end{aligned}$$

where
$$|\eta_\nu| < K/\nu.^*$$

Hence the series (19) may be written in the form

$$\sqrt{(2\pi)}\,\Sigma(-1)^{\nu-1}\nu^{-3s}(1+\eta_\nu); \tag{21}$$

and this series is absolutely convergent if $R(s) > \frac{1}{3}$, and conditionally convergent if $0 < R(s) \leqslant \frac{1}{3}$.

It remains to consider the series

$$\Sigma\int_{(j_\nu)} e^{-x}u(x)\,dx, \tag{22}$$

where j_ν is the interval $(\nu+\delta)^3 \leqslant x \leqslant (\nu+1-\delta)^3$. But it follows at once from the work in my note in the *Messenger of Mathematics* quoted above that, in j_ν,

$$|e^{-x}u(x)| < K_1 x^{K_2R(s)+K_3}e^{-K_4x^{\frac{1}{3}}},$$

where K_1, K_2, K_3 and K_4 are constants: and from this it follows that the series (22) is absolutely convergent for all values of s. And this completes the proof of the assertion made at the beginning of this section.

The same conclusions may be extended to the series

$$(1^k)^{-s}+0+\ldots-(2^k)^{-s}+0+\ldots+(3^k)^{-s}+0+\ldots,$$

where k is any integer greater than 2.

* A smaller limit may be assigned for $|\eta_\nu|$, but this one is sufficient for our present purpose.

8. The series considered in the last section are interesting as examples of the possibility of Borel's method proving less powerful than Cesàro's. For it follows from investigations of Cesàro and myself* that the summability $(C1)$ of the series

$$a_1+a_2+a_3+\dots$$

involves that of all the series

$$a_1+0+0+a_2+0+0+0+0+a_3+0+\dots,$$
$$a_1+0+0+0+0+0+0+a_2+0+0+0+0+\dots,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

in which a_ν occurs respectively in the ν^2-th, ν^3-th, ... position. Hence, for example, the series (12) is summable $(C1)$ for $-\frac{1}{3} < R(s) \leqslant 0$: it may be proved to be summable (Ck) for $-\frac{1}{3}k < R(s) \leqslant -\frac{1}{3}(k-1)$; and so summable by one or other of Cesàro's means all over the plane.

9. Let us call

$$\Sigma a_n n^{-s} \tag{23}$$

the Dirichlet's series *associated with* the power series

$$\Sigma a_n x^n. \tag{24}$$

If (24) has a radius of convergence greater than unity, (23) is absolutely convergent for all values of s, and represents an integral function of s. If the radius of convergence of (24) is less than unity, (23) is never convergent. If the radius of convergence of (24) is unity, (23) may converge for none or for some or for all values of s; examples are given by the series

$$\Sigma e^{\sqrt{n}} n^{-s}, \quad \Sigma n^{-s}, \quad \Sigma e^{-\sqrt{n}} n^{-s}.$$

I shall assume only that the radius of convergence of (24) is positive: the series then possesses a circle of convergence and a polygon of summability.† And I shall now prove the following theorem :—

If the point $x = 1$ lies within the polygon of summability of the series (24), *the associated Dirichlet's series is summable by Borel's method for all values of s, and represents an integral function of s.*

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* See *Quarterly Journal*, Vol. 38, pp. 269 *et seq.* : Bromwich, *Infinite Series*, pp. 386 *et seq.*

† Borel, *Leçons sur les séries divergentes*, p. 126 ; Bromwich, *Infinite Series*, p. 295.

Borel's sum is given by the integral

$$\int_0^\infty e^{-x} u_s(x)\, dx, \tag{25}$$

where

$$u_s(x) = \sum_1^\infty \frac{a_n x^{n-1}}{n^s (n-1)!}. \tag{26}$$

Now

$$n^{-s} = \frac{i\Gamma(1-s)}{2\pi} \int_W (-w)^{s-1} e^{-nw}\, dw,^* \tag{27}$$

where $$(-w)^{s-1} = \exp\{(s-1)\log(-w)\},$$

the logarithm being real when w is real and negative, and where W represents a contour beginning and ending at the infinitely distant end of the positive real axis and surrounding the positive real axis by a counter-clockwise circuit. We shall suppose that the point of W, where $R(w)$ has its algebraical minimum, is its point of intersection with the negative real axis, where $w = -\delta$, δ being a positive number as small as we please.

The equation (27) holds for all values of s save $s = 1, 2, 3, \ldots$: values which we shall at present omit from consideration.

From (27) we at once deduce

$$u_s(x) = \frac{i\Gamma(1-s)}{2\pi} \int_W (-w)^{s-1} e^{-w} u(xe^{-w})\, dw \tag{28}$$

$$= \frac{i\Gamma(1-s)}{2\pi} \int_T (\log t)^{s-1} u(xt)\, dt,$$

where T is a loop from the origin in the plane of T round the point $t = 1$, as shown in the figure. The term by term integration here employed is of a type whose justification presents no difficulty.

We shall now prove that

$$\int_0^\infty e^{-x} dx \int_T (\log t)^{s-1} u(xt)\, dt = \int_T (\log t)^{s-1} dt \int_0^\infty e^{-x} u(xt)\, dx\,; \tag{29}$$

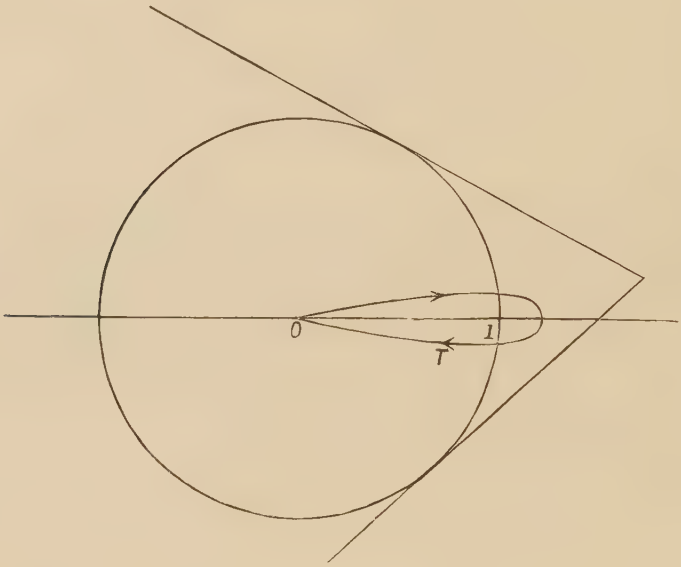
or, in other words, that the integral (25) is convergent, and may be calculated by an inversion of the order of integration.

The integral

$$\int_0^\infty e^{-x} u(xt)\, dx \tag{30}$$

* Whittaker, *Modern Analysis*, p. 182.

is uniformly (and absolutely) convergent throughout any region in the plane of t which lies entirely inside the polygon of summability of the series $\Sigma a_n t^n$. If $t = 1$ lies inside the polygon it is evident (see the figure) that we can draw W so that T lies entirely in the polygon, and



then (30) is uniformly convergent for all values of t on T. Our conclusion then follows at once from the auxiliary theorem of § 4.

The proof that the "sum" of the series represents an analytic function regular all over the plane may now be supplied as in § 4. Incidentally we see that the sum is

$$\frac{i\Gamma(1-s)}{2\pi}\int_T (\log t)^{s-1}\,\phi(t)\,dt,$$

where $\phi(t)$ is the function represented by $\Sigma a_n t^n$ and its continuation by Borel's integral.

So far we have assumed that s is not a positive integer: if it is we replace (27) by the equation

$$n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty w^{s-1}e^{-nw}\,dw,$$

and argue in precisely the same way.

10. The conclusions of the last section may be extended (cf. § 6) to more general series of the type $\Sigma a_n \{\phi(n)\}^{-s}$, such as

$$\sum_{n_0}^{\infty} \frac{a_n}{(n+a)^{\alpha_0 s}\{\log(n+a)\}^{\alpha_1 s}\dots}.$$

It is to be observed that they are quite independent of any assumption that the Dirichlet's series is ever *convergent:* thus they apply to such series as

$$\Sigma(-1)^{n-1}a^n n^{-s}, \quad \Sigma(-1)^{n-1}a^{2n+1}(2n+1)^{-s},$$

where a is any positive number, however large. But certainly the most interesting case of the theorem is:

If $\Sigma a_n x^n$ is convergent for $|x|<1$, and the function represented by the series is regular for $x=1$, then the associated Dirichlet's series is summable for all values of s, and represents an integral function of s.

The last assertion (having no reference to summability) is easily established directly. As examples of series which satisfy these conditions, we may mention

(31)
$$\Sigma \frac{\cos n\theta}{n^s}, \quad \Sigma \frac{\sin n\theta}{n^s},$$

$$\Sigma \frac{1}{n^{\beta_0}(\log n)^{\beta_1}\dots} \frac{e^{ni\theta}}{n^s},$$

where
$$\theta \not\equiv 0 \pmod{2\pi}.$$

With the help of the remarks made at the beginning of this section the conclusion may be extended to such series as, *e.g.*,

$$\Sigma \frac{\xi^n e^{ni\theta}}{(n+b)^{\beta_0}\{\log(n+b)\}^{\beta_1}\dots[(n+a)^{\alpha_0}\{\log(n+a)\}^{\alpha_1}\dots]^s}.$$

11. It is instructive to verify the results of §§ 9, 10 in the case of the simple series (31). Writing $2\pi\phi$ for θ, and using Hurwitz's formula*

(32)
$$\zeta(s,\phi) = 2\Gamma(1-s)(2\pi)^{s-1}\sum_1^{\infty} n^{-(1-s)}\sin\{(2n\phi+\tfrac{1}{2}s)\pi\},$$

where
$$0<\phi<1, \quad R(s)<1,$$

and $\zeta(s,\phi)$ is the generalised Riemann ζ-function defined by the series

$$\phi^{-s}+(\phi+1)^{-s}+(\phi+2)^{-s}+\dots,$$

* Lindelöf, *Le calcul des résidus*, p. 107.

and its continuations, we see that

$$C_s(\phi) = \sum_1^\infty \frac{\cos 2n\pi\phi}{n^s} = \frac{(2\pi)^{1-s}}{4\Gamma(1-s)\sin\frac{1}{2}s\pi}\{\xi(s,\phi)+\xi(s,1-\phi)\},$$

$$S_s(\phi) = \sum_1^\infty \frac{\sin 2n\pi\phi}{n^s} = \frac{(2\pi)^{1-s}}{4\Gamma(1-s)\cos\frac{1}{2}s\pi}\{\xi(s,\phi)-\xi(s,1-\phi)\}.$$

The functions $\xi(s,\phi)$, $\xi(s,1-\phi)$ are each regular save for a simple pole, with unit residue, at $s=1$. It is clear that the only possible singularities of C and S are simple poles at the points $s=0, -2, -4, \ldots$ and $s=-1, -3, -5, \ldots$ respectively. To verify that C and S are in fact regular at these points, we have to show that

$$\xi(-2k,\phi)+\xi(-2k,1-\phi)=0, \quad \xi(-2k-1,\phi)=\xi(-2k-1,1-\phi):$$

and these equations are easily verified by the help of known results in the theory of the Zeta and Bernoullian functions.*

12. If $\Sigma a_n x^n$ is regular for $|x|<1$, and has a simple pole for $x=1$, then $\Sigma a_n n^{-s}$ is regular all over the plane except for a simple pole at $s=1$. If $\Sigma a_n x^n$, at $x=1$, behaves like

$$(1-x)^\alpha\,\phi(x),$$

where $\phi(x)$ is regular, then $\Sigma a_n n^{-s}$ has simple poles for

$$s=-\alpha,\ -1-\alpha,\ -2-\alpha,\ \ldots,$$

unless α is integral. If α is a positive integer, there are no poles; if a negative integer, the poles are

$$1,\ 2,\ \ldots,\ -\alpha.$$

These results follow at once from the consideration of the equation

$$\Sigma a_n n^{-s} = \frac{i}{2\Gamma(s)\{\sin(\alpha+s)\pi\}}\int_W (-w)^{\alpha+s-1}\chi(e^{-w})\,dw,$$

where $\chi(e^{-w})$ is a function regular for $w=0$. I do not imagine that they are new: but they are worth stating here in connection with the results of §§ 9–11.

* See Barnes, *Messenger of Mathematics*, Vol. XXIX, pp. 74 *et seq*, 88 *et seq*.

THE ORDINAL RELATIONS OF THE TERMS OF A CONVERGENT SEQUENCE

By G. H. HARDY.

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1. Suppose that we are given a sequence

$$(a_n) \quad (n = 1, 2, 3, \ldots) \tag{1}$$

and are able, when any two terms of the sequence, a_p and a_q say, are taken, to determine which of the relations

$$a_p \begin{matrix} > \\ = \\ < \end{matrix} a_q \tag{2}$$

holds between a_p and a_q. We may express this by saying that we know the *ordinal relations* (as regards order of magnitude) which hold between the various terms of the sequence.

This note is devoted to a consideration of the following question, which seems to me of some logical interest: *How far is it possible to discriminate, on the ground* simply *of the ordinal relations that hold between their terms, between sequences which converge to a limit and sequences which do not?* And, in so far as this *is* possible, what is the simplest expression of the ordinal relations which characterise convergent sequences?

2. It is easy to recognise certain limitations to which the answers to these questions must be subject. In the first place, it is clearly hopeless to attempt to discriminate in this way between *convergent* and *properly divergent* sequences. This appears at once from a consideration of the simplest case of all, that of *monotonic* (say increasing) sequences. In this case

$$a_p \geqslant a_q,$$

if $p \geqslant q$. A classical theorem asserts that (a_n) is convergent or divergent to $+\infty$. But we cannot distinguish between the two cases: the sequences $(1-1/n)$ and (n) are characterised by exactly the same ordinal relations.

Thus we are obliged to regard (as, indeed, it is often convenient to regard) sequences which diverge to $+\infty$ or to $-\infty$ as a special case of convergent sequences, and to distinguish only between convergent and oscillatory series.

But even when this reservation is made, it is easy to see that the first of our two questions cannot be answered with a simple affirmative. Consider, for example, the sequence (a_n), where $a_n = (\sin \frac{1}{2}n\pi)/n$. Here the limit is zero. And now consider the sequence defined by the equations

$$b_n = 1 + \frac{1}{n} \sin \tfrac{1}{2}n\pi \qquad (n = 4k+1),$$

$$b_n = \frac{1}{n} \sin \tfrac{1}{2}n\pi \qquad (n = 4k,\ 4k+2),$$

$$b_n = -1 + \frac{1}{n} \sin \tfrac{1}{2}n\pi \qquad (n = 4k+3).$$

Each of the inequalities $a_p \gtreqless a_q$ involves the corresponding one of the inequalities $b_p \gtreqless b_q$: the ordinal relations of the terms of the two sequences are precisely the same; but (a_n) is convergent, while (b_n) has the *three* limiting values -1, 0, 1.

We shall see later that this last example is of genuine theoretical interest as being as simple an example as possible of the one type of oscillatory sequence that cannot be distinguished from a convergent sequence simply by means of its ordinal relations. The answer to our questions is, in fact, that the ordinal relations of convergent sequences do possess an exceedingly simple characteristic which distinguishes them from all other sequences except one very special type of oscillatory sequence; but that this special type of oscillatory sequence cannot be distinguished from a convergent sequence by *any* marks of its ordinal relations.

The Classes P_n, Q_n, R_n.

3. We choose a definite term a_n of the sequence (a_n) and divide the terms of the sequence (including a_n itself) into three classes P_n, Q_n, R_n, which comprise the terms a_m for which

$$a_m < a_n, \qquad a_m = a_n, \qquad a_m > a_n$$

respectively. Thus in a monotonic (increasing) sequence, no two terms of which are equal, P_n includes $a_1, a_2, \ldots a_{n-1}$, Q_n includes a_n only, and R_n includes $a_{n+1}, a_{n+2}, \ldots$.

Quasi-monotonic Sequences.

4. A *quasi-monotonic (increasing) sequence* is a sequence such that the classes P_n, Q_n are always finite (or such that P_n+Q_n is always finite).

THEOREM.—*The necessary and sufficient condition that a sequence should converge to a limit, greater than any of its terms, is that it should be quasi-monotonic (increasing).*

There are, of course, a corresponding definition and theorem for quasi-monotonic (decreasing) sequences.

Proof of the Theorem.—First, let the sequence converge to a limit greater than any a_n. If P_n+Q_n were infinite, for any particular value of n, we could find values of m as large as we please and such that $a_m \leqslant a_n$. This would imply $\lim a_m \leqslant a_n$, contrary to our hypothesis.

Secondly, let P_n+Q_n be always finite. Then, given n_1, we can choose n_2 so that

$$a_p > a_q \quad (p \geqslant n_2,\ q \leqslant n_1):$$

for the class formed by all the terms which belong to any of the classes P_q+Q_p $(q \leqslant n_1)$ is finite. Also we can suppose n_2 to be the *least* number satisfying this condition. Similarly we define n_3 as the least number such that

$$a_p > a_q \quad (p \geqslant n_3,\ q \leqslant n_2):$$

and so on. Thus we define a monotonic sequence (a_{n_ν}), which must tend to a limit a greater than any a_{n_ν} or to $+\infty$. If $a_{n_\nu} \to +\infty$, we have

$$a_{n_\nu} > G$$

by choice of ν, however large be G, and so

$$a_p > G \quad (p \geqslant n_{\nu+1}),$$

so that $a_n \to +\infty$.

On the other hand, if $a_{n_\nu} \to a$, a must be greater than any a_n. For, if $a_n \geqslant a$, and we choose ν so that $n_\nu > n$, we have $a_{n_{\nu+1}} > a$, which is impossible.

Finally, however small be ϵ, we can choose ν so that

$$a - a_{n_\nu} < \epsilon,$$

and so

$$a - a_n < a - a_{n_\nu} < \epsilon \quad (n \geqslant n_{\nu+1}).$$

Hence $a_n \to a$, and the theorem is established.

5. The proof may be shortened by the use of maximum and minimum limits of indetermination. If a_n does not converge to a limit, let G and H be its minimum and maximum limits: H may be $+\infty$, but neither can be $-\infty$, since only a finite number of terms are less than any a_n.

We can choose a subsequence (b_μ) whose limit is G, and a subsequence (c_ν) whose limit is H, and, as $G < H$, we can determine n_0 so that

$$b_\mu < c_\nu \quad (\mu, \nu \geqslant n_0).$$

Hence P_n is infinite for c_ν, in contradiction to our hypotheses.

6. The theorem of § 4 is verbally identical with one which I obtained some time ago,* but using a different definition. The definition of a quasi-monotonic (increasing) sequence that I adopted then is as follows: let ν be the largest number such that $a_1, a_2, \ldots, a_\nu$ are all less than a_n, so that ν is a function of n; then the sequence is quasi-monotonic (increasing) if $\nu \to \infty$ with n. It is easy to prove directly that the two definitions are equivalent. First, if the definition just stated is satisfied, then $P_n + Q_n$ is always finite. Otherwise we could, for some value of n, find an infinite number of terms $a_{m_1}, a_{m_2}, \ldots$, all less than or equal to a_n. But, by the definition, if ν is large enough,

$$a_m < a_{m_\nu} \quad (m \leqslant n),$$

which involves a contradiction.

Secondly, suppose that the definition just stated is *not* satisfied. Then we can find a number n_0 such that

$$a_{n_\nu} \geqslant a_{m_\nu},$$

where (m_ν) is an infinite ascending sequence, and $n_\nu \leqslant n_0$ for all values of ν. Since a_{n_ν} has at most n_0 possible values, there must be some one a_n $(n \leqslant n_0)$ which is not less than an infinity of the terms of the sequence (a_{m_ν}): and this shows that $P_n + Q_n$ is not always finite.

Quasi-monotonic Sequences in the wider sense.

7. A *quasi-monotonic (increasing) sequence in the wider sense* is a sequence such that P_n is always finite.

A quasi-monotonic increasing sequence in the wider sense has a limit not less than *any term of the sequence.* The proof is the same as that of the theorem of § 4: but there is one case in which the limit may be equal to terms of the sequence, viz., the trivial case in which all the terms of the sequence, from a certain rank onwards, are equal.

* The result is stated in substance, but without proof, as an example in Bromwich's *Infinite Series* (Ex. 14, p. 393).

The converse is not true. For let us take a quasi-monotonic (in the stricter sense) increasing sequence. It has a limit l greater than any term. In this sequence interpolate in any manner an infinity of terms equal to l. The sequence has still the limit l, not less than any of its terms: but it is no longer quasi-monotonic, even in the wider sense; for, if $a_n = l$, P_n is infinite.

Apart from this case of exception, it is easy to see that the theorem obtained from that of § 4 by changing *greater* to *not less* and *quasi-monotonic* to *quasi-monotonic in the wider sense* is true.

Convergent Sequences in General.

8. Now let us consider any convergent sequence (a_n). If its limit is $+\infty$ or $-\infty$, it is a quasi-monotonic sequence, increasing or decreasing.

Let us then suppose that it converges to a finite limit a. If Q_n is infinite for any value of n, then $a_n = a$, and the sequence contains an infinity of terms equal to a. The remaining terms may be divided into two classes, those less than and those greater than a. Hence *any convergent sequence is formed of* (i) *a sequence of equal terms,* (ii) *a quasi-monotonic* (*increasing*) *sequence, and* (iii) *a quasi-monotonic* (*decreasing*) *sequence.* Any of these classes, of course, may be finite or entirely absent. The terms of (ii) and (iii) are characterised by the facts that P_n+Q_n and Q_n+R_n are respectively finite: for the terms of (i) the classes P_n, Q_n, R_n may all be infinite.

Conversely, suppose that a sequence (a_n) possesses the properties (i) *that for at most one value of* a *are there infinitely many terms equal to* a, (ii) *that for any term of the sequence not equal to* a *either* P_n *or* R_n *is finite.*

For the latter terms Q_n is finite, and so either P_n+Q_n or Q_n+R_n. Let (b_μ) be the sequence formed by the terms for which P_n+Q_n is finite, and (c_ν) the sequence formed by the terms for which Q_n+R_n is finite. If P'_μ, Q'_μ, R'_μ are the classes corresponding to b_μ, but formed from terms of the sequence (b_μ) only, $P'_\mu+Q'_\mu$ is, *a fortiori*, finite, and so (b_μ) is a quasi-monotonic (increasing) sequence, and converges to a limit β greater than any of its terms. Similarly (c_ν) is a quasi-monotonic (decreasing) sequence, and converges to a limit γ less than any of its terms. Also

$$\beta \leqslant \alpha \leqslant \gamma.$$

For it is obvious that, if $\beta > \alpha$ or $\beta > \gamma$, then P_n would be infinite for some terms of (b_μ).

If α, β, γ are equal, say all equal to a, the sequence converges to

a limit, viz. α. But it is quite possible that $\beta < \alpha < \gamma$: thus, in the case of the sequence considered in § 2, $\beta = -1$, $\alpha = 0$, $\gamma = 1$.

If $\beta < \alpha < \gamma$, the sequence has precisely the same ordinal relations as the sequence (b_n) defined by the equations

$$b_n = a_n \qquad (a_n = \alpha),$$
$$b_n = a_n + (\alpha - \beta) \qquad (a_n < \alpha),$$
$$b_n = a_n - (\gamma - \alpha) \qquad (a_n > \alpha);$$

a sequence which plainly converges to the limit α.

Thus *the properties* (i) *and* (ii) *above characterise convergent sequences, in so far as these sequences can be characterised by their ordinal relations only. Every convergent sequence possesses them: and any sequence which possesses them is either convergent or is an oscillatory sequence of the special type formed by adding a constant to all the terms of a convergent sequence which are greater than the limit, and subtracting a constant from all those which are less than the limit. And this class of oscillatory sequences is indistinguishable from a convergent sequence by any test dependent on ordinal relations only.*

Oscillatory Sequences.

9. If (a_n) oscillates there must be terms for which both P_n and R_n are infinite (except in the special case discussed above). All of P_n, Q_n, R_n may always be infinite, as in the sequence

$$\tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{4},\ \tfrac{2}{4},\ \tfrac{3}{4},\ \tfrac{1}{5},\ \tfrac{2}{5},\ \tfrac{3}{5},\ \tfrac{4}{5},\ \tfrac{1}{6},\ \tfrac{2}{6},\ \tfrac{3}{6},\ \ldots;$$

or, again, P_n and R_n may always be infinite, and Q_n consist of one term only, as in the sequence deduced from that above by rejecting all fractions which are not in their lowest terms.

THEOREMS RELATING TO THE SUMMABILITY AND CONVERGENCE OF SLOWLY OSCILLATING SERIES

By G. H. HARDY.

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1. When, in this paper, I say that a series is summable, I mean "summable by mean values." The best known and most useful of methods of summation by mean values is that due to Cesàro. If

$$\begin{aligned} s_n &= a_1 + a_2 + \ldots + a_n, \\ s_n^{(1)} &= s_1 + s_2 + \ldots + s_n, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ s_n^{(r)} &= s_1^{(r-1)} + s_2^{(r-1)} + \ldots + s_n^{(r-1)}, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

and

$$(r!\, s_n^{(r)})/n^r \to s,$$

as $n \to \infty$, the series Σa_n is said to be "summable by Cesàro's mean of the r-th order." For the sake of brevity I shall say that such a series is "*summable* (Cr)." A series which is summable (Cr) is certainly summable (Ck) for all values of k greater than r.*

It is easy to see that if Σa_n is summable (Cr), then

$$a_n/n^r \to 0 :\dagger$$

and this result shows that the scope of Cesàro's method is somewhat restricted. For example, the series

$$a - a^2 + a^3 - \ldots \quad (a > 1)$$

cannot be summable (Cr) for any value of r.

2. Thus Cesàro's method is inapplicable if, to put it roughly, $|a_n|$ is *too large* when n is large. It does not, however, appear to have been observed that it is also inapplicable if $|a_n|$ is *too small* when n is large.

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* Bromwich, *Infinite Series*, p. 312.

† *Ibid.*, p. 318.

That this is so appears from the theorem:—

If $na_n \to 0$ as $n \to \infty$, then the series Σa_n cannot be summable by any of Cesàro's means unless it is convergent.

This theorem is an immediate corollary from well known theorems of Frobenius, Hölder, Cesàro, and Tauber. If Σa_n is summable (Cr), and s is its sum, then

$$\Sigma a_n x^n \to s,$$

as $x \to 1$.* But the last equation, together with $na_n \to 0$, involves the convergence of Σa_n.†

3. It is, however, possible to prove a more general theorem, viz.:

If $|na_n| < K$, then the series Σa_n cannot be summable by any of Cesàro's means unless it is convergent.

This theorem I shall now proceed to prove: the extension is of some importance in that it enables us to deal with a particularly interesting case, viz., that of the series

$$\Sigma n^{-1-\alpha i}$$

(where α is real).

Suppose first that $r = 1$. Then ...

$$(1) \qquad \frac{s_1+s_2+\ldots+s_n}{n} = \frac{na_1+(n-1)\,a_2+\ldots+a_n}{n}$$

$$= \frac{n+1}{n}\,s_n - \frac{a_1+2a_2+\ldots+na_n}{n}.$$

If we put

$$(2) \qquad na_n = b_n,$$

and denote by t_n, $t_n^{(1)}$, ... the functions of n formed from the b's as are the s's from the a's, we may write the equation (1) in the form

$$(3) \qquad s_n^{(1)}/n = (n+1)\,s_n/n - t_n/n.$$

Hence, if Σa_n is summable $(C1)$, the necessary and sufficient condition that it should be convergent is that

$$(4) \qquad t_n/n \to 0.‡$$

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* Bromwich, *Infinite Series*, p. 313.

† *Ibid.*, p. 251.

‡ This is a known result: see Bromwich, *l.c.*; Vivanti, *Theorie der Analytischen Funktionen*, p. 429; and the papers by Tauber and Pringsheim there quoted.

I shall now prove that the conditions

$$s_n^{(1)}/n \to s, \quad |b_n| < K,$$

necessarily involve (4), and so the convergence of the series.

We have $$\sum_1^n a_\nu = \sum_1^n \frac{b_\nu}{\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} + \frac{t_n}{n},$$

where $$\Delta\phi(\nu) \equiv \phi(\nu) - \phi(\nu+1).$$

Hence

(5) $$\sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} = \frac{ns_n - t_n}{n} = \frac{s_{n-1}^{(1)}}{n};$$

and so the summability $(C1)$ of the series involves the convergence (to the same sum) of the series

(6) $$\sum_1^\infty \frac{t_\nu}{\nu(\nu+1)}.$$

Now suppose that t_ν/ν does not tend to the limit zero. We may, without real loss of generality, restrict ourselves to the case in which a_n is real. For, if $a_n = \alpha_n + i\alpha_n'$, the series Σa_n is convergent or summable if, and only if, $\Sigma\alpha_n$ and $\Sigma\alpha_n'$ are both convergent or summable. If $|na_n| < K$, then α_n and α_n' satisfy the same condition. Finally, if t_ν/ν has not the limit zero, the same must be true of at least one of τ_ν/ν and τ_ν'/ν, where $t_\nu = \tau_\nu + i\tau_\nu'$, and τ_ν and τ_ν' are formed from α_ν and α_ν' just as is t_ν from a_ν.

Thus we suppose a_ν real. Then we can find values of N, as large as we like, and such that

(7) $$t_N > K_1 N$$

(or $t_N < -K_1 N$), K_1 being a positive constant which we may without loss of generality suppose less than K.

Let N_1 be the least integer such that

$$N_1 \geqslant \left(1 - \frac{K_1}{2K}\right) N = K_2 N,$$

say. Then also $N_1 < K_3 N$, where K_3 is another constant which we may suppose as nearly equal to K_2 as we like, and so certainly less than 1.

If $N_1 \leqslant n \leqslant N$, we have

$$|t_N - t_n| = |b_{n+1} + b_{n+2} + \ldots + b_N| < (N - N_1) K \leqslant \tfrac{1}{2} K_1 N,$$

and so, by (7), $$t_n \geqslant t_N - |t_N - t_n| > \tfrac{1}{2} K_1 N.$$

Hence $$\sum_{N_1}^{N} t_\nu \Delta \frac{1}{\nu} > \tfrac{1}{2}K_1 N \sum_{N_1}^{N} \Delta \frac{1}{\nu} = \tfrac{1}{2}K_1 N \left(\frac{1}{N_1} - \frac{1}{N+1}\right)$$

$$> \tfrac{1}{2}K_1 \left(\frac{1}{K_3} - 1\right) = K_4,$$

say: and this inequality is evidently inconsistent with the convergence of the series (6). Hence the theorem is proved when $r = 1$.

4. I shall now prove that *a series for which* $|na_n| < K$ *cannot be summable* (Cr) *without being summable* $(C, r-1)$.

It is easy to verify that

$$s_n^{(r)} = \binom{r+n-1}{r} a_1 + \binom{r+n-2}{r} a_2 + \ldots + \binom{r}{r} a_n,$$

$$s_n^{(r-1)} = \binom{r+n-2}{r-1} a_1 + \binom{r+n-3}{r-1} a_2 + \ldots + \binom{r-1}{r-1} a_n,$$

$$t_n^{(r-1)} = \binom{r+n-2}{r-1} a_1 + \binom{r+n-3}{r-1} 2a_2 + \ldots + \binom{r-1}{r-1} na_n.$$

By means of these equations it is easily proved that

$$rs_n^{(r)} + t_n^{(r-1)} = (n+r)\, s_n^{(r-1)}, \tag{8}$$

$$rs_n^{(r)} + t_{n+1}^{(r-1)} = (n+1)\, s_{n+1}^{(r-1)}. \tag{9}$$

The second of these equations is a generalisation of the second of the equations (5), to which it reduces if we put $r=1$ and write $n-1$ for n. The first may be written in the form

$$r\frac{s_n^{(r)}}{n^r} = \left(\frac{n+r}{n}\right)\frac{s_n^{(r-1)}}{n^{r-1}} - \frac{t_n^{(r-1)}}{n^r},$$

which is a generalisation of (3), and shows that *the necessary and sufficient condition that a series, known to be summable by* r *means, should be summable by* $r-1$ *means, is that*

$$t_n^{(r-1)} / n^r \to 0$$

as $n \to \infty$.

Again, $$s_n = \sum_1^n \frac{b_\nu}{\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\nu} + \frac{t_n}{n}$$

$$= \sum_1^{n-2} t_\nu^{(1)} \Delta^2 \frac{1}{\nu} + t_{n-1}^{(1)} \Delta \frac{1}{n-1} + \frac{t_n}{n}$$

$$= \ldots \quad \ldots \quad \ldots \quad \ldots,$$

or

$$\sum_{1}^{n-r} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} = s_n - \sum_{0}^{r-1} t_{n-k}^{(k)} \Delta^k \frac{1}{n-k}. \tag{10}$$

Now, by (5),

$$s_n - \frac{t_n}{n} = \frac{s_{n-1}^{(1)}}{n};$$

and it is easily verified that

$$s_n - \frac{t_n}{n} - t_{n-1}^{(1)} \Delta \frac{1}{n-1} = \frac{2 s_{n-2}^{(2)}}{n(n-1)}.$$

These equations suggest that

$$s_n - \sum_{0}^{r-1} t_{n-k}^{(k)} \Delta^k \frac{1}{n-k} = \frac{r!\,(n-r)!}{n!} s_{n-r}^{(r)}. \tag{11}$$

That this is so may be verified by induction. The equation (11), assumed true for $r = p$, will be true for $r = p+1$ also if only

$$\frac{p!\,(n-p)!}{n!} s_{n-p}^{(p)} - \frac{(p+1)!\,(n-p-1)!}{n!} s_{n-p-1}^{(p+1)} = t_{n-p}^{(p)} \Delta^p \frac{1}{n-p}.$$

But

$$\Delta^p \frac{1}{n-p} = \frac{p!\,(n-p-1)!}{n!},$$

so that the equation which we have to verify is

$$(n-p)\, s_{n-p}^{(p)} - (p+1)\, s_{n-p-1}^{(p+1)} = t_{n-p}^{(p)},$$

and if we write r for $p+1$, and then n for $n-r$, this reduces to (9). Thus (11) is established.

From (10) and (11) it follows that

$$\sum_{1}^{n-r} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} = \frac{r!\,(n-r)!}{n!} s_{n-r}^{(r)}. \tag{12}$$

Hence *the necessary and sufficient condition that the series* Σa_n *should be summable by Cesàro's mean of the r-th order is that the series*

$$\sum_{1}^{\infty} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} \tag{13}$$

should be convergent.

5. I shall now prove that, for the special class of series that we are considering, the convergence of (13) involves

$$t_n^{(r-1)} / n^r \to 0;$$

and therefore the summability of the series Σa_n by $r-1$ means. As before, we are at liberty to suppose a_n real.

Suppose that the last relation is untrue. Then we can find a value of N, as large as we like, and such that

$$t_N^{(r-1)} > K_1 N^r, \tag{14}$$

or

$$t_N^{(r-1)} < -K_1 N^r,$$

say the former. We may, as in § 3, suppose $K_1 < K$; and we choose N_1 as in § 3. Let $N_1 \leqslant n \leqslant N$. Then

$$\begin{aligned} t_N^{(r-1)} - t_n^{(r-1)} &= \sum_1^N \binom{r-1+N-s}{r-1} sa_s - \sum_1^n \binom{r-1+n-s}{r-1} sa_s \\ &= \sum_1^n \left\{ \binom{r-1+N-s}{r-1} - \binom{r-1+n-s}{r-1} \right\} sa_s \\ &\quad + \sum_{n+1}^N \binom{r-1+N-s}{r-1} sa_s. \end{aligned}$$

Both the coefficients of sa_s are positive; and so the sum is certainly increased numerically by writing K throughout in place of sa_s. Hence

$$\begin{aligned} |t_N^{(r-1)} - t_n^{(r-1)}| &< K \left[\sum_1^n \left\{ \binom{r-1+N-s}{r-1} - \binom{r-1+n-s}{r-1} \right\} \right. \\ &\qquad \left. + \sum_{n+1}^N \binom{r-1+N-s}{r-1} \right] \\ &= K \sum_1^{N-n} \binom{r-1+N-s}{r-1} \\ &< K(N-n) \binom{r+N-2}{r-1} < K(N-n)\, N(N+1) \dots (N+r-2) \\ &< \bar{K}(N-N_1)\, N^{r-1} \\ &< \tfrac{1}{2} K_1 \bar{K} N^r / K = \tfrac{1}{2} \bar{K}_1 N^r, \end{aligned} \tag{15}$$

where $\bar{K}$, $\bar{K}_1$ are constants which we may suppose greater than K, K_1 by as little as we please. From (14) and (15) it follows that

$$t_n^{(r-1)} > \tfrac{1}{2} K_1' N^r \quad (N_1 \leqslant n \leqslant N),$$

K_1' being a constant which we may suppose less than K_1 by as little as

we please. Thus

$$\sum_{N_1}^{N} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} > \tfrac{1}{2} K_1' N^r \sum_{N_1}^{N} \Delta^r \frac{1}{\nu}$$

$$= \tfrac{1}{2} K_1' N^r (r-1)! \left\{ \frac{1}{N_1(N_1+1)\dots(N_1+r-1)} - \frac{1}{(N+1)(N+2)\dots(N+r)} \right\}.$$

But

$$\frac{1}{N_1(N_1+1)\dots(N_1+r-1)} > \frac{K_5}{N_1^r} > \frac{K_5}{K_3^r N^r},$$

$$\frac{1}{(N+1)(N+2)\dots(N+r)} < \frac{1}{N^r},$$

K_5 being a constant less than 1 by as little as we please. Hence, finally,

$$\sum_{N_1}^{N} t_\nu^{(r-1)} \Delta^r \frac{1}{\nu} > \tfrac{1}{2} K_1' (r-1)! \left\{ \frac{K_5}{K_3^r} - 1 \right\} > K_6,$$

say: and this inequality is evidently incompatible with the convergence of the series (13).

6. It follows that, if $|na_n| < K$, the series Σa_n, if summable (Cr), is summable $(C, r-1)$: and so the series is summable $(C1)$, and so convergent. That is to say, *a series for which* $|na_n| < K$ *cannot be summable by Cesàro's means unless it is convergent.*

The following examples of this theorem are interesting.

(i) The series $\Sigma n^{-1-\alpha i} \quad (\alpha \gtrless 0)$

is known not to be convergent for any value of α. It follows that it is not summable by any of Cesàro's means: *i.e.*, that the region of definition of the Riemann Zeta-function cannot be extended by Cesàro's method of summation.

This result has been stated by Riesz,* who deduces it from a general theorem in analytic function-theory.

The simplest proof of this particular result, however, is obtained by considering the function

$$\Sigma n^{-1-\alpha i} x^n.$$

* *Comptes Rendus*, June 21, 1909.

As $x \to 1$, this function behaves like*

$$\Gamma(-\alpha i)\left(\log\frac{1}{x}\right)^{\alpha i},$$

and so does not tend to a limit. It follows, by well known theorems, that the series cannot be summable by Cesàro's means, nor yet by Borel's exponential method, since summability in any of these ways would involve the existence of such a limit.†

(ii) Fejér has shown that the Fourier's series corresponding to any continuous‡ function is summable (C1). Now suppose that $f(x)$ is not only continuous, but also monotonic. Then, by the second theorem of mean value,

$$\int_0^{2\pi} f(x)\cos nx\,dx = f(0)\int_0^{\xi}\cos nx\,dx + f(2\pi)\int_{\xi}^{2\pi}\cos nx\,dx,$$

which is numerically less than K/n. Hence the Fourier's series satisfies the condition $|na_n| < K$, and is therefore *convergent*. This result may be immediately extended to any case in which $f(x)$ is continuous and of limited total fluctuation.

In short, the theorem proved in the last section enables us to present Dirichlet's classical result as a mere corollary of Fejér's theorem.

7. The theorem of § 5 naturally suggests the following question: cannot Tauber's theorem itself be generalised by replacing the condition $na_n \to 0$ by the more general condition $|na_n| < K$; *i.e., is it not true that the conditions* (i) *that* $\Sigma a_n x^n$ *tends to a limit* s *as* $x \to 1$, *and* (ii) *that* $|na_n| < K$, *involve the convergence of the series* Σa_n?

I have not been able to prove this theorem, and I am inclined to think that it is not true. At the same time, I have been quite unsuccessful in my attempts to construct an example to the contrary.

8. The fact that such finitely oscillating series§ as

$$\Sigma n^{-1-\alpha i}, \quad \Sigma n^{-1}(\log n)^{-1-\alpha i}, \quad \ldots,$$

cannot be summed by Cesàro's means suggests a variety of interesting questions. The first is whether we can devise other analogous, purely

* *Proc. London Math. Soc.*, Vol. 3, p. 385.

† Bromwich, *Infinite Series*, pp. 291, 313.

‡ This hypothesis has been very widely generalised: see Hobson, *The Theory of Functions of a Real Variable*, pp. 707 *et seq.*

§ For proofs that these series oscillate finitely see a paper by Dr. Bromwich, *Proc. London Math. Soc.*, Vol. 6, pp. 327 *et seq.*, where further references are given.

arithmetical, methods of summation which will be more effective in such cases. This question has in part been answered by Riesz,* who has devised methods of summation which are generalisations of all of Cesàro's. In what follows we shall be concerned solely with the generalisation of Cesàro's first method.

Let μ_n be the general term of a divergent series of positive terms which, from a certain value of n onwards, tend steadily to zero, so that, in the notation of Du Bois Reymond's *Infinitärcalcül*,† we may write

$$\mu_n \prec 1.\ddagger$$

And let $$\lambda_n = \mu_1+\mu_2+\ldots+\mu_n,$$

so that $\lambda_{n+1}/\lambda_n \to 1$. Then we shall say that the series is *summable* $(R1, \mu_n)$ if

$$(\mu_2 s_1+\mu_3 s_2+\ldots+\mu_{n+1} s_n)/\lambda_{n+1}$$

tends to a limit s as $n \to \infty$, and we shall call s the *sum* of the series.

An equivalent way of stating the definition is to demand that

$$s_n - \frac{\lambda_1 a_1+\lambda_2 a_2+\ldots+\lambda_n a_n}{\lambda_{n+1}} \to s$$

as $n \to \infty$.

If we put $\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 1$, ..., $\lambda_n = n$ we are led to Cesàro's definition. The case of most interest after Cesàro's is that in which

$$\mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = \tfrac{1}{2}, \quad \ldots, \quad \mu_{n+1} = \frac{1}{n}, \quad \ldots,$$

when $\lambda_{n+1} \sim 1/n$. Our sum is then defined as

$$\lim_{n\to\infty} \left(\frac{s_1}{1} + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \Big/ \log n.\S$$

It is known‖ that, if Σb_n, Σc_n are two divergent series of positive terms, such that

$$c_n/c_{n+1} \geqslant b_n/b_{n+1}$$

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* *Comptes Rendus*, July 5, 1909.

† For an exposition of the ideas and notation of the *Infinitärcalcül*, see my "Orders of Infinity" (*Camb. Math. Tracts*, No. 12, 1910).

‡ This restriction on the definition is not essential: we might suppose $\mu_n = \log n, n, \ldots$. But a theorem quoted later in this section shows that if the definition is effective for such a form of μ_n it will also be effective when $\mu_n = 1$, when the definition reduces to Cesàro's. Thus nothing can be gained by considering such cases.

§ I had framed this definition before seeing Riesz's note.

‖ *Quarterly Journal*, Vol. 38, p. 271; Bromwich, *Infinite Series*, p. 386. Both parts of the theorem there proved (and not merely the first part, of which alone use is made here) are really due to Cesàro [*Rom. Acc. Lincei Rend.* (IV), 4, 1888, p. 452].

for all values of n, then the equation

$$\frac{b_1 s_1 + b_2 s_2 + \ldots + b_n s_n}{b_1 + b_2 + \ldots + b_n} \to s$$

implies the equation $\quad \dfrac{c_1 s_1 + c_2 s_2 + \ldots + c_n s_n}{c_1 + c_2 + \ldots + c_n} \to s.$

For practical purposes this is equivalent to the assertion: *if* Σa_n *is summable* $(R1, \mu_n)$ *it is summable* $(R1, m_n)$ *for any form of* m_n, *such that*

$$m_n \prec \mu_n.$$

For example, a series *summable* $(R1, 1)$ is certainly *summable* $(R1, 1/n)$: *i.e.*, a series summable $(C1)$ is summable by the special form of Riesz's method alluded to above. *The efficacy of the method rises as the rapidity of decrease of* μ_n *rises*—provided of course μ_n does not decrease so rapidly that $\Sigma\mu_n$ ceases to diverge. Thus, by taking

$$\mu_n = 1, \ \frac{1}{n}, \ \frac{1}{n \log n}, \ \frac{1}{n \log n \log\log n}, \ \ldots,$$

we obtain a series of methods of summation of increasing efficacy.

9. We have seen that $\Sigma n^{-1-\alpha i}$ is not summable $(C1)$. It is, however, as Riesz has pointed out, summable $(R1, 1/n)$; as is almost immediately deducible from the formula

$$\sum_1^n \nu^{-1-\alpha i} = -\frac{n^{-\alpha i}}{\alpha i} + \zeta(1+\alpha i) + O\left(\frac{1}{n}\right),$$

where, in Landau's notation, $O\{\phi(n)\}$ denotes a function of n whose modulus is less than $K\phi(n)$ for all values of n from a certain value onwards. Another simple example of a series summable $(R1, 1/n)$ is

$$1-1+0+1+0+0+0-1+0+\ldots,$$

where the non-zero terms occur in the 1st, 2nd, 4th, 8th, ... places. This series is not summable $(C1)$ or (Cr) or by Borel's method, since the series

$$x - x^2 + x^4 - x^8 + \ldots$$

oscillates as $x \to 1$.*

* *Quarterly Journal*, Vol. 38, pp. 276 *et seq.*; Bromwich, *Infinite Series*, p. 389.

10. The summability $(R1, \mu_n)$ of Σa_n does not, as we have seen, involve the existence of a limit for $\Sigma a_n x^n$ as $x \to 1$. But there is an interesting theorem with respect to summability $(R1, \mu_n)$ which is a generalisation of Frobenius's theorem concerning summability $(C1)$; and corresponding theorems which are generalisations of Tauber's theorem and the theorem of § 3 (so far as it applies to summation by Cesàro's *first* mean).

These theorems I shall now proceed to state and to prove.

(i) *Analogue of Frobenius's Theorem.—If* Σa_n *is summable* $(R1, \mu_n)$ *to the sum* s, *and*

$$s_n e^{-\lambda_n x} \to 0,$$

as $n \to \infty$, *for any positive value of* x, *then* $\Sigma a_n e^{-\lambda_n x}$ *is convergent for* $x > 0$, *and*

$$\Sigma a_n e^{-\lambda_n x} \to s$$

as $x \to 0$.*

(ii) *Analogue of Tauber's Theorem.—If* $\lambda_n a_n / \mu_n \to 0$, *and the series* $\Sigma a_n e^{-\lambda_n x}$ *(then certainly convergent for* $x > 0$*) tends to the limit* s *as* $x \to 0$, *then* Σa_n *is convergent to the sum* s.

(iii) *Analogue of the Theorem of* § 3.—*If* $|\lambda_n a_n / \mu_n| < K$, *the series* Σa_n *cannot be summable* $(R1, \mu_n)$ *without being convergent.*

The second of these theorems has been proved by Landau :† I shall now proceed to prove the others.

11. We have

$$\sum_1^n a_\nu e^{-\lambda_\nu x} = \sum_1^{n-1} s_\nu \Delta e^{-\lambda_\nu x} + s_n e^{-\lambda_n x}.$$

* In Frobenius's theorem there is no condition similar to the second of these conditions. In fact the existence of a limit for $(s_1 + s_2 + \ldots + s_n)/n$ *implies* that $s_n e^{-nx} \to 0$ for any positive x. This is not so when the divergence of $\Sigma\mu_n$ is very slow.

Suppose Σa_n summable $(R1, \mu_n)$. Then it is easy to see that

$$s_n = (\epsilon_{n+1}\lambda_{n+1} - \epsilon_n \lambda_n)/\mu_{n+1},$$

where $\epsilon_n \to 0$, and that the condition that $s_n e^{-\lambda_n x} \to 0$ will certainly be satisfied if (to put the matter roughly) λ_n/μ_n increases more slowly than any positive power of e^{λ_n}. This condition is satisfied if $\mu_n = 1$, $\lambda_n = n$: but not if $\mu_n = 1/n$, $\lambda_n \sim \log n$.

† *Monatshefte für Mathematik und Physik*, Jahrgang XVIII, p. 12.

The last term tends to zero. Also

$$\Delta e^{-\lambda_\nu x} = e^{-\lambda_\nu x} - e^{-\lambda_{\nu+1} x} \tag{16}$$

$$= x \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-xt}\, dt < x\mu_{\nu+1} e^{-\lambda_\nu x}.$$

Now $\Sigma \mu_{\nu+1} \lambda_\nu^{-\xi}$ is convergent for $\xi > 1$:* *a fortiori* $\Sigma \mu_{\nu+1} e^{-\lambda_\nu x}$ is convergent for any positive x. Also, for any positive value of r, $|s_\nu e^{-\lambda_\nu r}| < K$, where K depends only on r, and

$$|s_\nu \Delta e^{-\lambda_\nu x}| < K x \mu_{\nu+1} e^{-\lambda_\nu (x-r)},$$

so that $\Sigma s_\nu \Delta e^{-\lambda_\nu x}$ is absolutely convergent for $x > r$: as r is arbitrarily small, this condition may be replaced by $x > 0$. It follows that

$$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty s_\nu \Delta e^{-\lambda_\nu x}. \tag{17}$$

Hence

$$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty \mu_{\nu+1} s_\nu \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right)$$

$$= \sum_1^\infty (\mu_2 s_1 + \ldots + \mu_{\nu+1} s_\nu)\, \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right),$$

provided only that

$$(\mu_2 s_1 + \ldots + \mu_{n+1} s_n) \left(\frac{1}{\mu_{n+1}} \Delta e^{-\lambda_n x}\right) \to 0:$$

and as $\mu_2 s_1 + \ldots + \mu_{n+1} s_n \sim \lambda_{n+1} s$ and $\lambda_{n+1}/\lambda_n \to 1$, this follows at once from (16). Thus we have

$$\sum_1^\infty a_\nu e^{-\lambda_\nu x} = \sum_1^\infty (s + \epsilon_\nu)\, \lambda_{\nu+1} \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right), \tag{18}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

If we had applied the same analysis to the particular case in which

$$a_1 = 1, \quad a_2 = a_3 = \ldots = 0, \quad s_n = 1,$$

we should have arrived at the equation

$$e^{-\lambda_1 x} + \frac{\mu_1}{\mu_2} \Delta e^{-\lambda_1 x} = \sum_1^\infty \lambda_{\nu+1} \Delta \left(\frac{1}{\mu_{\nu+1}} \Delta e^{-\lambda_\nu x}\right). \tag{19}$$

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* Abel, *Œuvres*, t. II, p. 198; Bromwich, *Infinite Series*, p. 32.

Before proceeding further, we observe that

$$\frac{\mu_{\nu+2}\Delta e^{-\lambda_\nu x}}{\mu_{\nu+1}\Delta e^{-\lambda_{\nu+1}x}} = \frac{\mu_{\nu+2}(e^{\mu_{\nu+1}x}-1)}{\mu_{\nu+1}(1-e^{-\mu_{\nu+2}x})} \geqslant 1,$$

so that

$$\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right) \geqslant 0 \tag{20}$$

for all values of ν and x.

From (18) and (19) it follows that

$$\Sigma a_\nu e^{-\lambda_\nu x} - s e^{-\lambda_1 x} - \frac{\mu_1 s}{\mu_2}\Delta e^{-\lambda_1 x} = \sum_1^\infty \epsilon_\nu \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right),$$

$$\left|\Sigma a_\nu e^{-\lambda_\nu x} - s\right| < |s|(1-e^{-\lambda_1 x}) + \frac{\mu_1}{\mu_2}|s|\,\Delta e^{-\lambda_1 x}$$

$$+\sum_1^N \left|\epsilon_\nu \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right)\right|$$

$$+\delta_N \sum_1^\infty \lambda_{\nu+1}\Delta\left(\frac{1}{\mu_{\nu+1}}\Delta e^{-\lambda_\nu x}\right),$$

where δ_N is the greatest value of $|\epsilon_n|$ for $n > N$. The last term is less than $\delta_N\{1+(\mu_1/\mu_2)\}$, and so N can be chosen so as to make it as small as we please. After N is fixed we can choose x so as to make the remaining terms as small as we please: and so

$$\Sigma a_\nu e^{-\lambda_\nu x} \to s$$

as $x \to 0$.

12. We have now to prove the extension of the theorem of § 3. From the equation

$$\frac{\mu_2 s_1 + \mu_3 s_2 + \ldots + \mu_{n+1} s_n}{\lambda_{n+1}} = s_n - \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_{n+1}},$$

it follows that the necessary and sufficient condition that a series, known to be summable ($R1$), shall be convergent, is that

$$\frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_{n+1}} \to 0. \tag{21}$$

Let $$b_n = \lambda_n a_n, \quad t_n = b_1 + b_2 + \ldots + b_n.$$

Then $$s_n = \sum_1^n \frac{b_\nu}{\lambda_\nu} = \sum_1^{n-1} t_\nu \Delta \frac{1}{\lambda_\nu} + \frac{t_n}{\lambda_n},$$

$$\sum_1^{n-1} t_\nu \Delta \frac{1}{\lambda_\nu} = s_n - \frac{t_n}{\lambda_n} = \frac{\mu_2 s_1 + \ldots + \mu_{n+1} s_n}{\lambda_{n+1}} - \frac{\mu_{n+1} t_n}{\lambda_n \lambda_{n+1}}.$$

Now, *ex hypothesi*, $|b_n| < K\mu_n$, and so $|t_n| < K\lambda_n$: hence

$$\mu_{n+1} t_n / \lambda_n \lambda_{n+1} \to 0.$$

It follows that

(22) $$\sum_1^\infty t_\nu \Delta \frac{1}{\lambda_\nu}$$

is convergent: we have now to show that this necessarily implies (21), and so the convergence of Σa_n. As in §§ 3 and 5, we may suppose a_n real.

If (21) is not satisfied we can find values of N, as large as we please, and such that

$$t_N > K_1 \lambda_{N+1}$$

(or $t_N < -K_1 \lambda_{N+1}$): we can, without loss of generality, suppose $K_1 < K$. Let N_1 be the least integer, such that

$$\lambda_{N_1+1} \geqslant \left(1 - \frac{K_1}{2K}\right) \lambda_{N+1} = K_2 \lambda_{N+1},$$

say. Then, also $\lambda_{N_1+1} < K_3 \lambda_{N+1}$, where K_3 is another constant which we can suppose as nearly equal to K_2 as we please.

If $N_1 \leqslant n \leqslant N$, we have

$$\begin{aligned} |t_N - t_n| = |b_{n+1} + b_{n+2} + \ldots + b_N| &\leqslant K(\mu_{n+1} + \mu_{n+2} + \ldots + \mu_N) \\ &\leqslant K(\lambda_N - \lambda_{N_1}) \\ &< \bar{K}(\lambda_{N+1} - \lambda_{N_1+1}) \\ &< \tfrac{1}{2}\bar{K}_1 \lambda_{N+1}, \end{aligned}$$

$\bar{K}$ and $\bar{K}_1$ being constants which we may suppose as nearly equal as we please to K and K_1 respectively. Hence

$$t_n \geqslant t_N - |t_N - t_n| > \tfrac{1}{2} K_1' \lambda_{N+1},$$

where K_1' also is as nearly equal to K_1 as we please.

Thus
$$\sum_{N_1}^{N} t_\nu \Delta \frac{1}{\lambda_\nu} > \tfrac{1}{2}K_1' \lambda_{N+1} \left(\frac{1}{\lambda_{N_1}} - \frac{1}{\lambda_{N+1}}\right)$$
$$> \tfrac{1}{2}K_1' \left(\frac{1}{K_3} - 1\right) = K_4,$$

say: and this inequality is plainly inconsistent with the convergence of the series (22). Hence the theorem is established.*

13. Suppose, in particular, that

$$\mu_1 = 0, \quad \mu_n = 1/(n-1) \quad (n > 1).$$

Then the series is summable if

$$\left(\frac{s_1}{1} + \frac{s_2}{2} + \ldots + \frac{s_n}{n}\right) \Big/ \log n \to s,$$

and the condition $|\lambda_n a_n/\mu_n| < K$ takes the form $|n \log n a_n| < K$. If this condition is satisfied the series cannot be summable unless it is convergent.

The analogue of Frobenius's theorem in this case takes the form: *if the series Σa_n is summable, and $s_n n^{-x} \to 0$ for all positive values of x, then $\Sigma a_n n^{-x}$ is convergent, and $\Sigma a_n n^{-x} \to s$ as $x \to 0$.*

It follows from the theorem cited in § 8 that by making $\Sigma \mu_n$ diverge *more and more slowly*, we obtain more and more powerful methods of summation. The theorem just proved, however, assigns a limit to the efficacy of the method for any particular choice of μ_n.

Thus the series
$$\Sigma n^{-1-ai}$$
is not summable when $\mu_n = 1$: but it is summable if $\mu_n = 1/n$ and, *a fortiori*, if $\mu_n \prec 1/n$. The series

$$\Sigma \frac{1}{n(\ln)^{1+ai}}$$

is not summable when $\mu_n = 1/n$, since now $|n \ln n a_n| = 1$, but it is, as we shall see in a moment, summable when $\mu_n = 1/n \ln n$, and, *a fortiori*, if $\mu_n \prec 1/n \ln n$: and so on. In general, *the slower the oscillations of a series*,

* In giving our definition we restricted ourselves to the case in which $\mu_n \prec 1$. All that we have actually assumed in §§ 11, 12 is that $\lambda_{n+1}/\lambda_n \to 1$, a condition certainly satisfied when $\mu_n \prec 1$, but satisfied also if, *e.g.*, $\mu_n = n, n^2, \ldots$ (not if $\mu_n = e^n$). But, as we stated before, the case of real interest is that in which $\mu_n \prec 1$.

the harder it is to sum: this, at first sight, appears paradoxical. But the paradox disappears when we reflect that, if there are no oscillations, *i.e.*, if all the terms are positive, *no* method of summation by mean values can be effective except when the series is convergent.

14. Consider, in particular, the series

$$\Sigma \phi'(n) e^{ai\phi(n)}, \tag{23}$$

where $\phi(n)$ is a function which tends steadily to infinity, while $\phi'(n)$ tends steadily to zero: so that, assuming $\phi(n)$ to be in every sense *à croissance régulière*, we have

$$\phi(n) \prec n.$$

Suppose further that $\Sigma\{\phi'(n)\}^2$ is convergent, as will certainly be the case if

$$\phi(n) \prec n^{\frac{1}{2}-\delta}, \quad \phi(n) \prec n^{\frac{1}{2}}(ln)^{-\frac{1}{2}-\delta}, \quad \ldots, \quad (\delta > 0).$$

Then it is known* that

$$\sum_1^n \phi'(\nu) e^{ai\phi(\nu)} = \frac{1}{ai} e^{ai\phi(n)} + C + \epsilon_n, \tag{24}$$

where C is a constant, and $\epsilon_n \to 0$ as $n \to \infty$. Hence the condition that the series (23) should be summable $(R1, \mu_n)$ is that

$$\{\mu_2 e^{ai\phi(1)} + \mu_3 e^{ai\phi(2)} + \ldots + \mu_{n+1} e^{ai\phi(n)}\}/\lambda_{n+1}$$

should tend to a limit as $n \to \infty$.

Suppose that

$$\mu_{n+1} = \phi'(n).$$

Then a renewed application of the formula (24) shows at once that this condition is satisfied, and that the limit is 0. Hence we can certainly sum the series by taking

$$\mu_{n+1} \preccurlyeq \phi'(n).$$

If, *e.g.*, $\phi(n) = \log n$, we can sum the series by taking $\mu_{n+1} \preccurlyeq 1/n$, and so on.

It is easy to extend this conclusion to more general series of the type

$$\Sigma \chi(n) e^{ai\phi(n)},$$

where $\phi(n) \prec n$ and $\Sigma \chi(n) \phi'(n)$ is convergent. In many cases of this kind it may be shown that

$$s_n \sim \frac{\chi(n)}{ai\phi'(n)} e^{ai\phi(n)}.$$

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* Bromwich, *Proc. London Math. Soc.*, Vol. 6, p. 330.

If we take $$\mu_{n+1} = \phi'(n),$$

and suppose $$\chi(n) \prec \phi'(n)\, e^{x\phi(n)}$$

for any positive value of x, we see that

$$\Sigma \chi(n)\, e^{(\alpha i - x)\phi(n)}$$

has a limit as $x \to 0$: and, with appropriate restrictions on $\chi(n)$, this conclusion may be extended to more general series of the type

$$\Sigma \chi(n)\, e^{\alpha i \phi(n)} \{\psi(n)\}^{-x},$$

where $$\psi(n) \preccurlyeq e^{\phi(n)}.$$

15. Euler defined the sum of the oscillatory series Σa_n as the limit, if it exists, of $\Sigma a_n x^n$ as $x \to 1$. It is natural to generalise this definition by saying that the sum of the series is

$$\lim_{x \to 0} \Sigma a_n e^{-\lambda_n x}, \tag{25}$$

if it exists. The theorems of the preceding sections give us a number of criteria as to the existence or non-existence of this limit.

As λ_n is chosen so as to tend to infinity less and less rapidly, the greater becomes the probability that the series in (25) will not converge for all positive values of x—in which case there can, of course, be no question of the existence of the limit (25). This is obvious: the more interesting fact, which emerges from our previous analysis, is this—that, the *less* the increase of λ_n, then, if the series in (25) converges for all positive values of x, the *greater* the probability of the existence of the limit. Thus

$$\lim \Sigma n^{-1-\alpha i} e^{-nx}$$

does not exist, but $$\lim \Sigma n^{-1-\alpha i} n^{-x}$$

does: and $$\lim \Sigma n^{-1} (\log n)^{-1-\alpha i} n^{-x}$$

does not, but $$\lim \Sigma n^{-1} (\log n)^{-1-\alpha i} (\log n)^{-x}$$

does: and so on. As the definition becomes less effective in one direction, it becomes, by way of compensation, more effective in another.

As an illustration of this general principle, I shall prove the following theorem:*—

If $$\Sigma a_n n^{-x}$$

* This theorem is independent of any assumption as to the summability of Σa_n by mean values.

is convergent for $x > 0$, *and if the series* $\Sigma a_n x^n$ *tends to a limit as* $x \to 1$, *then the series* $\Sigma a_n n^{-x}$ *will tend to the same limit as* $x \to 0$.

That is to say, of the two series

$$\Sigma a_n x^n, \quad \Sigma a_n n^{-x},$$

the convergence of the second for $x > 0$ involves that of the first for $0 < x < 1$, but the converse is not true: but, if the second is convergent, then the existence of a limit for the first as $x \to 1$ involves that of a limit for the second as $x \to 0$, while again the converse is not true.

16. I shall base my proof on the following theorem, which I have proved elsewhere :*—

If
$$\phi(t) = \sum_1^\infty a_n e^{-nt},$$

and $R(x) > 0$, *then*

$$\Sigma a_n n^{-x} = \frac{1}{\Gamma(x)} \int_0^\infty \phi(t)\, t^{x-1}\, dt,$$

whenever the series on the left-hand side is convergent.

Since $x\Gamma(x) \to 1$ as $x \to 0$, and it is given that $\phi(t)$ tends to a limit, which we may denote by $\phi(0)$, as $t \to 0$, it is clear that what we have to show is that

$$\int_0^\infty x t^{x-1} \phi(t)\, dt \to \phi(0). \tag{26}$$

Here $\phi(t)$ is a function continuous for $t = 0$, and, as is easily seen, less than a constant multiple of e^{-t} when t is large.

Let
$$\phi(t) - \phi(0) = \psi(t),$$

so that $\psi(t) \to 0$ with t. Then we have

$$\int_0^\infty x t^{x-1} \phi(t)\, dt = \phi(0) + \int_0^1 x t^{x-1} \psi(t)\, dt + \int_1^\infty x t^{x-1} \phi(t)\, dt.$$

The last integral is numerically less than

$$Kx \int_1^\infty e^{-t} dt < Kx,$$

* *Messenger of Mathematics*, Vol. XXXIX, p. 136.

and has the limit 0 as $x \to 0$. Again, if we choose δ so that $|\psi(t)| < \epsilon$, for $0 \leqslant t \leqslant \delta$, we have

$$\left| \int_0^1 x t^{x-1} \psi(t)\, dt \right| < \epsilon \int_0^\delta x t^{x-1}\, dt + x \int_\delta^1 t^{x-1} |\psi(t)|\, dt.$$

The first term is less than ϵ; and when δ is fixed we can choose x_0 so that the second is less than ϵ for $0 \leqslant x \leqslant x_0$. Hence

$$\int_0^1 x t^{x-1} \psi(t)\, dt \to 0\,; \tag{27}$$

and the truth of (26) follows.

This result is capable of considerable generalisation. Thus in the note in the *Messenger of Mathematics*, quoted above, I proved that:—

If $\phi(n)$ is any positive function of n which tends steadily to infinity with n, and $R(x) > 0$, then

$$\int_0^\infty \{\Sigma a_n e^{-t\phi(n)}\}\, t^{x-1}\, dt = \Gamma(x)\, \Sigma\, \frac{a_n}{\{\phi(n)\}^x},$$

whenever the last series is convergent: and from this it may be deduced that:

If $\Sigma a_n e^{-x\phi(n)} \to s$ as $x \to 0$, then

$$\Sigma\, \frac{a_n}{\{\phi(n)\}^x} \to s$$

as $x \to 0$.

17. All the theorems that have been discussed in this paper suggest numerous generalisations.

(i) Riesz has given a definition of summability (Rk) related to summability ($R1$) as Cesàro's more general method is to its simplest case. It is suggested that we extend Theorems (i) and (iii) to cover this case: thereby obtaining analogues of Hölder's extension of Frobenius's theorem and of the result of § 5.

(ii) The theorems of the last section suggest more general theorems as to the cases in which the conditions

$$\Sigma a_n e^{-\lambda_n x} \to s, \qquad \Lambda_n \prec \lambda_n$$

imply

$$\Sigma a_n e^{-\Lambda_n x} \to s.$$

(iii) Our theorems are all capable of generalisations bringing in com-

plex values of x, and resembling in this respect Stolz's generalisation of Abel's original theorem on the continuity of power series.

(iv) All our theorems have analogues concerning integrals of the type

$$\int_1^\infty \phi(t)\,\{\psi(t)\}^{-x}\,dt.$$

It is not my object, however, to work out a whole hierarchy of theorems of a similar character: my desire is rather to take a definite text—"the arithmetic methods of summation of more or less slowly oscillating series"—and to prove just those theorems necessary to throw a general light upon the subject. There will be time to work out proofs of the more elaborate theorems when the need arises for their application to some special class of functions.

[*Added April*, 1910.—It is perhaps worth while pointing out that when, in § 3, $na_n \to 0$ (and not merely $|na_n| < K$), then a much simpler direct proof of the theorem can be given. For, if $b_n = na_n \to 0$, it follows at once that

$$t_n/n = (b_1+b_2+\ldots+b_n)/n \to 0,$$

so that the condition (4) of § 3 is obviously satisfied. A similar simplification can be effected in § 12, when $\lambda_n a_n/\mu^n \to 0$ (instead of merely $|\lambda_n a_n/\mu_n| < K$). For then $b_n/\mu_n \to 0$, and so

$$\frac{t_n}{\lambda_n} = \frac{b_1+b_2+\ldots+b_n}{\mu_1+\mu_2+\ldots+\mu_n} \to 0,$$

so that the condition (21) of § 12 is certainly satisfied.

The proof of §§ 4, 5 can also be simplified in the same way in this case. But the theorems obtained in this way would not be general enough to enable us to deal with any of the most interesting cases considered in the paper.]

ON THE REPRESENTATION OF A GROUP OF FINITE ORDER AS A GROUP OF LINEAR SUBSTITUTIONS WITH RATIONAL COEFFICIENTS

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THE theorem that a group of linear substitutions of finite order is either irreducible or is completely reducible was materially extended by Prof. Loewy in a memoir, "Über die Reducibilität der Gruppen linearer homogener Substitutionen," published in the *Transactions of the American Mathematical Society* for January, 1903.

The theorem, in its original form, implies that there is no limitation on the arithmetical nature of the coefficients. Herr Loewy considers a group of linear substitutions, the coefficients in which belong to a given field of rationality R. He shews that, if there is a definite Hermitian bilinear form which is invariant for the group, then (i) when the coefficients in the component groups are limited to the field R, the group is either irreducible or completely reducible; and, (ii) if the reduction is carried out in two different ways, a one-to-one correspondence can be established between the two sets of irreducible components, such that corresponding components are equivalent to each other. Since, for a group of linear substitutions of finite order there is always an invariant definite bilinear Hermitian form, Herr Loewy's extension of the theorem of complete reducibility is, as he points out, always applicable to groups of finite order.

The simplest field of rationality that can be chosen is the field of the rational numbers. Every group of finite order must admit a certain number of distinct representations in which all the coefficients are rational numbers, such that no one of them is reducible without introducing irrational numbers. These may be described as rationally irreducible rational representations. If these are denoted by H_1, H_2, ..., H_ρ, Herr Loewy's theorem, as applied to the rational representations, is that every rational representation can be denoted by such a symbol as $\sum_{i=1}^{i=\rho} c_i H_i$, and that this can be done in one way only.

The number of distinct (absolutely) irreducible representations of a group of finite order is known to be equal to the number of distinct sets of conjugate operations in the group; and the question arises as to whether the number of distinct rationally irreducible rational representations is given by any similar simple law. It is shewn in what follows that this is the case, the number in question being the same as the number of distinct sets of conjugate cyclical subgroups contained in the group.

1. If $$S_1,\ S_2,\ \ldots,\ S_h$$

is a set of conjugate operations in a group G of finite order N, and if μ is any number that is prime to the order of S_1, then the operations

$$S_1^{\mu},\ S_2^{\mu},\ \ldots,\ S_h^{\mu}$$

are all distinct and constitute a conjugate set which will, for certain values of μ, be the same as the previous set.

Let the given set contain

$$S_1,\ S_1^{\mu_2},\ S_1^{\mu_3},\ \ldots,\ S_1^{\mu_s},$$

where each index represents a distinct number less than and prime to m, the order of S_1, and no other distinct power of S_1. Then, when, in the above process, the values $1, \mu_2, \mu_3, \ldots, \mu_s$ are used for μ, the new conjugate set is the same as the old; but, if for μ any other of the $\phi(m)$ numbers less than and prime to m is used, the new set will be different from the old. Moreover, if μ_{s+1} is any number less than and prime to m, then

$$\mu_{s+1},\ \mu_2\mu_{s+1},\ \mu_3\mu_{s+1},\ \ldots,\ \mu_s\mu_{s+1},$$

when used for μ, all lead to the same conjugate set, and no other value of μ, less than and prime to μ, leads to this set. Hence s must be a factor of $\phi(m)$, and if

$$\phi(m) = st,$$

just t distinct sets of conjugate operations arise in this way from the original one. The r conjugate sets in G fall in this way into a number of systems such that, if S belongs to a system of sets, the system of sets consists of all operations of G which are conjugate to S or to any power of S which is of the same order as S.

Let ρ denote the number of such systems. The system containing S consists of all the operations belonging to the set of groups conjugate to $\{S\}$ and not belonging to the subgroups of these groups. Hence ρ is the number of distinct conjugate sets of cyclical subgroups contained in G.

Let n be the least common multiple of the orders of the operations of G; and ν any number less than and prime to n. Then, when every operation S is replaced by S^ν, the conjugate sets of operations of G will be permuted among themselves in such a way that the conjugate sets in each of the ρ systems are permuted among themselves.

To each value of ν will correspond a permutation of the sets belonging to each system, and to $\nu\nu'$ will correspond the product of the permutations given by ν and ν' respectively. The permutation group of the sets of each system that thus arises is therefore transitive and Abelian.

If

$$1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_s$$

are the distinct values of ν which leave every conjugate set unchanged, then the values

$$\nu',\ \nu_2\nu',\ \nu_3\nu',\ \ldots,\ \nu_s\nu'$$

of ν, and no others, give the same permutation of the conjugate sets. Hence s must be a factor of $\phi(n)$, and the order of the Abelian permutation group on the conjugate sets is $\phi(n)/s$. It is intransitive and permutes the sets in ρ transitive systems.

Let

$$D_1,\ D_2,\ \ldots,\ D_\rho$$

represent the sums of the operations of G belonging to the ρ systems of conjugate sets. (It is convenient to take D_1 to be the system containing E, the identical operation, only.) Then it may be shewn that the D's combine by multiplication in a manner similar to the conjugate sets. Since each D consists of the sum of a number of conjugate sets, the same must be true of the product of any two D's. Assume, then, that

$$D_i D_j = \sum_{k=1}^{k=r} c_k C_k,$$

where the C's denote the conjugate sets. [This relation, being an identity when the multiplication table of the group is taken account of, must hold between the corresponding substitutions in any irreducible representation. Now, if

$$D_i = C_{i_1} + C_{i_2} + \ldots + C_{i_s},$$

the operation that corresponds to D_i in any irreducible representation replaces each symbol by μ_i times itself, where

$$\mu_i = h_{i_1}(\chi_{i_1} + \chi_{i_2} + \ldots + \chi_{i_s})/\chi_1,$$

and is therefore a rational integer. Hence the operation corresponding to $\Sigma c_k C_k$ replaces each symbol by a rational integral multiple of itself. Therefore

$$\sum_k c_k \frac{h_k \chi_k^p}{\chi_1^p} = \nu_p,$$

where, for each p, ν_p is a rational integer. Hence

$$c_k = \frac{1}{N} \sum_p \chi_1^p \nu_p \chi_{k'}^p.$$

Now, if ω is a primitive m-th root of unity, where m is the order of the operation of the k-th set, $\chi_{k'}^p$ is the sum of χ_1^p powers of ω; and since $\sum_p \chi_1^p \nu_p \chi_{k'}^p$ is rational it must be unaltered when in each $\chi_{k'}^p$, ω^μ is written for ω, μ being relatively prime to m.]* Hence, if C_p and C_q belong to the same system, c_p and c_q are equal. The relation may therefore be written

$$D_i D_j = \sum_{k=1}^{k=\rho} d_{ijk} D_k,$$

where the coefficients are zeroes or positive integers.

If h_i is the number of operations in the system D_i, it may be shewn exactly as with conjugate sets that $d_{ijk}\mathsf{h}_k$ is a multiple both of h_i and h_j, and that

$$d_{ijk}\mathsf{h}_k = d_{ikj}\mathsf{h}_j.$$

2. If ω is a primitive n-th root of unity, each characteristic is a sum of powers of ω. Now it has been shewn† that, if

$$\chi_1, \ \chi_2, \ \ldots, \ \chi_r$$

are the characteristics in an irreducible representation Γ of G, then $\chi_1\Gamma$ can be expressed in a form in which the coefficients are rational functions of the χ's, and therefore of ω. Denote this form of $\chi_1\Gamma$ by H. If

$$s_1, \ s_2, \ \ldots, \ s_N$$

are the substitutions of H, and if $s_i^{(\nu)}$ is what s_i becomes when in each of its coefficients ω^ν is written for ω (ν being relatively prime to n), then

$$s_1^{(\nu)}, \ s_2^{(\nu)}, \ \ldots, \ s_N^{(\nu)}$$

constitute a group H_ν simply isomorphic with H, s_i and s_i^ν being corresponding substitutions. Since H has χ_1 irreducible components each equivalent to Γ, H_1 must have χ_1 irreducible components equivalent to each other; and the characteristics in one of these are

$$\chi_1^{(\nu)}, \ \chi_2^{(\nu)}, \ \ldots, \ \chi_r^{(\nu)},$$

where $\chi_i^{(\nu)}$ is what χ_i becomes when ω^ν is written for ω.

If, then, in the characteristics of an irreducible representation of G, expressed as rational functions of ω, we replace ω by any other primitive n-th root of unity, the new set of quantities are again a set of characteristics of an irreducible representation of G.

* This passage has been altered since the paper was communicated to the Society.

† *Proc. London Math. Soc.*, Series 2, Vol. IV, p. 4.

Moreover, if

$$1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_\sigma$$

are the only values of ν (prime to and less than n) such that, writing ω^ν for ω, leave each characteristic of the set

$$\chi_1,\ \chi_2,\ \ldots,\ \chi_r$$

unchanged, then

$$\nu',\ \nu_2\nu',\ \nu_3\nu',\ \ldots,\ \nu_\sigma\nu'$$

and no others change the set into another definite set. Hence σ must be a factor of $\phi(n)$; and in this way, by using all values of ν, just $\phi(n)/\sigma$ distinct sets of characteristics arise. These (or, more shortly, the corresponding irreducible representations) are permuted among themselves when ω is replaced by ω^ν; and the group of permutations that arises is a transitive Abelian group.

Suppose, now, that

$$1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_{s'}$$

are the only distinct values of ν which leave each characteristic in each set unchanged. Then, since no two different conjugate sets have the same characteristics in each of the r representations, S, S^{ν_2}, S^{ν_3}, ..., $S^{\nu_{s'}}$ (not necessarily all distinct) are conjugate operations, and therefore the numbers

$$1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_{s'}$$

all occur among

$$1,\ \nu_2,\ \nu_3,\ \ldots,\ \nu_s.$$

On the other hand, since, for each S,

$$S,\ S^{\nu_2},\ S^{\nu_3},\ \ldots,\ S^{\nu_s}$$

are conjugate operations, each of the latter set of numbers occurs among the former. Hence the two sets are identical.

The Abelian group of permutations of the r conjugate sets that arises when S^ν is written for S is therefore simply isomorphic with the Abelian group of permutations of the r sets of characteristics (or of the r irreducible representations) that arises when ω^ν is written for ω in the characteristics.

Let ρ' denote the number of transitive systems in which the r irreducible representations are permuted by the Abelian group; and let τ_j ($j = 1, 2, \ldots, \rho'$) be the number in each system. It will now be shewn that ρ' is the same as ρ.

Let

$$\Gamma_1,\ \Gamma_2,\ \ldots,\ \Gamma_{\tau_j}$$

be the irreducible representations in the j-th system,* and

$$\chi_1^1,\ \chi_2^1,\ \ldots,\ \chi_r^1$$

* There is here a slight departure from the usual notation in which Γ_1 denotes the identical representation.

the group characteristics of Γ_1. Then

$$\chi_1^i,\ \chi_2^i,\ \ldots,\ \chi_r^i \quad (i = 1, 2, \ldots, \tau_j)$$

arise from $\chi_1^1,\ \chi_2^1,\ \ldots,\ \chi_r^1,$

on replacing ω by all the other primitive n-th roots of unity.

For each i, $\sum_{s=1}^{s=\tau_j} \chi_i^s$ is a rational integer, and, if the p-th and q-th conjugate sets belong to the same system,

$$\sum_{s=1}^{s=\tau_j} \chi_p^s = \sum_{s=1}^{s=\tau_j} \chi_q^s.$$

Hence, in the representation of G denoted by $\sum_{s=1}^{s=\tau_j} \Gamma_s$, all the operations belonging to a given system of conjugate sets have the same characteristic. For the i-th system this will be denoted by ψ_i^j.

Let D^i be the sum of t_i conjugate sets with the suffixes $i_1, i_2, \ldots, i_{t_i}$. Then

$$\begin{aligned}\psi_i^j &= \chi_{i_1}^1 + \chi_{i_1}^2 + \ldots + \chi_{i_1}^{\tau_j},\\ &= \chi_{i_2}^1 + \chi_{i_2}^2 + \ldots + \chi_{i_2}^{\tau_j},\\ &= \ldots \quad \ldots \quad \ldots\\ &= \chi_{i_{t_i}}^1 + \chi_{i_{t_i}}^2 + \ldots + \chi_{i_{t_i}}^{\tau_j}.\end{aligned}$$

Now $\chi_{i_1}^s + \chi_{i_2}^s + \ldots + \chi_{i_{t_i}}^s$ is rational, and is therefore the same for each s from 1 to τ_j. Hence

$$t_i \psi_i^j = \tau_j (\chi_{i_1}^s + \chi_{i_2}^s + \ldots + \chi_{i_{t_i}}^s)$$

for each s, and

$$\begin{aligned}t_i^2 (\psi_i^j)^2 &= \tau_j^2 (\chi_{i_1}^s + \chi_{i_2}^s + \ldots + \chi_{i_{t_i}}^s)^2\\ &= \tau_j \sum_{s=1}^{s=\tau_j} (\chi_{i_1}^s + \chi_{i_2}^s + \ldots + \chi_{i_{t_i}}^s)^2.\end{aligned}$$

Hence

$$\begin{aligned}t_i^2 \sum_{j=1}^{j=\rho'} \frac{1}{\tau_j} (\psi_i^j)^2 &= \sum_{j=1}^{j=\rho'} \sum_{s=1}^{s=\tau_j} (\chi_{i_1}^s + \chi_{i_2}^s + \ldots + \chi_{i_{t_i}}^s)^2\\ &= \sum_{\sigma=1}^{\sigma=r} (\chi_{i_1}^\sigma + \chi_{i_2}^\sigma + \ldots + \chi_{i_{t_i}}^\sigma)^2,\end{aligned}$$

where the last summation is extended to the r different irreducible representations of G.

Now

$$\sum_{\sigma=1}^{\sigma=r} \chi_i^\sigma \chi_{i'}^\sigma = \frac{N}{h_i}$$

and $$\sum_{\sigma=1}^{\sigma=r} \chi_i^\sigma \chi_j^\sigma = 0 \quad (j \neq i').$$

Hence, since among the i_1-th, i_2-th, ..., i_{t_i}-th sets the set conjugate to the i_1-th occurs just once, while h_i is the same for each of them,

$$\sum_{\sigma=1}^{\sigma=r} (\chi_{i_1}^\sigma + \chi_{i_2}^\sigma + \ldots + \chi_{i_{t_i}}^\sigma)^2 = t_i \frac{N}{h_i},$$

and $$\sum_{j=1}^{j=\rho'} \frac{1}{\tau_j} (\psi_i^j)^2 = \frac{N}{t_i h_i} = \frac{N}{\mathsf{h}_i}. \tag{1}$$

In precisely the same way it may be shewn that

$$\sum_{j=1}^{j=\rho'} \frac{1}{\tau_j} \psi_i^j \psi_k^j = 0 \quad (k \neq i). \tag{2}$$

Further, $$\sum_{i=1}^{i=\rho} \mathsf{h}_i (\psi_i^j)^2 = \sum_{i=1}^{i=r} h_i (\chi_i^1 + \chi_i^2 + \ldots + \chi_i^{\tau_j})^2.$$

Now, among the representations Γ_1, Γ_2, ..., Γ_{τ_j} the representation inverse to Γ_1 occurs just once. Moreover,

$$\sum_{i=1}^{i=r} h_i \chi_i^j \chi_i^{j'} = N,$$

$$\sum_{i=1}^{i=r} h_i \chi_i^j \chi_i^k = 0 \quad (k \neq j').$$

Hence $$\sum_{i=1}^{i=\rho} \mathsf{h}_i (\psi_i^j)^2 = \tau_j N. \tag{3}$$

In just the same way it may be shewn that

$$\sum_{i=1}^{i=\rho} \mathsf{h}_i \psi_i^j \psi_i^k = 0 \quad (k \neq j). \tag{4}$$

The relations (1), (2), (3), (4) express that the $\rho\rho'$ quantities

$$\sqrt{\frac{\mathsf{h}_i}{\tau_j N}}\, \psi_i^j, \quad (i = 1, 2, \ldots, \rho\,;\ j = 1, 2, \ldots, \rho'),$$

form an orthogonal system, and therefore ρ and ρ' must be the same. This result is equivalent to the following statement:—

The irreducible representations of a group of finite order fall into ρ sets, where ρ is the number of distinct conjugate sets of cyclical subgroups contained in the group. The characteristics of all members of a set are obtained from these of any one, expressed as rational functions of a primitive n-th root of unity, on replacing the primitive n-th root by all other primitive n-th roots; where n is the least common multiple of the orders of the operations of the group.

3. A rational representation of a group of finite order, *i.e.*, a representation in which the coefficients are rational numbers, must have rational characteristics.

Let $\sum_{s=1}^{s=r} c_s\Gamma_s$ be the completely reduced form of a representation with rational characteristics. Each characteristic is unaltered on replacing ω by any other primitive root of unity; and this process permutes transitively the irreducible representations belonging to any given system. Hence, if Γ_s and Γ_t belong to the same system, then $c_s = c_t$.

Let H_i denote the representation which, when completely reduced, has the τ_i representations of the i-th system, each once, for its irreducible components. Then the above result is equivalent to saying that all the representations in which the characteristics are rational are given by the form

$$\sum_{i=1}^{i=\rho} c_i H_i,$$

where the c's are zeroes or positive integers. It is not, however, necessarily the case that these representations can be expressed as rational representations. In fact, the irreducible representation of the quaternion group in two variables has rational characteristics, while it cannot be expressed with rational coefficients.

Though Γ cannot necessarily be expressed in a form in which the coefficients are rational in the characteristics, $\chi_1\Gamma$ can always be so expressed; and therefore, if $\Gamma_1, \Gamma_2, \ldots, \Gamma_\tau$ form a system,

$$\chi_1(\Gamma_1+\Gamma_2+\ldots+\Gamma_\tau) \quad \text{or} \quad \chi_1 H$$

can certainly be expressed with rational coefficients. It may, and generally will, be the case that this group is rationally reducible. When this is the case, it is completely reducible rationally by Herr Loewy's theorem, and each rational irreducible component having rational characteristics must be of the form aH. Moreover, so long as, in the reduced form, the a's are not all equal the reduction can certainly be carried further.

Hence, corresponding to each of the ρ systems of (absolutely) irreducible representations there is a single rationally irreducible rational representation that may be denoted by $a_i H_i$. In general, the numbers represented by a_i are unity, but there are exceptions. These ρ rational representations are certainly distinct, *i.e.*, no one can be transformed into another. In fact, if they were not so, the r (absolutely) irreducible representations would not be distinct.

The only other possible rationally irreducible representations would be denoted by the symbol $\Sigma c_i H_i$, where the c's are not multiples of the corresponding a's. But, if there were a rationally irreducible representa-

tion H denoted by $c_iH_i+c_jH_j$, then a_ia_jH would admit the two distinct complete reductions $a_ia_j.H$ and $a_jc_i.a_iH_i+a_ic_j.a_jH_j$. This is not possible by Herr Loewy's theorem; and therefore the ρ representations a_iH_i $(i=1, 2, \ldots, \rho)$ are the only rationally irreducible rational representations. Hence:

A group of finite order admits just ρ distinct rationally irreducible rational representations, where ρ is the number of distinct conjugate sets of cyclical subgroups contained in the group.

4. In addition to the relations (1), (2), (3), and (4), the numbers ψ_i^j satisfy the relations

$$\mathsf{h}_i\mathsf{h}_j\psi_i^s\psi_j^s = \psi_1^s \sum_{k=1}^{k=\rho} d_{ijk}\mathsf{h}_k\psi_k^s. \tag{5}$$

These, in fact, may be established precisely in the same way as the corresponding relations among the characteristics for the (absolutely) irreducible representations.

Moreover, if

$$\xi = \sum_{i=1}^{i=\rho} A_i \frac{\mathsf{h}_i\psi_i}{\psi_1^s},$$

then, by multiplying (5) by A_i/ψ_1^s and summing with respect to i,

$$\frac{\mathsf{h}_j\psi_j^s}{\psi_1^s}\xi = \sum_{i,\,k=1}^{i,\,k=\rho} A_i d_{ijk} \frac{\mathsf{h}_k\psi_k^s}{\psi_1^s}.$$

Hence, since the determinants of the ρ^2 quantities $\mathsf{h}_k\psi_k^s/\psi_1^s$ is not zero,

$$\begin{vmatrix} \sum_i A_i d_{i11}-\xi, & \sum_i A_i d_{i12}, & \ldots, & \sum_i A_i d_{i1\rho} \\ \sum_i A_i d_{i21}, & \sum_i A_i d_{i22}-\xi, & \ldots, & \sum_i A_i d_{i2\rho} \\ \ldots & \ldots & \ldots & \ldots \\ \sum_i A_i d_{i\rho 1}, & \sum_i A_i d_{i\rho 2}, & \ldots, & \sum_i A_i d_{i\rho\rho}-\xi \end{vmatrix} = 0.$$

This is a homogeneous equation of degree ρ in ξ, A_1, A_2, ..., A_ρ; and, since the ψ's are rational integers, it falls into ρ linear factors with rational coefficients which are determinable by rational processes. The ratios of the characteristics in the ρ rationally irreducible rational representations can therefore be calculated by rational processes when the d's are known.

ON THE DETERMINATION OF A SEMI-CONTINUOUS FUNCTION FROM A COUNTABLE SET OF VALUES

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1. The remarkable extent to which semi-continuous functions possess properties which are the same as, or analogous to, those of continuous functions, is now well known. The object of the present note is to add yet another to the list of such properties.

Not the least important of the properties of a continuous function is the property that, if its values are known at a countable everywhere-dense set of points, the function is uniquely determined, viz., we have only to take the unique limit with respect to the given set of points at each point at which the value of the function is unknown, and this is the required value.

The corresponding property of, for example, an upper semi-continuous function is here found to be that a countable set of points dense everywhere *can always be found* such that, when the values of the function at these points are known, the value at any remaining point can be determined, being, in fact, the upper limit at that point of the known values of the function.

We first (§ 2) shew that this is the utmost that can be stated, that the theorem is, in fact, not in general true if the countable everywhere-dense set of points be taken at random. This is an important distinction between functions that are continuous and those that are only semi-continuous, and carries with it as a consequence that in the applications whole classes of derived theorems which can be stated about continuous functions cease to hold for semi-continuous ones.

In § 3 we use the principle that a semi-continuous function can be expressed as the limit of a monotone sequence of continuous functions to shew that a knowledge of a suitable countable set of values may, *a priori*, be expected to enable us to determine the function. In § 4 we give one possible choice of these points, the essential feature of the chosen set being

that *it includes the points at which the function assumes its upper (lower) bound in each of the intervals obtained by continued bisection of the original interval.*

It is clear that any set of points suitable for the unique determination of a semi-continuous function must include the set S_1 of points, known* to be countable, at which the function is greater (less) than all the limits of values in the neighbourhood, and it might be supposed that any countable everywhere-dense set which included that set S_1 would serve the purpose. This is, however, not true; indeed it is not even sufficient if the chosen countable set includes, besides S_1, another set known to be countable, viz., the set S_2 of points at which the limits differ as regards right and left. If this were the case it would follow that an upper (lower) semi-continuous function for which there is no distinction of right and left at points of discontinuity, and which is also such that its value is at no point greater (less) than all the limits of values in the neighbourhood, would be uniquely determined by its values at any countable set of points dense everywhere.

This is shewn not to be the case, and at the same time the existence of this class of semi-continuous functions is demonstrated, by the actual construction, in § 6, of such a function, whose value is zero at all its points of continuity. If this function is doubled, the properties in question are unaltered, and we get in this way two functions of the type considered, agreeing only at the common zeros, that is at the points of a set which is not only everywhere dense, but is complementary to a set of the first category.

We may add that it follows from a theorem given in the *Rendiconti di Palermo* (1907), quoted below, that such semi-continuous functions belong to the class of functions which assume in every interval all values between their upper and lower bounds in that interval.

As regards the method of choice of the determining set of points previously shewn to be possible, we may remark that it is applicable to functions of n variables.

2. If we assign at any chosen countable set of points dense everywhere particular values, we can always find an upper (lower) semi-continuous function having those values at those points, provided those

* W. H. Young, "Sulle due funzioni a più valori costituite dai limiti d'una funzione d'una variabile reale a destra e a sinistra di ciascun punto," *Rend. dell. R. Acc. dei Lincei*, Tomo XVII, Ser. 2 (1908), pp. 582–587; "On the Distinction of Right and Left at Points of Discontinuity," 1907, *Quarterly Journal of Pure and Applied Mathematics*, Vol. XXXIX, pp. 67–83.

values are upper (lower) semi-continuous with respect to the chosen countable set. In this case, in fact, we only have to define the value of the function at any point not belonging to the chosen set as the upper (lower) limit of the assigned values.

To prove that the function so defined is upper (lower) semi-continuous, we have only to remember that a limit of limits is itself a limit, so that the new values in the neighbourhood of any point, being limits of the assigned values, introduce no new limits at the point, and therefore leave the upper (lower) limit at the point unaltered. Hence the relation of the value of the function to its upper (lower) limit at any point of the chosen set is unaltered, and at each remaining point the value of the function is precisely its upper (lower) limit, so that the function is upper (lower) semi-continuous.

That the semi-continuous function so determined is not necessarily the only such function is obvious. We could, of course, assign any higher (lower) value at any isolated point. That it is not even the only upper (lower) semi-continuous function whose value at each point not belonging to the chosen set is precisely the upper (lower) limit of all the values in the neighbourhood is seen by the following example :—

Ex.—Let the assigned values be zero at all the rational points of an interval, and let S be any perfect set of irrational points. S is, of course, dense nowhere.

The function determined in the manner above indicated is zero everywhere. Another upper semi-continuous function of the type mentioned is, however, zero everywhere except at the points of S, where it has the value unity. It should be noticed that this latter function presents a difference of right and left as regards its limits at each of the end-points of the black intervals* of S.

3. Consider, for definiteness, an upper semi-continuous function. This may be expressed as the limit of a monotone descending sequence of continuous functions. Such a set of functions

$$f_1 \geqslant f_2 \geqslant \ldots \geqslant f_n \geqslant \ldots,$$

as has been shewn,† may be obtained as follows. To determine the

* See *The Theory of Sets of Points*, by W. H. Young and Grace Chisholm Young, 1907, Camb. Univ. Press, p. 19.

† R. Baire, "Sur les fonctions de variables réelles," *Ann. di Mat.*, Ser. III A, Vol. III, 1899. Other references in Hobson's *Functions of a Real Variable*, Camb. Univ. Press, 1906,

function f_n, divide the segment under consideration into 2^n equal parts, and at each point of division erect an ordinate equal to the upper bound of the upper semi-continuous function f in the segment composed by the two small intervals abutting at that point of division. At the end-points we erect ordinates equal to the upper bound in the respective small intervals ending at the points in question. Joining the tops of these ordinates by a broken line, drawn from each to the next, the equation of this broken line is taken to be

$$y = f_n(x).$$

As the whole succession of these curves, for all values of n, is known from a knowledge of the values of f at a certain countable everywhere-dense set of points, it is plain that their limiting function, which necessarily exists, is also so determined. But this limiting function is f.

Here we have supposed f to be a function of only one variable, but the method is general.

4. THEOREM.—*Given any upper semi-continuous function, there is a certain countable set S of points, dense everywhere, such that the value of the function at any point not belonging to the set is the upper limit at the point of the values of the function at points of the set in the neighbourhood.*

Let (a, b) be the interval in which the upper semi-continuous function is defined. Then $f(x)$ attains its upper bound; let p_1 be a point at which this occurs, which may be a or b or some internal point of the interval.

Bisect (a, b) at m_1 (which may by chance coincide with p_1) and by the like process determine the points in each of the two half-intervals (a, m_1), (m_1, b) at which f attains its upper bound in that half-interval. One of these points is bound to be p_1; the other will, unless p_1 coincides with m_1, certainly be another point, say p_2.

We then bisect each of the half-intervals and repeat the process,

p. 525. See also W. H. Young, "On Monotone Sequences of Continuous Functions" (1908), *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 523, where the monotony of Baire's sequence is pointed out, and the converse theorem is proved, that every monotone sequence of continuous functions defines a semi-continuous function. A more extended theorem will be found in a paper by the same author, "On Functions defined by Monotone Sequences and their Upper and Lower Bounds," 1908, *Mess. Math.*, Vol. XXXVII, pp. 148–154.

and so on, *ad infinitum*. In this way we obtain the points of bisection $m_1, m_2, \ldots$ which form a countable set of points dense everywhere in (a, b), and all are distinct from one another. We also obtain the points $p_1, p_2, \ldots, p_n, \ldots$ at which f assumes its upper bound in the successive intervals determined by $m_1, m_2, \ldots$ as indicated. These points $p_1, p_2, \ldots$ are such that in each of these successive intervals there lies one of them, and it may happen by chance that a second is an end-point of the same interval; thus there certainly is a countably infinite set of the points p_n, and they are dense everywhere in (a, b). Certain of the points p_n may coincide with certain of the points m_i, but this will not cause any confusion.

First we proceed to shew that, if there is any point Q at which f has a value greater than its upper limit, this point Q is one of the points p_n. For in this case, supposing that the value of $f(x)$ at Q exceeds its upper limit at Q by more than e, there is an interval d, with Q as centre, such that at every point of this interval, except Q, $f(x)$ is less than $f(Q)-e$. Thus the point in the interval d at which $f(x)$ attains its upper bound is Q, and the same is true of any interval inside d and containing Q. But, when n is sufficiently large, the ratio of the length of each of the intervals between the points $a, m_1, m_2, \ldots, m_n, b$, taken in order from a to b, to the length (a, b), being $1/2^n$, is less than $\frac{1}{2}d$, so that one of these intervals certainly lies inside d and contains Q, and in this interval Q is the point at which f attains its upper bound, so that Q is, by their definition, one of the points p_n, as was stated.

Let the countable set S be taken to consist of the points $p_1, p_2, \ldots, p_n, \ldots$ and the points $m_1, m_2, \ldots, m_n, \ldots$. Then S is dense everywhere.

Next we proceed to shew that at any point not belonging to the set S (at such a point, by what precedes, the value of f is its upper limit at the point) the value of f is its upper limit at that point with respect to the set of points S.

For, let q be such a point and $q_1, q_2, \ldots$ a sequence of points with q as limiting point, passing along which $f(x)$ has an unique limit which is its upper limit at q, viz., $f(q)$.

Then, since q is not a point of S, it is an internal point of one of the intervals (a, m_1), (m_1, b), say of the former. In this interval (a, m_1) there is one of the points q_n, since these form a sequence with q as limiting point; let the first of these in the interval be q_{i_1}, and let, say, p_{j_1} be the point of (a, m_1) at which f attains its upper bound in (a, m_1). Then p_{j_1} is a point of S, and we have

$$f(p_{j_1}) \geqslant f(q_{i_1}), \tag{1}$$

where, if q_{i_1} is a point of S, it may coincide with p_{j_1}.

Again, at a later stage of bisection, the interval containing the point q will be so small that it contains neither p_{j_1} nor q_{i_1}, and we get from it in the same way two points p_{j_2} and q_{i_2}, of which the first is certainly a point of S, such that

$$f(p_{j_2}) \geqslant f(q_{i_2}). \tag{2}$$

Proceeding thus, *ad infinitum*, we get in the limit

$$\underset{n=\infty}{\text{Llt}}\, f(p_{j_n}) \geqslant \underset{n=\infty}{\text{Lt}}\, f(q_{i_n}) \geqslant f(q) \ldots. \tag{3}$$

But $f(q)$ is the upper limit of $f(x)$ at q, so that this is only possible if along the sequence $p_{j_1}, p_{j_2}, \ldots$ $f(x)$ has an unique limit, and this limit is $f(q)$. But, since the points of this sequence are all points of S, this shews that the upper limit at a as we approach the point a by means of the set S is $f(a)$. This proves the theorem.

If we attempt to extend this reasoning, which has so far tacitly assumed that there is only one independent variable, to functions of real variables generally, we shall be met with the apparent difficulty that the points of division are now replaced by straight lines, so that these points have the potency of the continuum. In the above proof, however, it was only for the sake of simplicity in the wording that the points of division $m_1, m_2, \ldots$ were included in the set S. If we do not include them, the point q may be a point of division at some stage, and will then not determine uniquely the corresponding interval. Nevertheless we can choose out one of the intervals meeting at q as containing points of the sequence $q_1, q_2, \ldots$. The argument given above then applies, shewing that the value $f(q)$ is the upper limit of $f(x)$ for approach to q along the chosen set S.

When this modification has been made, the proof given is at once generalisable for any number of dimensions, the division into intervals by continual bisection being replaced, in the usual way, by division of the plane into rectangles, of space of three dimensions into parallelepipeds, &c.

5. THEOREM. — *An upper (lower) semi-continuous function is not necessarily completely determined even when its values are known at a countable set of points everywhere dense, which includes the two sets, each of which is known to be countable,* (1) *at which its limits differ as regards right and left,* (2) *at which the function is greater (less) than all the limits of values in the neighbourhood.*

To prove this theorem it is sufficient to shew that upper (lower) semi-continuous functions exist, having no distinction of right and left at points of discontinuity, and whose value at each point is not greater (less) than the limits of values in the open neighbourhood of the point, and which are not determined by their values at a countable set of points dense everywhere, unless that set be suitably chosen. In particular it will be sufficient to prove the existence of an upper semi-continuous function, which, without being identically zero, is zero at an everywhere-dense set of points, and has the above-mentioned characteristics : (1) it has no distinction of right and left as regards limits, and (2) its value at any point is not greater than the upper limit of the values in the neighbourhood.

It is plain that a knowledge of the zeros of such a function will not give us the function ; moreover, by, for example, doubling such a function, we should get a second function of the type considered, agreeing with the former at an everywhere-dense set of points.

6. We now proceed to construct such a function. It will be noticed that we follow somewhat closely the method used in another paper for another purpose.*

The function $f_1(x)$, the first of the monotone descending sequence of continuous functions about to be constructed, is the familiar function which effects the (1, 1)-correspondence, maintaining the order, of Cantor's typical ternary set with the continuum. We form Cantor's typical ternary set, as is well known, by trisecting the segment (0, 1) blackening the middle part, and repeating this process *ad infinitum* in each white part at each stage. Cantor's set consists of the end-points and the external points of the black intervals. The distances of the points of Cantor's set from the origin are conveniently expressed in the ternary scale: they consist of all the ternary fractions involving only the figures 0 and 2, and, in particular, the terminating fractions are the right-hand end-points of the black intervals. If N denote any terminating combination of 0's and 2's, the left-hand end-point of the black interval whose right-hand end-point is $\cdot N2$, is expressed as the pure circulator $\cdot N\dot{0}\dot{2}$, involving only the figures 0 and 2, or, in another way, it is $\cdot N1$.

The abscissa of the point of the continuum (0, 1) which is familiarly

* W. H. Young, "On the Construction of a Pointwise Discontinuous Function all of whose Continuities are Infinities and which has a Finite Generalised Integral," *Quart. Jour. of Pure and Applied Math.*, Vol. xxxix, pp. 217–221.

taken to correspond to any point of Cantor's set is formed from the abscissa of the corresponding point by changing every 2 into 1 and interpreting in the binary scale. In this way each point x of Cantor's set has a different correspondent from any other point of Cantor's set, except in the case of the end-points a and b of any black interval of Cantor's set, which have the same correspondent, since in the binary scale any number ending in 1 is the same as the number formed by changing that 1 into 01.

If x denote any point of the interval (0, 1), then, if x belong to Cantor's set, and y denote the corresponding point of the continuum (so that, formally, x is twice y, but x is interpreted in the ternary and y in the binary scale), we put

$$f_1(x) = y.$$

In each black interval (a, b) of Cantor's set, however, $f_1(x)$ is to be constant and equal to $f_1(a)$ and to $f_1(b)$, which, as we saw, are equal.

At the origin $f_1(0) = 0$, and we put $f_1(1) = 1$; also we assign to f_1 the same value at points equidistant from the origin. Thus $f_1(x)$ is defined from -1 to $+1$, is never negative, and is obviously continuous, as it is monotone throughout each of the intervals $(-1, 0)$, $(0, 1)$, assumes in each of those intervals all values between 0 and 1, and is continuous at the end-points.

The values assigned to f_1 at the points of the set C_1, consisting of Cantor's set and its reflexion in the origin, are retained in forming $f_2, f_3, \ldots, f_n, \ldots$, and are therefore the values of the limiting function f at those points.

In any black interval (a_1, b_1) of the set C_1 of length $2k_1$, f_2 is defined in a manner similar to that used in defining f_1 in the interval $(-1, 1)$, and so as to make f_2 a continuous function from -1 to 1. Thus, if x denote any point of the segment (0, 1),

$$f_2(a_1+k_1-k_1x) = f_2(a_1+k_1+k_1x) = f(a_1)f_1(x) = f(b_1)f_1(x).$$

Denoting by C_2 the perfect set got by adding to C_1 sets similar to itself in each black interval of C_1, we retain the values of f_2 at the points of C_2 in forming $f_3, f_4, \ldots, f_n, \ldots$, so that these are also the values of f at these points.

The principle is now evident on which we construct the successive functions $f_3, f_4, \ldots, f_n$. We get successive perfect sets $C_1, C_2, C_3, \ldots, C_n, \ldots$, by inserting in each black interval of these sets a set similar to C_1. The values assigned to f_n at the points of C_n are retained at all subsequent stages, and in each of the black intervals of C_n, say (a_n, b_n) of length $2k_n$, we have

$$f_{n+1}(a_n+k_n-k_nx) = f_{n+1}(a_n+k_n+k_nx) = f(a_n)f_1(x) = f(b_n)f_1(x).$$

Then all the f_n's are continuous non-negative functions, f_n being zero at the middle point of each black interval of C_{n-1}, and constant in each black interval of C_n. In fact, if a_n and b_n are the end-points of a black interval of C_n, and x is any point inside that interval, we have, by what has been pointed out,

$$f(a_n) = f_n(x) = f(b_n).$$

From this last relation it follows that at any point x internal to a black interval at each stage, that is, at any point of the set complementary to the set of the first category which is the outer limiting set of $C_1, C_2, \ldots, C_n, \ldots$, $f(x)$ is one of the limits of its values in the neighbourhood both on the right and also on the left, viz., of $f(a_n)$ and $f(b_n)$, where a_n and b_n both move up to x as limiting point as the integer n increases indefinitely.

That the same is true at any point x of the set of the first category above mentioned is evident from the mode of construction of the functions f_n. For, if x is a limiting point on both sides of C_n, $f(x) = f_n(x)$ is the unique limit of the common values of f and f_n at points of C_n, since f_n is a continuous function. But if x is an end-point of a black interval of C_n, it is, by construction, a limiting point on both sides of C_{n+1}, and therefore, by the same argument, $f(x) = f_{n+1}(x)$ is the unique limit of the common values of f and f_{n+1} at points of C_{n+1}.

Thus the function f has the property that at every point its value is one of the limits of its values in the neighbourhood both on the right and also on the left.

That *f is an upper semi-continuous function* follows from the fact that the sequence of continuous functions $f_1, f_2, \ldots, f_n, \ldots$, of which it is the limit, is, by construction, a monotone descending sequence.

Hence *f is at every point equal to the upper limit of the values in the neighbourhood on either side.*

Again, since f_n and therefore f is zero at the middle points of the black intervals of C_n, it follows that *f is zero at a set of points dense everywhere.* Hence *f is zero at all its points of continuity.* Moreover, since f, like f_n, is never negative, *zero is its lower bound in any interval.* Hence, in the first place, *f, being upper semi-continuous, is continuous at all its zeros,* and in the second place *zero is its lower limit at every point, whether approach be from the right or from the left.*

Now, since f is the limit of a continuous function, and is a limit of values in the neighbourhood on either side,* its limits on either side fill up the interval between its upper and lower limits on that side. Hence

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* W. H. Young, "A Theorem in the Theory of Functions of a Real Variable," 1907, *Rend. di Palermo*, Tomo XXIV, pp. 1–6.

the limits of the function f on the right of any point are the same as those on the left, and they fill up the interval between zero and the value of f at the point.

7. The function constructed in § 6 possesses an interest of its own, unconnected with the subject of the present paper. Continuous functions assume in every interval all values between their upper and lower bounds in that interval, and also assume the upper and lower bounds themselves. Differential coefficients assume in every interval all values between their upper and lower bounds in that interval, but not, in general, their upper or lower bounds. The function we have constructed is obviously not a continuous function, and it is not a differential coefficient, since its integral is zero. Yet it assumes in every interval all values between its upper and lower bounds in that interval, and also assumes its upper and lower bounds themselves. Our function is, moreover, like a differential coefficient, the limit of a continuous function. Like a continuous function, our function has the property that, if k denote any value between its upper and lower bounds, the points at which $f \geqslant k$ and those at which f assumes its upper bound form closed sets. Unlike a continuous function, f has the property that the points at which $f \leqslant k$, and those at which f assumes its lower bound form an open set, and, in fact, a set dense everywhere. The assumption of all values between its upper and lower bounds is in strict accordance with the theorem already referred to.*

It follows, in fact, from that theorem as a particular case, that *an upper (lower) semi-continuous function, having no distinction of right and left at points of discontinuity, and whose value at any point is not greater (less) than the limits in the open neighbourhood of that point, necessarily assumes in every interval all values between its upper and lower bounds.* One other result closely connected with the subject of the paper is worthy of remark. It follows, namely, from the theorem of § 4, that *two upper (lower) semi-continuous functions which possess in every interval the same upper (lower) bounds necessarily coincide.*†

* See preceding reference.

† §§ 5–7 have been modified during the passage of the paper through the press. The detection by the authors, while reading the first proofs, of a flaw in the reasoning originally given led to the discovery of the class of functions constructed in § 6, and the modifications in question then became necessary.

THE ELIMINANT OF THE EQUATIONS OF FOUR QUADRIC SURFACES

By A. L. Dixon.

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In the first four paragraphs of this paper I find the eliminant of the equations of four surfaces of the second degree in the form of a determinant of order 20.

The method of finding the eliminant is one that I have employed recently in the solution of the problem of finding the eliminant of three ternary quantics,* for this method, though not applicable to the case of four quaternary quantics in general, can be applied without difficulty to four quaternary quadratics.

In the rest of the paper I show how, as a deduction from the previous work, certain symmetric functions of the coordinates of the eight common points of a system of three quadric surfaces may be expressed in terms of known invariants of the system, and I also find expressions for the eliminant of four quadrics in terms of the coordinates of the points of intersection of sets of three out of the four.

The conception of invariants of binary quantics as symmetrical functions of differences of roots is, of course, fundamental, but, so far as I know, little has been done in respect to the corresponding functions for systems of ternary and quaternary quantics.

I hope therefore that results such as (13) and (15) in §§ 9, 10 of this paper, and those given for ternary quantics in §§ 36–41 of my paper already quoted, may be of interest.

For the determinant (a_1, b_2, c_3, d_4) whose principal diagonal is a_1, b_2, c_3, d_4, I shall often use the notation $|a_1, b_1, c_1, d_1|$, or $|a, b, c, d|$, where a_1, b_1, c_1, d_1 is the first row.

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* *Vide Proc. London Math. Soc.*, Ser. 2, Vol. 7, p. 49 and p. 473.

1. Take four quadric surfaces,

$$\phi_1(x, y, z) = 0, \quad \phi_2(x, y, z) = 0, \quad \phi_3(x, y, z) = 0, \quad \phi_4(x, y, z) = 0.$$

Consider the surface $F(x, y, z, a, b, c) = 0,$

where $F(x, y, z, a, b, c)$ is defined by

$$F \times (x-a)(y-b)(z-c)$$

$$= \begin{vmatrix} \phi_1(x, y, z), & \phi_1(a, y, z), & \phi_1(a, b, z), & \phi_1(a, b, c) \\ \phi_2(x, y, z), & \phi_2(a, y, z), & \phi_2(a, b, z), & \phi_2(a, b, c) \\ \phi_3(x, y, z), & \phi_3(a, y, z), & \phi_3(a, b, z), & \phi_3(a, b, c) \\ \phi_4(x, y, z), & \phi_4(a, y, z), & \phi_4(a, b, z), & \phi_4(a, b, c) \end{vmatrix}.$$

Since this determinant obviously vanishes when $x = a$, or $y = b$, or $z = c$, F is a rational integral function of the third degree in x, y, z, and also in a, b, c.

Further, if $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 0$ have a common point, this point obviously lies on $F = 0$, so that, as a, b, c are arbitrary parameters, we see that, when F is expanded in terms of a, b, c, the coefficient of each term must vanish separately at the common point. Thus $F = 0$ is equivalent to as many equations as there are different terms in a, b, c, that is to sixteen equations, each containing the coefficients of each ϕ in the first degree.

Take, then, four of these equations, and associate with them the four sets of four equations

$$\phi = 0, \quad x\phi = 0, \quad y\phi = 0, \quad z\phi = 0,$$

thus getting twenty equations in all. Since the number of terms in the equation of a cubic surface is also twenty, we can eliminate linearly the quantities x^3, y^3, z^3, x^2y, x^2z, ..., and the resulting determinant must either be the eliminant of the four ϕ's, or else vanish identically, since its degree in the coefficients of each surface is eight, as it should be.

2. To obtain the eliminant, the four equations, derived from $F = 0$, which are to be taken, are those got by equating to zero the coefficients of a, b, c, and the term independent of a, b, c. I will show at once that this is necessary; that it is sufficient can be established by taking each quadric to be of the form

$$Ax^2+By^2+Cz^2+D = 0.$$

For, consider the determinant of § 1, which is equal to

$$F \times (x-a)(y-b)(z-c).$$

Subtract the third column from the fourth, the second from the third, and the first from the second. Then, in its new form, the second column is divisible by $x-a$, the third by $y-b$, and the fourth by $z-c$, the quotients being linear both in x, y, z and a, b, c.

Thus F can be written in the form

$$P_1\phi_1(x, y, z)+P_2\phi_2(x, y, z)+P_3\phi_3(x, y, z)+P_4\phi_4(x, y, z),$$

where P_1, P_2, P_3, P_4 are functions of x, y, z and a, b, c, of the third degree in both sets together. From this we see that the coefficients of $a^2, b^2, ab, ac, \ldots$ in F are all of the form

$$Q_1\phi_1(x, y, z)+Q_2\phi_2(x, y, z)+Q_3\phi_3(x, y, z)+Q_4\phi_4(x, y, z),$$

where, now, Q_1, Q_2, Q_3, Q_4 are linear functions of x, y, z, so that these coefficients are not linearly independent of the sixteen functions $\phi, x\phi, y\phi, z\phi$. Thus the only equations which can be used in conjunction with $\phi = 0, x\phi = 0, \ldots$ to form a non-vanishing determinant are, as I have said, those got by equating to zero the coefficients of a, b, c and the term independent of a, b, c, in the expansion of $F(x, y, z, a, b, c)$.

3. It is obvious that the function F, as I have defined it in § 1, is not unique, since the coordinates x, y, z may be interchanged in any manner, corresponding interchanges being made among a, b, c; and, further, taking the equations to the surfaces ϕ to be homogeneous in four variables x, y, z, w, we may select any three of these as independent variables, corresponding to x, y, z in the definition of F of § 1, the remaining fourth variable being put equal to unity.

If it be required to write down the eliminant of four quaternary quadrics, the simplest way would seem to be, to take in addition to the four sets of four cubics $x\phi, y\phi, z\phi, w\phi$, the four which are given by

$$\left|\begin{matrix}\phi(x, y, z, w), & \phi(0, y, z, w), & \phi(0, 0, z, w), & \phi(0, 0, 0, 1)\end{matrix}\right| / xyz,$$

$$\left|\begin{matrix}\phi(x, y, z, w), & \phi(x, 0, z, w), & \phi(x, 0, 0, w), & \phi(1, 0, 0, 0)\end{matrix}\right| / yzw,$$

$$\left|\begin{matrix}\phi(x, y, z, w), & \phi(x, y, 0, w), & \phi(x, y, 0, 0), & \phi(0, 1, 0, 0)\end{matrix}\right| / xzw,$$

$$\left|\begin{matrix}\phi(x, y, z, w), & \phi(x, y, z, 0), & \phi(0, y, z, 0), & \phi(0, 0, 1, 0)\end{matrix}\right| / xyw.$$

If we take $\phi(x, y, z, w)$ to be

$$Ax^2+By^2+Cz^2+Dw^2+Fyz+Gzx+Hxy+Lxw+Myw+Nzw,$$

the first of these is at once seen to be

$$|Ax+Hy+Gz+Lw,\ By+Fz+Mw,\ Cz+Nw,\ D|,$$

so that the same result would be obtained by an extension of Sylvester's method of elimination (Salmon, *Higher Algebra*, p. 86, § 91) to the case of four quaternary quadrics.

4. I return to the consideration of the determinant which is obtained by taking the sixteen cubics $x\phi$, $y\phi$, $z\phi$, ϕ and the additional four which are the coefficients of 1, a, b, c in $F(x, y, z, a, b, c)$ of § 1.

The determinant of order 20 whose constituents are the coefficients of these cubics, I call E, and I will now establish the relation

$$\Delta = DD'E,$$

where D is the determinant $|1,\ a_r,\ b_r,\ c_r|$,

and D' the determinant of order 20,

$$|1,\ a_r,\ \beta_r,\ \gamma_r,\ a_r^2,\ \ldots,\ \beta_r\gamma_r^2,\ \gamma_r^3|,$$

the constituents of each row being the terms of a general cubic surface, whilst Δ is the determinant of order 20 given by the scheme,

$$\left.\begin{matrix}4\text{ rows}\\ r=1,2,3,4\end{matrix}\right\}\ F(a_r, b_r, c_r, a_1, \beta_1, \gamma_1),\ F(a_r, b_r, c_r, a_2, \beta_2, \gamma_2),\ \ldots,\ F(a_r, b_r, c_r, a_{20}, \beta_{20}, \gamma_{20}),$$

$$,,\qquad \phi_r(a_1, \beta_1, \gamma_1),\quad \phi_r(a_2, \beta_2, \gamma_2),\ \ldots,\quad \phi_r(a_{20}, \beta_{20}, \gamma_{20}),$$

$$,,\qquad a_1\phi_r(a_1, \beta_1, \gamma_1),\ a_2\phi_r(a_2, \beta_2, \gamma_2),\ \ldots,\ a_{20}\phi_r(a_{20}, \beta_{20}, \gamma_{20}),$$

$$,,\qquad \beta_1\phi_r(a_1, \beta_1, \gamma_1),\ \beta_2\phi_r(a_2, \beta_2, \gamma_2),\ \ldots,\ \beta_{20}\phi_r(a_{20}, \beta_{20}, \gamma_{20}),$$

$$,,\qquad \gamma_1\phi_r(a_1, \beta_1, \gamma_1),\ \gamma_2\phi_r(a_2, \beta_2, \gamma_2),\ \ldots,\ \gamma_{20}\phi_r(a_{20}, \beta_{20}, \gamma_{20}).$$

The four points (a, b, c) and the twenty points (α, β, γ) are quite arbitrary, except that the four (a, b, c) must not lie in a plane, nor the twenty (α, β, γ) on a cubic surface.

This relation is easy to prove, for, in the first four rows of Δ, the terms in a, b, c, of order higher than the first, may be crossed out, since they are linear functions of the other sixteen rows of Δ, and then the factor D may be taken out of the first four rows. After this has been done, the resulting determinant is merely the product of E and D', by

the ordinary rule for the multiplication of determinants. The relation

$$\Delta = DD'E$$

is therefore established.

Expressions for the eliminant of four quadrics, and for the invariants I and J of three quadrics, in terms of the coordinates of the common intersections of three quadrics.

5. Consider the result of putting

$$a_r = \alpha_r, \quad b_r = \beta_r, \quad c_r = \gamma_r \quad (r = 1, 2, 3, 4),$$

and taking $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$. ..., $(\alpha_8, \beta_8, \gamma_8)$,

as the intersections of the three surfaces ϕ_2, ϕ_3, ϕ_4. Also for the sake of clearness, change the notation for $(\alpha_9, \beta_9, \gamma_9)$, ..., writing $(\lambda_r, \mu_r, \nu_r)$ for $(\alpha_{8+r}, \beta_{8+r}, \gamma_{8+r})$.

Then it is obvious that $F(\alpha_r, \beta_r, \gamma_r, \alpha_s, \beta_s, \gamma_s)$ will vanish unless $r = s$, and that $F(\alpha_r, \beta_r, \gamma_r, \alpha_r, \beta_r, \gamma_r)$ becomes

$$\phi_1(\alpha_r, \beta_r, \gamma_r) \left[\frac{\partial(\phi_2, \phi_3, \phi_4)}{\partial(x, y, z)}\right]_{x=\alpha_r,\ y=\beta_r,\ z=\gamma_r},$$

which I will write $\phi_1(\alpha_r, \beta_r, \gamma_r)\, J(\alpha_r, \beta_r, \gamma_r)$.

Since all the constituents of Δ which contain $\phi_r(\alpha, \beta, \gamma)$ $[r = 2, 3, 4]$ are now zero, Δ becomes the product of three parts

(i) $\phi_1(\alpha_1, \beta_1, \gamma_1)\, \phi_1(\alpha_2, \beta_2, \gamma_2)\, \phi_1(\alpha_3, \beta_3, \gamma_3)\, \phi_1(\alpha_4, \beta_4, \gamma_4)$
$\times J(\alpha_1, \beta_1, \gamma_1)\, J(\alpha_2, \beta_2, \gamma_2)\, J(\alpha_3, \beta_3, \gamma_3)\, J(\alpha_4, \beta_4, \gamma_4)$,

(ii) $\phi_1(\alpha_5, \beta_5, \gamma_5)\, \phi_1(\alpha_6, \beta_6, \gamma_6)\, \phi_1(\alpha_7, \beta_7, \gamma_7)\, \phi_1(\alpha_8, \beta_8, \gamma_8) \begin{vmatrix} 1, & \alpha_5, & \beta_5, & \gamma_5 \\ 1, & \alpha_6, & \beta_6, & \gamma_6 \\ 1, & \alpha_7, & \beta_7, & \gamma_7 \\ 1, & \alpha_8, & \beta_8, & \gamma_8 \end{vmatrix}$,

(iii) a determinant Λ of order 12 whose constituents are

$$\phi_r(\lambda, \mu, \nu),\ \lambda\phi_r(\lambda, \mu, \nu),\ \mu\phi_r(\lambda, \mu, \nu),\ \nu\phi_r(\lambda, \mu, \nu) \quad [r = 2, 3, 4],$$

so that I have the result

$$\prod_{r=1}^{r=8} \phi_1(\alpha_r, \beta_r, \gamma_r) \prod_{r=1}^{r=4} J(\alpha_r, \beta_r, \gamma_r)\, \Lambda D(5, 6, 7, 8) = D(1, 2, 3, 4)\, D'E, \quad (1)$$

where D' has eight rows in α, β, γ, and twelve in λ, μ, ν, and $D(1, 2, 3, 4)$.

denotes the value of D when the suffixes are 1, 2, 3, 4. But now, if I write R for the eliminant of the terms of the second degree, homogeneous in x, y, z, of the quadrics ϕ_2, ϕ_3, ϕ_4, I have the known relation

$$R^2 \prod_{r=1}^{r=8} \phi_1(a_r, \beta_r, \gamma_r) = E,$$

so that I get

$$\Lambda D(5, 6, 7, 8) \prod_{r=1}^{r=4} J(a_r, \beta_r, \gamma_r) = R^2 D(1, 2, 3, 4)\, D'. \qquad (2)$$

6. In this very general relation, in which (λ, μ, ν) are all arbitrary, equate the coefficients, on each side, of the determinant of order 12, whose elements are $\lambda^r \mu^s \nu^t$, where either r, s, or t is greater than unity. The coefficient of this in D' is

$$| 1, \alpha, \beta, \gamma, \beta\gamma, \gamma\alpha, \alpha\beta, \alpha\beta\gamma |,$$

which I will call T.

The coefficient in Λ will be found to be the product of two determinants, one of the third order and the other of the ninth, the first being that formed by the first three columns of the array

x^2	y^2	z^2	yz	xz	xy
A_2,	B_2,	C_2,	F_2,	G_2,	H_2,
A_3,	B_3,	C_3,	F_3,	G_3,	H_3,
A_4,	B_4,	C_4,	F_4,	G_4,	H_4,

and the second that obtained by leaving out the last column of the array

x^3	y^3	z^3	x^2y	x^2z	y^2x	y^2z	z^2x	z^2y	xyz
A_2,			H_2,	G_2,	B_2,		C_2,		F_2
A_3,			H_3,	G_3,	B_3,		C_3,		F_3
A_4,			H_4,	G_4,	B_4,		C_4,		F_4
	B_2,		A_2,		H_2,	F_2,		C_2,	G_2
	B_3,		A_3,		H_3,	F_3,		C_3,	G_3
	B_4,		A_4,		H_4,	F_4,		C_4,	G_4
		C_2,		A_2,		B_2,	G_2,	F_2,	H_2
		C_3,		A_3,		B_3,	G_3,	F_3,	H_3
		C_4,		A_4,		B_4,	G_4,	F_4,	H_4

The product of these two determinants, which is a function of the coefficients of terms of the second degree only in ϕ_2, ϕ_3, ϕ_4, I will call H, and now I get

$$HD(5, 6, 7, 8) \prod_{r=1}^{r=4} J(\alpha_r, \beta_r, \gamma_r) = TR^2D(1, 2, 3, 4). \tag{3}$$

Interchanging the two sets of common intersections (1, 2, 3, 4) and (5, 6, 7, 8), I have

$$HD(1, 2, 3, 4) \prod_{r=5}^{r=8} J(\alpha_r, \beta_r, \gamma_r) = TR^2D(5, 6, 7, 8); \tag{4}$$

and therefore

$$H^2 \prod_{r=1}^{r=8} J(\alpha_r, \beta_r, \gamma_r) = T^2R^4, \tag{5}$$

and

$$\prod_{r=1}^{r=4} J(\alpha_r, \beta_r, \gamma_r) \Big/ \prod_{r=5}^{r=8} J(\alpha_r, \beta_r, \gamma_r) = [D(1, 2, 3, 4)]^2/[D(5, 6, 7, 8)]^2. \tag{6}$$

7. We have obviously from (2) and (3),

$$HD' = \Lambda T, \tag{7}$$

and this relation is easy to prove directly by forming the product HD', in such a way that twelve of the columns of the resulting determinant consist of

$$\phi_1(\lambda, \mu, \nu), \quad \lambda\phi_1(\lambda, \mu, \nu), \quad \mu\phi_1(\lambda, \mu, \nu), \quad \nu\phi_1(\lambda, \mu, \nu), \quad \ldots.$$

We see then, either in this way or directly from the last paragraph, that T may be taken to be any determinant which has four columns 1, α, β, γ, with three columns taken from the set

$$| \beta\gamma, \ \gamma\alpha, \ \alpha\beta, \ \alpha^2, \ \beta^2, \ \gamma^2 |,$$

and one column of the third degree in α, β, γ, and that the corresponding values of H are got by leaving out the corresponding columns in the arrays of § 6.

Thus any function of the ratios of the determinants

$$(A_2, B_3, C_4), \quad (A_2, B_3, F_4), \quad (F_2, G_3, H_4), \quad \ldots,$$

may be expressed in terms of the coordinates of the intersections of

$$\phi_2 = 0, \quad \phi_3 = 0, \quad \phi_4 = 0,$$

and in particular the ratio of R to H may be so expressed as a sum of

six terms of the form T, since R can be obtained by adding to the array of § 6 a tenth row*

$$0,\ 0,\ (C_2, G_3, F_4),\ 0,\ (C_2, A_3, H_4),\ 0,\ 0,\ (C_2, G_3, H_4)+(C_2, A_3, F_4),$$
$$(C_2, G_3, B_4),\ (C_2, A_3, B_4).$$

8. Interchanging 1 and 5 in (3) or (4), I get

$$\frac{J(\alpha_1, \beta_1, \gamma_1)}{D(1, 2, 3, 4)\, D(1, 6, 7, 8)} = -\frac{J(\alpha_5, \beta_5, \gamma_5)}{D(5, 2, 3, 4)\, D(5, 6, 7, 8)}; \tag{8}$$

and therefore writing $J(1)$ for $J(\alpha_1, \beta_1, \gamma_2)$ I shall have

$$[J(1)]^{10}/\prod_{20} D(1, p, q, r) = [J(5)]^{10}/\prod_{20} D(5, p, q, r), \tag{9}$$

the products being for all values of p, q, r except 1 and 5, so that $\Pi(1, p, q, r)$ has twenty factors.

It is not difficult to verify that this relation is not altered if we suppose that in the factors $D(s, p, q, r)$ the four numbers in the bracket are always written in some definite order, say the natural order, so that, for instance, $D(5, 3, 2, 6)$ is replaced by $D(2, 3, 5, 6)$. This order being assumed, the relation just obtained may be written,

$$[J(1)]^{10}/\prod_{35} D(1, p, q, r) = [J(5)]^{10}/\prod_{35} D(5, p, q, r), \tag{10}$$

the products on each side being for all the values of p, q, r except 1 and 5 respectively, so that each product Π has thirty-five factors.

Thus $[J(1)]^{10}/\prod_{35} D(1, p, q, r)$ has the same value if we replace 1 by any of the other numbers 2, 3, 4, 5, 6, 7, 8; and therefore we have, by continued multiplication,

$$[J(1)]^{80}/[\prod_{35} D(1, p, q, r)]^8 = \prod_{s=1}^{s=8} [J(s)]^{10}/[\prod_{70} D(p, q, r, s)]^4, \tag{11}$$

the product on the right-hand side being for the set of 70 possible values of (p, q, r, s).

* The eliminant of three ternary quadratics $\psi_1(x, y, z)$, $\psi_2(x, y, z))$ $\psi_3(x, y, z)$ may be obtained by forming the determinant of the coefficients of the nine cubics $x\psi$, $y\psi$, $z\psi$ and a tenth cubic

$$(\psi_1(x, y, z),\ \psi_2(x, 0, z),\ \psi_3(0, 0, 1))\, z/xy \quad \text{(cf. § 3)}.$$

Taking notice of the fact that $\prod_{70} D(p, q, r, s)$ changes its sign if two of the numbers 1, 2, 3, 4, 5, 6, 7, 8 are interchanged, but is otherwise unaltered, we see that $R^{70}[\prod_{70} D(p, q, r, s)]^2$ is an invariant of the three quadrics, which is of weight 420, and of degree 280 in the coefficients of each quadric.

9. Now, in the notation used by Salmon (*Geometry of Three Dimensions*, pp. 211, 212), the tact-invariant of three quadrics, which is of degree 16 in the coefficients of each quadric, is called J, so that J vanishes whenever two of the eight points of intersection are coincident; and the invariant which vanishes whenever four of the points of intersection lie on a plane, which is of degree 10 in the coefficients of each quadric, is called I.

Also it is clear that $R^3 \prod_{s=1}^{s=8} J(\alpha_s, \beta_s, \gamma_s)$, which is the eliminant of

$$\phi_2 = 0, \quad \phi_3 = 0, \quad \phi_4 = 0,$$

and

$$\begin{vmatrix} \frac{\partial \phi_2}{\partial x}, & \frac{\partial \phi_2}{\partial y}, & \frac{\partial \phi_2}{\partial z} \\ \frac{\partial \phi_3}{\partial x}, & \frac{\partial \phi_3}{\partial y}, & \frac{\partial \phi_3}{\partial z} \\ \frac{\partial \phi_4}{\partial x}, & \frac{\partial \phi_4}{\partial y}, & \frac{\partial \phi_4}{\partial z} \end{vmatrix} = 0,$$

will contain J as a factor, and that, since

$$2\phi \equiv x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} + w\frac{\partial \phi}{\partial w},$$

this eliminant will contain a factor R as well as the factor J, for it must vanish if $w = 0$ goes through a common point of

$$\phi_2 = 0, \quad \phi_3 = 0, \quad \phi_4 = 0.$$

Thus we may put $$R^3 \prod_{r=1}^{r=8} J(\alpha_r, \beta_r, \gamma_r) = RJ, \tag{12}$$

and so, from (5), $$H^2 J = T^2 R^6. \tag{13}$$

10. From a consideration of the zeroes of $\prod D(p, q, r, s)$, we have also

$$R^{70}[\prod_{70} D(p, q, r, s)]^2 = J^h I^k, \tag{14}$$

where $$16h+10k = 280.$$

The possible solutions are

$$h = 5, \quad h = 10, \quad h = 15,$$

$$k = 20, \quad k = 12, \quad k = 4.$$

Of these the second one is rejected at once, since, on taking the square root of each side of (14), we have an impossible result.

The first solution is also impossible, for on substituting from it and (12) in (11), we should get

$$[\prod_{35} D(1, p, q, r)]^8/[J(\alpha_1, \beta_1, \gamma_1)]^{80} = I^{40}/R^{120},$$

from which, if we make the points $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ coincide, the left-hand side must vanish whilst the right-hand side does not.

We are left then with the relation

$$R^{70}[\prod_{70} D(p, q, r, s)]^2 = J^{15}I^4, \qquad (15)^*$$

and, from (11), $$[\prod_{35} D(1, p, q, r)]^2/[J(\alpha_1, \beta_1, \gamma_1)]^{20} = J^5I^2/R^{30}. \qquad (16)$$

11. The identical relation $HD' = \Lambda T$, (7) of § 7, may be made to yield expressions for E, E^2, E^3, in terms of the coordinates of the sets of common points of three of the quadrics. Keeping (α, β, γ) to denote an intersection of ϕ_2, ϕ_3, ϕ_4, let

$(\alpha', \beta', \gamma')$ denote an intersection of ϕ_1, ϕ_3, ϕ_4,

$(\alpha'', \beta'', \gamma'')$,, ,, ϕ_1, ϕ_2, ϕ_4,

$(\alpha''', \beta''', \gamma''')$,, ,, ϕ_1, ϕ_2, ϕ_3,

and also let the same notation be used to distinguish the other functions of a set of three of the quadrics, so that, for instance, J' denotes the tact-invariant J of ϕ_1, ϕ_3, ϕ_4.

Take four out of the twelve arbitrary points (λ, μ, ν) to be

$$(\alpha_r', \beta_r', \gamma_r') \quad (r = 1, 2, 3, 4).$$

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* This formula would seem to be the simplest possible analogue, for quaternary quadrics, of the theorem, for a binary quadratic, that the discriminant of $a_0(x-\alpha)(x-\beta)$ is $a_0^2(\alpha-\beta)^2$.

Then, from Λ, we may obviously take out a factor

$$\prod_{r=1}^{r=4} \phi_2(\alpha'_r, \beta'_r, \gamma'_r) \begin{vmatrix} 1, & \alpha'_1, & \beta'_1, & \gamma'_1 \\ 1, & \alpha'_2, & \beta'_2, & \gamma'_2 \\ 1, & \alpha'_3, & \beta'_3, & \gamma'_3 \\ 1, & \alpha'_4, & \beta'_4, & \gamma'_4 \end{vmatrix},$$

which I will write $\prod_{r=1}^{r=4} \phi_2(\alpha'_r, \beta'_r, \gamma'_r)\, D(1', 2', 3', 4')$.

D' will now have eight rows in α, β, γ, four rows in α', β', γ', and eight in λ, μ, ν.

Picking out on each side the coefficient of the determinant of order 8, whose elements are $\lambda^r \mu^s \nu^t$, where either s or t is greater than unity, I shall get from D' a determinant whose first row is

$$| 1, \alpha_1, \beta_1, \gamma_1, \beta_1\gamma_1, \gamma_1\alpha_1, \alpha_1\beta_1, \alpha_1\beta_1\gamma_1, \alpha_1^2, \alpha_1^2\beta_1, \alpha_1^2\gamma_1, \alpha_1^3 |,$$

which has eight rows in α, β, γ, and four in α', β', γ'. Calling this $\Delta(1', 2', 3', 4')$, I have

$$H\Delta(1', 2', 3', 4') = kTD(1', 2', 3', 4') \prod_{r=1}^{r=4} \phi_2(\alpha'_r, \beta'_r, \gamma'_r), \qquad (17)$$

where k is the product of

$$\begin{vmatrix} B_3, & C_3 \\ B_4, & C_4 \end{vmatrix}^2 \quad \text{and} \quad \begin{vmatrix} B_3, & 0, & F_3, & C_3 \\ 0, & C_3, & B_3, & F_3 \\ B_4, & 0, & F_4, & C_4 \\ 0, & C_4, & B_4, & F_4 \end{vmatrix}.$$

From this I have at once, on taking the set of the other four values of $(\alpha', \beta', \gamma')$,

$$H\Delta(5', 6', 7', 8') = kTD(5', 6', 7', 8') \prod_{r=5}^{r=8} \phi_2(\alpha'_r, \beta'_r, \gamma'_r). \qquad (18)$$

Now $\prod_{r=1}^{r=8} \phi_2(\alpha'_r, \beta'_r, \gamma'_r)$ is E/R'^2,

where R' is the eliminant of the terms of the second degree in ϕ_1, ϕ_3, ϕ_4,

so that I get

$$R'^2H^2\Delta(1', 2', 3', 4')\,\Delta(5', 6', 7', 8') = k^2ET^2D(1', 2', 3', 4')\,D(5', 6', 7', 8'). \tag{19}$$

An expression for E^2 will be obtained in the same manner, by taking another set of four values of (λ, μ, ν) to be $(\alpha'', \beta'', \gamma'')$.

12. If we take four of the rows of D' in (7) to contain the coordinates of four points $(\alpha', \beta', \gamma')$, four to contain $(\alpha'', \beta'', \gamma'')$, and four $(\alpha''', \beta''', \gamma''')$, Λ will become

$$\prod_{r=1}^{r=4}\phi_2(\alpha'_r, \beta'_r, \gamma'_r)\prod_{r=1}^{r=4}\phi_3(\alpha''_r, \beta''_r, \gamma''_r)\prod_{r=1}^{r=4}\phi_4(\alpha'''_r, \beta'''_r, \gamma'''_r)$$

$$\times D(1', 2', 3', 4')\,D(1'', 2'', 3'', 4'')\,D(1''', 2''', 3''', 4'''),$$

so that denoting the value of D' in this case by

$$\begin{vmatrix} 1', & 2', & 3', & 4' \\ 1'', & 2'', & 3'', & 4'' \\ 1''', & 2''', & 3''', & 4''' \end{vmatrix}$$

I get $\dfrac{E^3T^2}{R'^2R''^2R'''^2H^2}D(1', 2', 3', 4')\,D(5', 6', 7', 8')\,D(1'', 2'', 3'', 4'')$

$$\times D(5'' \;\; 6'', 7'', 8'')\,D(1''', 2''', 3''', 4''')\,D(5''', 6''', 7''', 8''')$$

$$= \begin{vmatrix} 1', & 2', & 3', & 4' \\ 1'', & 2'', & 3'', & 4'' \\ 1''', & 2''', & 3''', & 4''' \end{vmatrix} \begin{vmatrix} 5', & 6', & 7', & 8' \\ 5'', & 6'', & 7'', & 8'' \\ 5''', & 6''', & 7''', & 8''' \end{vmatrix} \tag{20}$$

Thus, if we form the symmetric function P, which is the square of the product of all the 70^3 values of

$$\begin{vmatrix} p', & q', & r', & s' \\ p'', & q'', & r'', & s'' \\ p''', & q''', & r''', & s''' \end{vmatrix}$$

we shall get $$P^{1/70^2} = \frac{E^{210} J^{70} J'^{15} J''^{15} J'''^{15} I'^4 I''^4 I'''^4}{R^{420} R'^{210} R''^{210} R'''^{210}}, \tag{21}$$

and the completely symmetric function $PP'P''P'''$ is given by

$$(PP'P''P''')^{1/70^2} = \frac{E^{840} (JJ'J''J''')^{115} (II'I''I''')^{12}}{(RR'R''R''')^{1050}}. \tag{22}$$

The corresponding theorem for two binary quadratics is that, if E is the eliminant of the two quadratics $a_0(x-\alpha)(x-\beta)$ and $a_0'(x-\alpha')(x-\beta')$ whose discriminants are D and D',

$$\frac{E^2 DD'}{a_0^6 a_0'^6} = \begin{vmatrix} 1, & \alpha, & \alpha^2 \\ 1, & \alpha', & \alpha'^2 \\ 1, & \beta', & \beta'^2 \end{vmatrix} \begin{vmatrix} 1, & \beta, & \beta^2 \\ 1, & \alpha', & \alpha'^2 \\ 1, & \beta', & \beta'^2 \end{vmatrix} \begin{vmatrix} 1, & \alpha', & \alpha'^2 \\ 1, & \alpha, & \alpha^2 \\ 1, & \beta, & \beta^2 \end{vmatrix} \begin{vmatrix} 1, & \beta', & \beta'^2 \\ 1, & \alpha, & \alpha^2 \\ 1, & \beta, & \beta^2 \end{vmatrix}.$$

The meaning of a formula, such as (21), is, I think, easier to grasp, if it is expressed in the language of geometry. $P = 0$ is the condition that, taking all the eight common points of ϕ_2, ϕ_3, ϕ_4, four of the common points of ϕ_1, ϕ_3, ϕ_4, four of the common points of ϕ_1, ϕ_2, ϕ_4, four of the common points of ϕ_1, ϕ_2, ϕ_3, these twenty points should all lie on a cubic surface, and the interpretation of (21) is that no such surface can be drawn unless (i) the surfaces ϕ_1, ϕ_2, ϕ_3, ϕ_4 have a common point, or (ii) three of the surfaces have two coincident points of intersection, or (iii) four of the points of intersection of ϕ_1 with two of the other three surfaces lie on a plane.

ON HOMOGENEOUS OSCILLATION OF SUCCESSIONS OF FUNCTIONS

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1. In considering successions of functions which oscillate, we may either require the property which we are discussing to be true of the succession as a whole, or we may require it also to be true for every succession contained in the original succession, say of every *sub-succession* or *sub-set*. Thus uniform oscillation has been defined* with respect to the whole succession, while Arzelà's celebrated postulate of equal continuity from its very nature applies to every sub-succession. The theorem† of Arzelà's which gives the result of this postulate of equal continuity has recently attracted so much notice that it has seemed to me worth while to shew how it follows naturally from the theory of uniform oscillation, as developed by myself. I am thus led to give a number of theorems which may perhaps from their simplicity, as well as from the light they throw on the theorem of Arzelà, be worthy of notice.

I use the word "homogeneous" in connection with an oscillation to denote that the property under consideration is possessed not only by the succession itself, but by every one of its sub-successions. By the *upper* (*lower*) *function* of a succession I denote the function whose value at every point is the upper (lower) limit of the succession at that point. This is the upper (lower) function *par excellence;* when we are considering the succession as the ensemble of all its sub-successions, we shall use the expression *all the upper and all the lower functions of the succession* to denote all the upper and lower functions of every sub-succession. When the upper and lower function of a succession, or sub-succession, coincide, each will be called the *unique limiting function*‡ of the succession, or sub-succession, and such a succession, or sub-succession, will be called a *sequence*.

* "On Oscillating Successions of Continuous Functions." *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 298–320.

† C. Arzelà, "Sulle Funzioni di Linee," 1895, *Mem. della R. Acc. delle Scienze dell' Ist. di Bologna*, Serie 5, Vol. v, pp. 3–22. It is no longer necessary to emphasise the importance of the theorem in question. It has been found to be of use not only in the theory of integrals and of differential equations and of infinite series, but also in the calculus of variations, as well as in more abstract speculations.

‡ This usage is different from that of Arzelà.

The main results arrived at are as follows :—

If all the lower functions of a succession are lower semi-continuous on the right (left), so are the upper ; and if all the upper functions are upper semi-continuous on the right (left), so are the lower.

If all the upper, or all the lower, functions are continuous on the right (left), then a sequence of the functions can be found, having an unique limiting function which is continuous on the right (left).

If the succession of continuous functions oscillates uniformly and homogeneously, then there is a sequence of the functions which converges uniformly to a continuous limiting function, and all the upper as well as all the lower functions of the succession are continuous.

More generally if a succession of continuous functions oscillates uniformly and homogeneously on the right (left), then all the upper and all the lower functions are continuous on the right (left), and in every sub-succession there is a sequence of the functions, converging towards an unique limiting function which is continuous on the right (left), the convergence to this limiting function being, moreover, uniform on the right (left).

As a corollary to this last theorem we have a theorem which includes Arzelà's celebrated theorem as a particular case, viz., we have the theorem that, if given any positive quantity e and any point P in the range of the independent variable, we can find an interval d, the same for all the functions but depending in general on P, such that the oscillation of each of the functions inside an interval of length d with P as left (right) hand end-points is less than e, then the same consequences follow as in the theorem last stated.

The paper terminates with some results regarding the upper and lower bounds of successions of continuous functions.

2. THEOREM 1.—*If all the lower (upper) functions of an oscillating succession are lower (upper) semi-continuous, then all the upper (lower) functions of the succession are also lower (upper) semi-continuous.*

For example, suppose all the lower functions to be lower semi-continuous, and take any one of the upper functions. Without any loss of generality we may, if we please, suppose it to be the upper function *par excellence* of the whole succession, and suppose P to be a point at which it is not lower semi-continuous. Then there must be a sequence of points $P_1, P_2, \ldots$, having P as limiting point, and such that the value of the upper function $u(x)$ at P is *greater* than the limit, which we may, if we please, suppose unique, of $u(x)$ along this sequence.

Now choose, as we evidently can, a sub-set of the functions which has the value of $u(x)$ at P as unique limit at that point. Then, since the

lower function of this sub-set is lower semi-continuous at P, this value is less than or equal to the limit or limits of this lower function as we move along the points $P_1, P_2, \ldots$; and therefore less than or equal to the limit or limits of the upper function of the sub-set along the same points; and therefore *a fortiori* less than or equal to the limit of the original upper function $u(x)$ along the same set.

But, by hypothesis, this value is greater than this latter limit, which is absurd. This proves the theorem.

Cor.—*The above holds if the semi-continuity assumed is on the right (left) only, the semi-continuity proved being then of the same type.*

3. Theorem 2.—*If either all the upper, or all the lower, functions of a succession of functions, or both sets, are continuous functions, then in any sub-succession of the functions a sequence of the functions can be found, having an unique limiting function, and this limiting function is continuous.*

Suppose all the upper functions are continuous.

Take any countable set of points dense everywhere, $P_1, P_2, \ldots, P_n, \ldots$. Select any one of the limits at P_1; we can then choose out a sub-set of the functions having, for the value of x corresponding to P_1, this selected limit as unique limit. Omitting now the first function $f_{1,1}(x)$ of this sub-set, let us, in like manner, choose out a sub-set of the remaining succession, having at P_2 an unique limit, which we select to be the value at P_2 of the lower function of the first sub-set. We then repeat this process, that is, we omit the first function $f_{2,1}(x)$ of this new succession, and determine a sub-set of the remaining succession having at P_3 an unique limit, which we select to be the value at P_3 of the lower function of the second sub-set. Proceeding thus *ad infinitum*, we take the succession

$$f_{1,1}(x),\ f_{2,1}(x),\ \ldots,\ f_{n,1}(x),\ \ldots, \tag{1}$$

formed by the successive omitted first members. As a sub-set of the first sub-set, this succession has an unique limit at P_1. As all but its first member belong to the second sub-set, it has an unique limit at P_2, and so on for the remaining points $P_3, \ldots, P_n, \ldots$. But these points are dense everywhere, so that the values of these unique limits determine at most one continuous function having those values at those points; this is, of course, the upper function of the succession (1), since we know that this is continuous. The lower function of the succession (1) is, by Theorem 1, upper semi-continuous, so that its value at any point is not less than the unique limit of its values as we approach the point in question

by means of the points P_n, that is, it is not less than the value of the upper function of the same succession at the same point. But the lower function is never greater than the upper function; hence the upper and lower functions are equal at each point, that is to say, the succession (1) has an unique limiting function, and this function is continuous.

Cor. 1.—*If all the upper (lower) functions are continuous, then there is an, in general, infinite set of lower (upper) functions which are continuous.*

Cor. 2.—*If the unique limiting functions, which are proved in Theorem 2 to exist, all coincide, then the upper and lower functions of the whole succession coincide, so that the whole succession becomes a sequence, having a continuous function as limiting function.*

For, repeating the argument in the first few lines of the proof of the theorem, we can shew that the upper and lower functions in question, each of them coincide at any assigned point we please, and therefore at every point, with this unique limiting function.

Cor. 3.—*If there be a finite number of distinct limiting functions, then the succession breaks up into the same number of sequences.*

Take the case of two limiting functions $f(x)$ and $g(x)$; it will be seen that the argument is quite general; it only needs repetition to apply to any finite number of limiting functions.

Let x be a point at which $f(x)$ and $g(x)$ differ, and let $2e$ be any positive quantity less than their difference. Divide the functions of the succession into two classes, the first containing all those functions which at the point x differ from $f(x)$ by less than e, and the second containing all the remaining functions. Then the functions of the first class differ from $g(x)$ at the point x by more than e; therefore the first class cannot yield a sequence with $g(x)$ as limit; hence $f(x)$ is the only possible unique limiting function of the first class, which is therefore, by Cor. 2, a sequence. Similarly, the second class is a sequence with $g(x)$ as limiting function.

Cor. 4.—*The above theorem and all the previous corollaries apply equally if the continuity assumed is on the right (left) only; in this case the continuity proved is also of the same type.*

4. So far no limitations have been imposed directly on the character of the functions of the succession, only those indirectly involved in the assumptions as to the upper or lower functions, or both, of the succession. Let us now suppose the functions of the succession to be continuous.

Consider the function of the two variables x and n given by

$$F(x, n) = f_n(x),$$

where, of course, n has only integral values. Then we call the upper double limit of $F(x, n)$ at the point (a, ∞) *the peak function* of the succession at the point a, and the lower double limit *the chasm function.** If the peak function is at any point equal to the upper function of the succession, we say that *the succession oscillates uniformly above*, and if the chasm function is equal to the lower function, we say that *the succession oscillates uniformly below*. If both these take place, we say *the succession oscillates uniformly*, or more fully, oscillates uniformly both above and below. When there is uniform oscillation above (below) the upper function is necessarily upper (lower) semi-continuous, instead of being only upper (lower) semi-continuous at all points except those of a set of the first category, which is the general case.

Tests for uniform oscillation, analogous to those for uniform convergence, have been devised.†

It may happen that a succession that oscillates uniformly above (below) is such that every sub-succession without exception has the same property; we may then say that *the succession oscillates uniformly and homogeneously above* (*below*). It follows from Theorem 1 that, in this case, if the oscillation is uniform and homogeneous above, all the lower as well as all the upper functions are upper semi-continuous, and that, if the oscillation is uniform and homogeneous below, all the upper as well as all the lower functions are lower semi-continuous. It hence follows further that, if the oscillation is uniform and homogeneous both above and below,

* When we distinguish *right* from *left*, we, of course, do so by confining x to be always greater or always less than a. That the above is equivalent to the original definition which I gave in the present *Proceedings*, Ser. 2, Vol. 6, p. 300, may be shewn as follows :—

Let U be this upper double limit when $a < x$, and $f_1(x_1), f_2(x_2), \ldots, f_n(x_n), \ldots$ a sequence of values having U as unique limit, while $x_1, x_2, \ldots, x_n, \ldots$ have a as unique limiting point.

Take any interval PQ on the right of P; then, if n is sufficiently large, all the points x_n lie inside this interval; hence, denoting as usual by $M_{n,Q}$ the upper bound of $f_n(x)$ in this interval, and by M_Q the upper limit of $M_{n,Q}$ as n increases indefinitely, we have

$$M_{n,Q} \geqslant f_n(x_n);$$

and therefore

$$M_Q \geqslant U.$$

Letting Q move up to P, we get, by the original definition of the right-hand peak function,

$$\pi_R(a) \geqslant U.$$

But, since $f_n(x)$ is a continuous function it assumes $M_{n,Q}$ as one of its values, so that $\pi_R(a)$ is a repeated limit, and therefore a double limit of $f_n(x)$, whence

$$\pi_R(a) \leqslant U.$$

The two last inequalities prove the equality of $\pi_R(a)$ and U. Q.E.D.

† "On Uniform Oscillation," *Camb. Phil. Trans.*, 1909.

all the upper and all the lower functions are continuous. Hence it follows further by Theorem 2 that, in every sub-succession of functions there is a sequence of functions having a continuous function as unique limiting function. Moreover, as uniform convergence is a special case of uniform oscillation, the convergence of each such sequence to its limiting function must be uniform.

In other words, we have the following theorem :—

THEOREM.—*If a succession of functions oscillates uniformly and homogeneously, then, every sub-succession of the functions contains a sequence converging uniformly to a continuous function as unique limit, and all the upper and all the lower functions of the succession are continuous.*

COR.—*If all the unique limiting functions to which the various sequences converge coincide, the whole succession is itself a sequence which converges uniformly.*

5. It may be remarked that, so far, there has been no restriction that the functions concerned should be bounded, except where the words "convergence" and "continuous" occur. If, moreover the word "continuous" be interpreted in the extended sense, the two infinities, $+\infty$ and $-\infty$, being distinguished, and the word "convergence" be interpreted to mean, where necessary, divergence to an unique infinite limit, this implicit restriction is removed.

It should also be noticed that the definitions and statements of the preceding article apply equally if we restrict our attention to the neighbourhood of a single point, instead of to a whole interval, in so far as these statements then have any meaning.

More important than this is the fact that all the reasoning and all the results apply equally whatever be the number of variables, and equally if we substitute for the range of the variables any perfect set of points.

Finally, we may remark that *in the case of a single variable, with obvious extensions in the case of more variables, we may distinguish between uniform and homogeneous oscillation on the right and on the left, and so get more general forms of our theorems involving continuity and semi-continuity on one side only.*

6. We now shew that *if, given any quantity e, however small, we can find an interval d, having the point P for its left-hand extremity, such that the excess of the maximum of $f_n(x)$ in that interval over its value at P, is less than e, for all values of n (so that d is the same for all values of n), then the succession oscillates uniformly above on the right.*

Denote the upper right-hand peak function of the succession at P [that is, the upper double limit at P of $f_n(x)$ as n increases indefinitely and x always lies to the right of P] by $\pi_R(P)$. This can at most exceed the upper limit of $f_n(P)$ as n increases indefinitely by e. Hence, if $u(P)$ be the upper function, we have

$$\pi_R(P) - u(P) \leqslant e.$$

But the left-hand side of this inequality is never negative, and e is as small as we please. Therefore

$$\pi_R(P) - u(P) = 0.$$

Thus the succession oscillates uniformly above on the right at P. But the same argument applies to any sub-succession, whence we see that the uniform oscillation above on the right at P is also homogeneous.

7. The result of the preceding article enables us to give the following theorem, which includes Arzelà's celebrated theorem as a particular case :—

Theorem.—*If a succession of functions is such that, given any positive quantity e, and any point P in the range of variation of the variable, we can find an interval d (this interval being the same for all the functions, but depending in general* on P), such that the oscillation of each of the functions inside an interval of length d with P as left-hand (right-hand) end-point is less than e, then* (1) *all the upper and all the lower functions are continuous on the right (left), and* (2) *there is in every sub-succession at least one sequence of the functions which converges towards an unique limiting function which is continuous on the right (left), the convergence to this limiting function being, moreover, uniform on the right (left).*

This theorem is a corollary to the theorem virtually proved above, viz.,

Theorem.—*If a succession of functions oscillates uniformly and homogeneously on the right (left), then all the upper and all the lower functions are continuous on the right (left), and in every sub-succession there is a sequence of the functions converging towards an unique limiting function which is continuous on the right (left), the convergence to this limiting function being, moreover, uniform on the right (left).*

8. It should be remarked that the difference between the alternatives in the former of the two theorems in § 7, and therefore between either of them and the special case of both, including Arzelà's theorem, in which no distinction is made between right and left, has reference at most

* When we consider right and left simultaneously, we can, of course, pass from the statement for the point P to that for the whole interval, by using the Heine-Borel theorem.

to a countable set of points, as there is no distinction of right and left as regards both discontinuities of a function or uniformity of oscillation except at a countable set of points.

It is also worthy of notice that the condition of uniform and homogeneous oscillation appears to involve fewer restrictions than Arzelà's assumption of equal continuity. This is not in contradiction with Arzelà's statement of his theorem which gives a sufficient condition that in a bounded succession of functions there should be at least one sequence converging uniformly, while it asserts that the condition is necessary only when the given succession is a sequence.

9. One of the most remarkable cases where we actually obtain successions of the kind contemplated is when integrating other successions. The results arrived at above enable us to go some steps further in some of the theorems stated and proved in my paper on "Term-by-Term Integration of Oscillating Series."* Thus, in Theorem 7, p. 113, of that paper, the restriction that the upper function $u(x)$ must have a finite integral may now be removed, and the first of the statements on p. 110 takes the more general form :—

If the point P be such that, in its right-hand neighbourhood $s_n(x)$ regarded as a function of (x, n) is bounded above, then the integral series oscillates uniformly and homogeneously above on the right, and below on the left, at P, so that, moreover, all the upper functions and all the lower functions are upper semi-continuous on the right and lower semi-continuous on the left at P.

We get, of course, similar statements when we substitute "left" for "right," and also if we substitute "lower" and "below" for "upper" and "above." In particular, *if $s_n(x)$ is bounded above throughout the whole interval, all the upper and all the lower functions of the integral series are upper semi-continuous on the right and lower semi-continuous on the left, the integral series oscillating everywhere uniformly and homogeneously above on the right, and uniformly and homogeneously below on the left, and a corresponding statement holds if $s_n(x)$ is bounded below.* If $s_n(x)$ is bounded both above and below, we can add nothing to the statements already made in the paper in question, except that it should be noticed that the results of the present paper with regard to the existence of sequences in the succession apply to the succession of integrals. This case has been virtually considered by

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 99–116.

Arzelà, *loc. cit.*, for he considers successions of functions whose derivates are in their ensemble bounded. Interpreting, therefore, integration in the generalised sense unknown to Arzelà at the time of writing his paper, the identity of his series in essentials with mine becomes evident. Formally, however, there is still a difference, for my bounded functions are not necessarily derivates; they may be any whatever. The upper and lower derivates of the Lebesgue integral of a bounded function lie between the bounds of the function and therefore differ from these, which are also bounded, only at a set of points of content zero (cp. also § 8, p. 107, of "Term-by-Term Integration of Oscillating Series"). Hence the fact stated follows.

The following is an alternative proof of this theorem:—

$$S_{n_i}(x_{n_i}) = \int_a^{x_{n_i}} s_{n_i}(t)\,dt = \int_a^x s_{n_i}(t)\,dt + \int_x^{x_{n_i}} s_{n_i}(t)\,dt \leqslant S_{n_i}(x) + M_x(x_{n_i} - x),$$

supposing x_{n_i} to lie to the right of x, and M_x to be the finite upper bound of $s_n(x)$ regarded as a function of n and x in some interval to the right of x inside which x_{n_i} lies.

Now, since the peak function of the integrated succession $S_1, S_2, \ldots$ is the upper double limit of $S_n(t)$ at $t = x$, $n = \infty$, we can choose such a sequence of integers n_i and corresponding points x_{n_i} with x as limiting point, that S_{n_i} has the right-hand peak function $\Pi_R(x)$ as unique limit. The first term on the right then has a limit, or limits, $\leqslant U(x)$, the upper function of the integrated succession, and the second term the limit zero; hence

$$\Pi_R(x) \leqslant U(x),$$

which, since the upper function is never greater than the peak function, gives

$$\Pi_R(x) = U(x),$$

shewing that the integrated succession oscillates uniformly above on the right at the point x. Q.E.D.

Similarly, if x_{n_i} lies on the left of x, we have

$$S_{n_i}(x_{n_i}) + \int_{x_{n_i}}^x s_{n_i}(t)\,dt = S_{n_i}(x),$$

whence, by the above argument, using "less" instead of "greater,"

$$X_L(x) = L(x),$$

shewing that the integrated succession oscillates uniformly below on the left at the point x. Q.E.D.

Since the property that $s_n(x)$ has of being bounded above involves the same property for every sub-succession, it follows that the uniformity proved is also homogeneous, as stated above.

10. Another application which may be made of the theorems of the present paper is to the theory of the upper and lower bounds of successions of continuous functions. It is often of importance to know when the upper or lower bound of the limit of a sequence of functions is the limit of the upper or lower bound of the constituent functions of the sequence.* In the more general case of an oscillating succession, two questions present themselves :—

(1) Is the upper (lower) bound of the upper (lower) function the upper (lower) limit of the upper (lower) bounds of the constituent functions of the succession ?

(2) Has the succession a limiting function, which itself possesses this property ?

These questions are answered by the following theorems.

THEOREM.—*If the succession oscillates uniformly above (below), or, more generally, if it everywhere either oscillates uniformly above (below), or has a non-uniformity of the upper (lower) oscillation which is completely visible, then the upper (lower) limit of the upper (lower) bounds of the succession is the upper (lower) bound of the upper (lower) function.*

I have already in a previous paper explained the difference between a visible and an invisible point of non-uniform oscillation of a succession of continuous functions. The non-uniformity of the upper oscillation is said to be invisible if the point is one at which the upper function is upper semi-continuous. I here use the expression "completely visible" to denote that the peak function is the upper limit of the upper function at the point in question.

To prove this theorem, take a sequence of values of n, for which the upper bounds of the corresponding functions $f_n(x)$ have for unique limit the upper limit U of all the upper bounds. The points at which the upper bounds of these selected functions are assumed as values by those functions will have one or more limiting points. At each such limiting point x, the peak function is evidently equal to this limit U. Hence there is at least one point at which the upper function, if the oscillation at that point is uniform above, or the associated upper limiting function of the upper function if the non-uniformity of the upper oscillation be every-

* "On Functions Defined by Monotone Sequences and their Upper and Lower Bounds," *Messenger of Math.* (1908), pp. 148–154.

where completely visible, assumes the value U. This proves the theorem, since it is evident that the upper bound of the upper function cannot be greater than U.

It by no means follows, however, that there is any unique limiting function which has this value for upper bound. To be sure that we get an unique limiting function in every sub-succession we must, if we are to apply the theorems of the present paper, assume that the upper functions and the lower functions are at least continuous on one side, say on the right. This will, as we have seen, be insured, if we require the whole succession to oscillate uniformly and homogeneously on the right. In this case, then, we could assert the existence of a function which for some sub-succession is the unique limiting function, and whose upper bound is the limit of the upper bounds of the constituent functions of the original succession. We have accordingly the following theorem :—

Theorem.—*If the succession oscillates uniformly and homogeneously above on the right, and it has a non-uniformity above on the left which, wherever it occurs, is completely visible, and where there is uniformity above on the left, this is not necessarily homogeneous, then the succession contains a sequence of the functions the upper bound of whose unique limiting function is the upper limit of the upper bounds of the whole succession of functions.*

If we attempt to get a more symmetrical theorem, we have to hypothecate that the upper and lower functions are all continuous. In this case there can be no non-uniformity, for there is evidently no visible non-uniformity. The theorem now takes the comparatively special form :—

If the succession oscillates uniformly and homogeneously both above and below, then there is an unique limiting function of some subsequence of the functions which assumes both the upper limit of the upper bounds and the lower limit of the lower bounds of all the constituent functions of the succession.

This last result corresponds precisely to that obtained by Arzelà, except that he requires the apparently more stringent condition of equal continuity of the succession of functions.

It may be noted finally that from the first theorem of the present article the result of the paper quoted " On Monotone Sequences," in the special case in which the constituent functions are continuous, may be deduced. In a monotone sequence, in fact, the non-uniformity of the convergence is always completely visible.

11. When the points of non-uniform oscillation are invisible, or only partially visible, it is evident that the argument used in the preceding article fails. Even when the succession of functions consists of a sequence of continuous functions having a continuous function as unique limiting function, it by no means follows that either the upper bounds or the lower bounds of the constituent functions have an unique limit, still less that they have unique limits which are respectively the upper and lower bounds of the unique limiting function. All that we can assert is that the upper bound of the limiting function is not greater than any one of the limits of the upper bounds, and that the lower bound of the former is not less than any of the limits of the lower bounds.

An interesting and important case of this in practice occurs when we have a function of (x, y), continuous with respect to x, and continuous with respect to y, but not continuous with respect to (x, y), and we are concerned with the maxima (minima) of the function for fixed values of y. These form, of course, a function of y, the maxima forming a lower semi-continuous and the minima an upper semi-continuous function of y. We cannot assert the continuity of these functions unless $f(x, y)$ is a continuous function of (x, y). In the contrary case these functions, far from being continuous, are not even necessarily bounded.*

* For further developments of the theory of oscillating successions of functions, see a forthcoming paper by the author entitled "On Semi-Integrals and Successions of Functions."

THE GREEN'S FUNCTION FOR A WEDGE OF ANY ANGLE, AND OTHER PROBLEMS IN THE CONDUCTION OF HEAT

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Introduction.

In a former paper* I have obtained certain multiform solutions of the partial differential equations of conduction of heat and sound, with the period $2n\pi$: and I have shown how these solutions can be applied to problems in which the boundaries are planes including an angle $n\pi/m$, m and n being any integers. This work involved the use of the theory of images in a Riemann's space, and the object of the investigation was to establish the application of that theory to these physical equations.

In this paper I shall give the solution for the case of a wedge of any angle, proving that this solution can be obtained without the introduction of the Riemann's space. That this is possible for the potential function was shown by Sommerfeld in his original paper.† The discussion which I give for the equation of conduction can be readily adapted to the problem of diffraction of sound.

The Green's function for the wedge of any angle having been obtained, it is easy to give the corresponding solutions for the various cases in two and three dimensions, in which the solid is bounded by axial planes, planes perpendicular to the axis, and circular cylinders. This extension involves the use of a method employed by me elsewhere.‡ These results, without proof, are given in the last article of this paper. Much of this work applies also to the theory of diffraction. As cylindrical coordinates are used throughout, it will be obvious that the cases in which the boundaries are spheres, cones, and axial planes do not follow, at any rate directly, from this work. However, it will be found that the solutions

* *Proc. London Math. Soc.*, Vol. xxx, p. 121 (1899).

† *Ibid.*, Vol. xxviii, p. 417 (1897).

‡ Cf. Carslaw, *Fourier's Series*, Ch. xviii.

for these cases can be obtained by methods not far removed from those here employed.*

1. *Multiform Solution of period* 2β.

In my former paper the solution

$$v_0 = \frac{1}{4\pi\kappa t} e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}$$

is transformed into a contour integral over the path (A), the two curved parts of Fig. 1, in the α-plane.

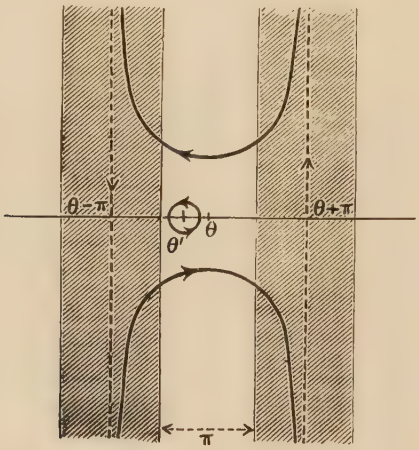

FIG. 1.—Breadth of strip, π; deformation of circuit round $\alpha = \theta'$; $|\theta-\theta'| < \pi$. In the shaded portions $\cos(\alpha-\theta)$ has a negative real part.

In this way we have

$$v_0 = \frac{1}{8\pi^2\kappa t} e^{-(r^2+r'^2)/4\kappa t} \int e^{[rr'\cos(\alpha-\theta)]/2\kappa t} \frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta'}} d\alpha,$$

over this path (A).

Consider the expression

$$v = \frac{1}{8\pi\kappa\beta t} e^{-(r^2+r'^2)/4\kappa t} \int e^{[rr'\cos(\alpha-\theta)]/2\kappa t} \frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta}-e^{i\pi\theta'/\beta}} d\alpha,$$

the integral being taken over the path (A), corresponding to the current coordinate θ.

* Cf. Dougall, *Proc. Edinburgh Math. Soc.*, Vol. XVIII, p. 69 (1900), where the corresponding problems in the theory of potential are examined.

(i) *This expression is a solution of the differential equation of conduction of heat*, since every element of the integrand satisfies this equation.

(ii) *It is periodic in θ, and of period 2β.* A change in θ, *e.g.*, from θ to ϕ, simply translates the path along the real axis of α, and the term $e^{[rr'\cos(\alpha-\theta)]/2\kappa t}$ is unaltered. Further, if the change in θ is equal to $2n\beta$ (n being any positive or negative integer), the other factor of the integrand also remains unaltered.

(iii) *It vanishes, when $t \to 0$, in the interval $-\beta < \theta < \beta$, except when $r \to r'$ and $\theta \to \theta'$, where it takes the form*

$$\operatorname*{Lt}_{t\to 0,\ r\to r',\ \theta\to\theta'} \left(\frac{e^{-[r^2+r'^2-2rr'\cos(\theta-\theta')]/4\kappa t}}{4\pi\kappa t} \right).$$

To prove this we have only to note that the path (A) can be changed into the two straight lines of Fig. 1, together with the small circuits surrounding such of the poles as lie in the interval $\theta \pm \pi$. The integrals over these straight lines vanish in the limit as $t \to 0$, since these lie in the shaded portions of the figure. We are thus left with the circuits round the poles; and if $-\beta < \theta' < \beta$, and $-\beta < \theta < \beta$, there is only one of these integrals which does not vanish in the limit as $t \to 0$, even when r and θ both tend to r' and θ'. Also this particular circuit gives the above result.

(iv) *It vanishes for r infinite*, owing to the presence of the factor $e^{-r^2/4\kappa t}$.

2. *Reduction of this Solution to the Form*

$$v = \frac{1}{2\beta} \sum_{-\infty}^{\infty} \cos \frac{n\pi}{\beta} (\theta-\theta') \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda.$$

The solution given in § 1 can be transformed into a series involving Bessel's functions by the aid of the contour integral which defines the Bessel's function of the first kind.*

We start with

$$v = \frac{e^{-(r^2+r'^2)/4\kappa t}}{8\pi\kappa\beta t} \int e^{[rr'\cos(\alpha-\theta)]/2\kappa t} \frac{e^{i\pi\alpha/\beta}}{e^{i\pi\alpha/\beta} - e^{i\pi\theta'/\beta}}\, d\alpha,$$

over the path (A).†

* Cf. Whittaker, *Modern Analysis*, p. 274.

† It is not difficult to put this result in the form

$$\frac{i}{8\pi\beta\kappa t} e^{-(r^2+r'^2)/4\kappa t} \int e^{(rr'\cos\alpha')/2\kappa t} \frac{\sin \pi\alpha'/\beta}{\cos \pi\alpha'/\beta - \cos \pi(\theta-\theta')/\beta}\, d\alpha',$$

over the upper portion of the path (A') of Fig. 2: and to deduce from this the expansion obtained

Putting $\alpha - \theta = \alpha'$, this becomes

$$v = \frac{e^{-(r^2+r'^2)/4\kappa t}}{8\pi\kappa\beta t} \int e^{(rr'\cos\alpha')/2\kappa t} \frac{e^{i\pi(\theta+\alpha')/\beta}}{e^{i\pi(\theta+\alpha')/\beta} - e^{i\pi\theta'/\beta}} d\alpha',$$

over the path (A') of Fig. 2.

In expanding the term

$$\frac{e^{i\pi(\theta+\alpha')/\beta}}{e^{i\pi(\theta+\alpha')/\beta} - e^{i\pi\theta'/\beta}},$$

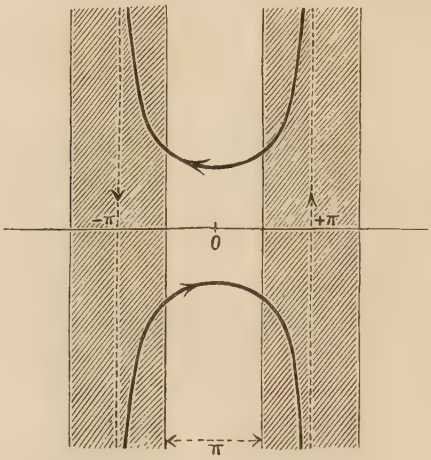

FIG. 2.—The path (A′) in the α'-plane.

we must proceed differently for the upper and lower parts of the path (A').

From the upper part we obtain

$$-\sum_{1}^{\infty} e^{in\pi(\theta-\theta')/\beta} e^{in\pi\alpha'/\beta},$$

since for the positive imaginary infinity in the α'-plane $e^{in\pi\alpha'/\beta}$ vanishes, and the series is convergent.

Also from the lower part we have

$$\sum_{0}^{\infty} e^{-in\pi(\theta-\theta')/\beta} e^{-in\pi\alpha'/\beta}.$$

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below. From this form of the result it can also be shown that the cases in which β is a sub-multiple of π, or an odd sub-multiple of 2π, admit of special treatment. [Cf. Macdonald's paper, in which he obtains the Green's Function for a Wedge of any Angle in Potential, *Proc. London Math. Soc.*, Vol. XXVI, pp. 161, 162 (1895).]

Now, if we change the sign of α' in the lower part, we bring this integral to the upper part, and have, finally,

$$v = -\frac{e^{-(r^2+r'^2)/4\kappa t}}{8\pi\kappa\beta t}\left[\int e^{(rr'\cos\alpha')/2\kappa t}\,d\alpha' + 2\sum_1^\infty \frac{\cos n\pi(\theta-\theta')}{\beta}\int e^{(rr'\cos\alpha')/2\kappa t}\,e^{in\pi\alpha'/\beta}\,d\alpha'\right]$$

over the upper portion of the path (A').

Now put* $u = rr'e^{-i\alpha'}/4\kappa t$. In the u-plane, the path proceeds from the negative end of the real axis, on the lower side, makes a circuit round the origin, and ends at the negative end of the real axis on the upper side, as in Fig. 3.

FIG. 3.

Also the integral $\int e^{(rr'\cos\alpha')/2\kappa t}\,e^{in\pi\alpha'/\beta}\,d\alpha'$

is replaced by $i\left(\frac{rr'}{4\kappa t}\right)^{n\pi/\beta}\int e^{u-\frac{1}{4}(irr'/2\kappa t)^2 1/u}\,u^{-(n\pi/\beta+1)}\,du,$

over this path in the u-plane. That is, it is replaced by

$$-2\pi e^{-in\pi^2/2\beta}\,J_{n\pi/\beta}\left(\frac{irr'}{2\kappa t}\right),$$

on using the contour integral for $J_n(z)$.†

Hence we have

$$v = \frac{e^{-(r^2+r'^2)/4\kappa t}}{4\kappa\beta t}\sum_{-\infty}^{\infty} e^{-in\pi^2/2\beta}\,J_{n\pi/\beta}\left(\frac{irr'}{2\kappa t}\right)\cos\frac{n\pi}{\beta}(\theta-\theta').$$

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* Cf. Sommerfeld, *Math. Ann.*, Bd. XLVII, p. 328 (1896).

† This result could have been obtained from Macdonald's integral for $I_n(x)$. Cf. Macdonald, *Electric Waves*, App. D, p. 191.

But it is known that

$$\int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_n(\lambda r)\, J_n(\lambda r')\, d\lambda = \frac{1}{2\kappa t}\, e^{-(r^2+r'^2)/4\kappa t}\, e^{-\frac{1}{2}in\pi}\, J_n\left(\frac{ir r'}{2\kappa t}\right)$$

for any value of n for which the real part is greater than -1.*

Therefore our solution, with period 2β, can be written

$$v = \frac{1}{2\beta} \sum_{-\infty}^{\infty} \cos\frac{n\pi}{\beta}(\theta-\theta') \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda.\dagger$$

3. *The Green's Function for a Solid bounded by the Planes* $\theta = 0$ *and* $\theta = \beta$ *is*

$$\frac{2}{\beta} \sum_{1}^{\infty} \sin\frac{n\pi\theta}{\beta} \sin\frac{n\pi\theta'}{\beta} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda.$$

By the Green's function in the conduction of heat we mean, in this connection, the temperature in the solid at the time t, due to a source at $(r',\ \theta')$ at time $t = 0$, the surface of the solid being kept at zero temperature.

If we take the solution of § 2, namely,

$$v = \frac{1}{2\beta} \sum_{-\infty}^{\infty} \cos\frac{n\pi(\theta-\theta')}{\beta} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda,$$

which has the value required by a source at the points $(r',\ \theta' \pm 2s\beta)$, s being zero or any positive integer, and subtract from it the corresponding solution with a source at the points $(r',\ -\theta' \pm 2s\beta)$, we have the Green's function for this solid.

For it is obvious that

$$v = \frac{1}{2\beta} \sum_{-\infty}^{\infty} \left\{ \cos\frac{n\pi(\theta-\theta')}{\beta} - \cos\frac{n\pi(\theta+\theta')}{\beta} \right\} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda$$

satisfies the initial conditions, having only the critical value at the point

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* Cf. Nielsen, *Handbuch der Theorie der Cylinderfunktionen*, p. 184 (Leipzig, 1904).

† This result resembles Sommerfeld's solution with period 4π (cf. *Math. Ann.*, Bd. XLV, p. 276); but it will be seen that for period $2m\pi$, we obtain

$$v = \frac{1}{2m\pi} \sum_{-\infty}^{\infty} \cos\frac{n}{m}(\theta-\theta') \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n/m}(\lambda r)\, J_{n/m}(\lambda r')\, d\lambda,$$

and not

$$\frac{1}{4\pi} \sum_{-\infty}^{\infty} \cos\frac{n}{m}(\theta-\theta') \int_{-\infty}^{\infty} \lambda e^{-\kappa\lambda^2 t} J_{n/m}(\lambda r)\, J_{n/m}(\lambda r')\, d\lambda,$$

the solution inferred by him for that case.

(r', θ'), when we regard the space as bounded by $\theta = 0$ and $\theta = \beta$. It also satisfies the boundary conditions, and it vanishes at infinity.

This solution can be written in the form

$$v = \frac{2}{\beta} \sum_1^\infty \sin \frac{n\pi\theta}{\beta} \sin \frac{n\pi\theta'}{\beta} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r) J_{n\pi/\beta}(\lambda r') d\lambda.$$

We have been dealing up till now with a line source through (r', θ'). It is not difficult to show that for a point source at (r', θ', z'), we have

$$v = \frac{e^{-(z-z')^2/4\kappa t}}{\beta\sqrt{(\pi\kappa t)}} \sum_1^\infty \sin \frac{n\pi\theta}{\beta} \sin \frac{n\pi\theta'}{\beta} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} J_{n\pi/\beta}(\lambda r) J_{n\pi/\beta}(\lambda r') d\lambda.*$$

4. *The Bessel's Function of the Second Kind used in the results which follow.*

In the remainder of this paper I shall require a solution of Bessel's equation which vanishes at the positive imaginary infinity. For this purpose I use

$$\frac{\pi}{2 \sin n\pi} \left[J_{-n}(z) - e^{-in\pi} J_n(z)\right],$$

and denote it by $U_n(z)$.† This differs by a constant factor from the solution usually denoted by $K_n(-i\lambda z)$, and also from Hankel's solution referred to in various parts of Nielsen's *Cylinderfunktionen* by the symbol $H_1^n(z)$.

To make $J_n(z)$ uniform, the argument of z is restricted to a range of 2π, and in this paper it is supposed to vary from $-\frac{1}{2}\pi$ to $\frac{3}{2}\pi$.

From the definition of $U_n(z)$ it follows that

$$i\pi J_n(z) = U_n(z) - e^{in\pi} U_n(ze^{i\pi}),$$

when the real part of z is positive.

When the real part of z is negative, we replace this equation by

$$i\pi J_n(z) = e^{in\pi} U_n(ze^{-i\pi}) - e^{2in\pi} U_n(z).$$

The approximate values of these Bessel's functions are known,‡ and we have

$$U_n(z) = e^{-\frac{1}{2}in\pi} e^{i(z+\frac{1}{4}\pi)} \sqrt{\frac{\pi}{2z}},$$

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \cos \left\{z - \tfrac{1}{2}(n+\tfrac{1}{2})\,\pi\right\},$$

* Cf. *Proc. London Math. Soc.*, Vol. xxx, p. 159.

† This notation is used to bring the results which follow into line with those given in my *Fourier's Series*, §§ 161–4.

‡ Cf. Nielsen, *loc. cit.*, Ch. x.

when the real part of z is positive. Also when the real part of z is negative, we can find the approximate value of $J_n(z)$ from the equation

$$J_n(z) = e^{in\pi} J_n(ze^{-i\pi}).$$

It follows that, at infinity in the upper part of the plane, $J_n(z)$ is infinite, $U_n(z)$ vanishes, and $J_n(az)\, U_n(bz)$ vanishes, if $0 < a < b$.

5. *The Green's Functions for a Solid bounded by Axial Planes, Concentric Cylinders, or Planes Perpendicular to the Axis.*

With the aid of a method employed in another place,* and using the Green's function found in § 3 of this paper, it is possible to give the results for the solids in which the boundaries are axial planes, concentric cylinders, or planes perpendicular to the axis. From these results, as also from that for the solid bounded by two axial planes, we can immediately obtain proofs of the possibility of the expansion of an arbitrary function in various forms.

(i) *Line Source in the Portion of an Infinite Cylinder $r = a$, cut off by the Two Axial Planes $\theta = 0$ and $\theta = \beta$.*

Here we have

$$v = \frac{2}{a^2\beta} \sum_1^\infty \sin\frac{n\pi\theta}{\beta} \sin\frac{n\pi\theta'}{\beta} \sum_\lambda e^{-\kappa\lambda^2 t} \frac{J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')}{[J'_{n\pi/\beta}(\lambda a)]^2},$$

the λ's being the roots of the equation

$$J_{n\pi/\beta}(\lambda a) = 0.$$

(ii) *Line Source in a Solid bounded internally by the Cylinder $r = a$ and the Two Planes $\theta = 0$ and $\theta = \beta$.*

Here we have

$$v = \frac{2}{i\beta\pi} \sum_1^\infty \sin\frac{n\pi\theta}{\beta} \sin\frac{n\pi\theta'}{\beta} \int_{-\infty}^{\infty} \lambda e^{-\kappa\lambda^2 t} \frac{U_{n\pi/\beta}(\lambda r')}{U_{n\pi/\beta}(\lambda a)} \times \{J_{n\pi/\beta}(\lambda r)\, U_{n\pi/\beta}(\lambda a) - J_{n\pi/\beta}(\lambda a)\, U_{n\pi/\beta}(\lambda r)\}\, d\lambda,$$

for $a < r < r'$.

The letters r, r' have to be interchanged for $r > r'$.

* Cf. *Fourier's Series*, Ch. XVIII.

(iii) *Line Source in a Solid bounded by the Cylinders $r = a$, $r = b$, and the Two Planes $\theta = 0$ and $\theta = \beta$.*

Here we have

$$v = \frac{2}{\beta} \sum_{1}^{\infty} \sin \frac{n\pi\theta}{\beta} \sin \frac{n\pi\theta'}{\beta} \sum_{\lambda} \lambda e^{-\kappa\lambda^2 t}$$

$$\times \frac{\{J_{n\pi/\beta}(\lambda a)\, U_{n\pi/\beta}(\lambda r') - J_{n\pi/\beta}(\lambda r')\, U_{n\pi/\beta}(\lambda a)\} \times \{J_{n\pi/\beta}(\lambda b)\, U_{n\pi/\beta}(\lambda r) - J_{n\pi/\beta}(\lambda r)\, U_{n\pi/\beta}(\lambda b)\}}{\frac{d}{d\lambda}\{J_{n\pi/\beta}(\lambda a)\, U_{n\pi/\beta}(\lambda b) - J_{n\pi/\beta}(\lambda b)\, U_{n\pi/\beta}(\lambda a)\}},$$

the λ's being the roots of the equation

$$J_{n\pi/\beta}(\lambda a)\, U_{n\pi/\beta}(\lambda b) - J_{n\pi/\beta}(\lambda b)\, U_{n\pi/\beta}(\lambda a) = 0.$$

(iv) *Line Source in a Solid bounded by the Cylinders $r = a$ and $r = b$.*

Here we have

$$v = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \cos n(\theta - \theta') \sum_{\lambda} \lambda e^{-\kappa\lambda^2 t}$$

$$\times \frac{\{J_n(\lambda a)\, U_n(\lambda r') - J_n(\lambda r')\, U_n(\lambda a)\} \{J_n(\lambda b)\, U_n(\lambda r) - J_n(\lambda r)\, U_n(\lambda b)\}}{\frac{d}{d\lambda}\{J_n(\lambda a)\, U_n(\lambda b) - J_n(\lambda b)\, U_n(\lambda a)\}},$$

the λ's being the roots of the equation

$$J_n(\lambda a)\, U_n(\lambda b) - J_n(\lambda b)\, U_n(\lambda a) = 0.$$

From this result we obtain the theorem that

$$f(r) = \operatorname*{Lt}_{t \to 0} \int_a^b f(r')\, r'$$

$$\times \left\{ \sum_{\lambda} \lambda e^{-\kappa\lambda^2 t} \frac{\{J_0(\lambda a)\, U_0(\lambda r') - J_0(\lambda r')\, U_0(\lambda a)\} \{J_0(\lambda b)\, U_0(\lambda r) - J_0(\lambda r)\, U_0(\lambda b)\}}{\frac{d}{d\lambda}\{J_0(\lambda a)\, U_0(\lambda b) - J_0(\lambda b)\, U_0(\lambda a)\}} \right\} dr',$$

the λ's being the roots of the equation

$$J_0(\lambda a)\, U_0(\lambda b) - J_0(\lambda b)\, U_0(\lambda a) = 0.$$

This is a special case of a general result obtained by W. McF. Orr.*

* Cf. W. McF. Orr, *Proc. Roy. Irish Acad.*, Vol. XXVII (A), No. 11, p. 238, § 2 (1909).

(v) *Point Source in the Wedge bounded by the Planes* $z = 0$, $z = h$, $\theta = 0$ *and* $\theta = \beta$.

Here we have

$$v = \frac{4}{h\beta} \sum_1^\infty e^{-m^2\pi^2\kappa t/h^2} \sin\frac{m\pi z}{h} \sin\frac{m\pi z'}{h} \sum_1^\infty \sin\frac{n\pi\theta}{\beta} \sin\frac{n\pi\theta'}{\beta} \int_0^\infty \lambda e^{-\kappa\lambda^2 t} \times J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')\, d\lambda.$$

(vi) *Point Source in a Solid bounded by the Cylinder* $r = a$, *and the Planes* $z = 0$, $z = h$, $\theta = 0$ *and* $\theta = \beta$.

Here we have

$$v = \frac{4}{a^2 h\beta} \sum_1^\infty e^{-m^2\pi^2\kappa t/h^2} \sin\frac{m\pi z}{h} \sin\frac{m\pi z'}{h} \sum_1^\infty \sin\frac{n\pi\theta}{\beta} \sin\frac{n\pi\theta'}{\beta} \sum_\lambda e^{-\kappa\lambda^2 t} \times \frac{J_{n\pi/\beta}(\lambda r)\, J_{n\pi/\beta}(\lambda r')}{\left[J_{n\pi/\beta}(\lambda a)\right]^2},$$

the λ's being the roots of the equation

$$J_{n\pi/\beta}(\lambda a) = 0.$$

KUMMER'S QUARTIC SURFACE AS A WAVE SURFACE

By H. BATEMAN.

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1. It has been known for some time that Fresnel's wave surface is a case of Kummer's sixteen nodal quartic surface; also it is known that the wave surface of a dynamical medium possessing certain general properties is a type of Kummer's surface which can be derived from Fresnel's surface by means of a homogeneous strain.*

The Kummer's surfaces which occur in this way are of a particular type in which the six nodes on any singular conic belong to an involution. The lines joining corresponding points of the involution meet in a point which is one of the vertices of a certain tetrahedron, each face of which cuts the surface in two conics intersecting in four nodes.† This type of surface is called the tetrahedroid.‡

The general Kummer's surface appears to be the wave surface for a medium of a purely ideal character in which the disturbance at any point is specified by means of four vectors, whose components are connected with one another by six linear relations of a certain type and satisfy a system of partial differential equations analogous to the electrodynamical equations.

These conditions may not correspond to anything occurring in nature; nevertheless their investigation was thought to be of some mathematical interest on account of the connection which is established between line-geometry and the theory of partial differential equations.

The system of partial differential equations under consideration is of special interest on account of the fact that when the four vectors occurring in them are regarded as independent, the equations are invariant for any transformation of coordinates.

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* T. J. I'A. Bromwich, *Proc. London Math. Soc.*, Ser. 1, Vol. xxxiv (1902), p. 307. A less general case had been studied previously by Macdonald. *Ibid.*, Vol. xxxii (1900), p. 311.

† Hudson's *Kummer's Quartic Surface*, Ch. ix.

‡ Cayley, *Coll. Papers*, i, 302; *Liouville's Journal*, t. xi (1846), p. 291.

2. Let a line in space be specified by six coordinates $(l, m, n, \lambda, \mu, \nu)$ satisfying the relations

$$\left.\begin{aligned} \lambda+mz-ny &= 0, \\ \mu+nx-lz &= 0, \\ \nu+ly-mx &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} \lambda x+\mu y+\nu z &= 0 \\ l\lambda+m\mu+n\nu &= 0 \end{aligned}\right\}, \tag{1}$$

where (x, y, z) are the coordinates of any point on the line.

Let the equation of a quadratic complex be

$$\Phi = \kappa\,(l, m, n, \lambda, \mu, \nu)^2 = 0,$$

where κ is a symmetrical matrix of six rows and columns. The point (x, y, z) is a singular point if the cone of lines belonging to the complex which pass through this point consists of two plane pencils. Let

$$(l, m, n, \lambda, \mu, \nu), \quad (l', m', n', \lambda', \mu', \nu')$$

be two lines belonging to the same pencil, then the line

$$(l+\theta l',\ m+\theta m',\ n+\theta n',\ \lambda+\theta\lambda',\ \mu+\theta\mu',\ \nu+\theta\nu')$$

must belong to the complex for all values of θ.

Substituting in the equation of the complex, we obtain

$$l'\frac{\partial\Phi}{\partial l}+m'\frac{\partial\Phi}{\partial m}+n'\frac{\partial\Phi}{\partial n}+\lambda'\frac{\partial\Phi}{\partial\lambda}+\mu'\frac{\partial\Phi}{\partial\mu}+\nu'\frac{\partial\Phi}{\partial\nu}=0. \tag{2}$$

Now let $(l, m, n, \lambda, \mu, \nu)$ be the line common to the two pencils having their vertices at the point (x, y, z); then we may obtain three linearly independent equations of the type (2), by choosing successively for $(l', m', n', \lambda', \mu', \nu')$, two lines belonging to one pencil and one line belonging to the other. The solution of these three linear equations is necessarily a linear combination of three particular solutions. Now we may obtain three particular solutions by taking

$$\left(\frac{\partial\Phi}{\partial\lambda}, \frac{\partial\Phi}{\partial\mu}, \frac{\partial\Phi}{\partial\nu}, \frac{\partial\Phi}{\partial l}, \frac{\partial\Phi}{\partial m}, \frac{\partial\Phi}{\partial n}\right)$$

successively proportional to the six coordinates of any three lines through the point (x, y, z), for then equations (2) are satisfied identically in virtue of the fact that two lines intersect. A linear combination of the coordinates of three lines through the point (x, y, z), however, represent the coordinates of some line through this point, and so every set of values of

$$\left(\frac{\partial\Phi}{\partial\lambda}, \frac{\partial\Phi}{\partial\mu}, \frac{\partial\Phi}{\partial\nu}, \frac{\partial\Phi}{\partial l}, \frac{\partial\Phi}{\partial m}, \frac{\partial\Phi}{\partial n}\right)$$

which satisfy the three equations of the type (2) represent the coordinates of a line through the point (x, y, z).

Putting $$l_0 = \frac{\partial \Phi}{\partial \lambda}, \qquad \lambda_0 = \frac{\partial \Phi}{\partial l}, \qquad \ldots,$$

we have the system of equations

$$\begin{aligned} \lambda + mz - ny &= 0, & \lambda_0 + m_0 z - n_0 y &= 0, \\ \mu + nx - lz &= 0, & \mu_0 + n_0 x - l_0 z &= 0, \\ \nu + ly - mx &= 0, & \nu_0 + l_0 y - m_0 x &= 0. \end{aligned}$$

These are the conditions that (x, y, z) should be a singular point of the complex, and $(l, m, n, \lambda, \mu, \nu)$ a singular line.

The equation of the complex to which $(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)$ belongs may be obtained by solving the equations

$$l_0 = \frac{\partial \Phi}{\partial \lambda}, \qquad \lambda_0 = \frac{\partial \Phi}{\partial l}, \qquad \ldots,$$

for $(l, m, n, \lambda, \mu, \nu)$, and substituting in the equation

$$l\lambda_0 + l_0\lambda + m\mu_0 + m_0\mu + n\nu_0 + n_0\nu = 0.$$

Denoting the equation of this complex by Ω, we have

$$\frac{\partial \Omega}{\partial \lambda_0} = l + \lambda_0 \frac{\partial l}{\partial \lambda_0} + l_0 \frac{\partial \lambda}{\partial \lambda_0} + \mu_0 \frac{\partial m}{\partial \lambda_0} + m_0 \frac{\partial \mu}{\partial \lambda_0} + \nu_0 \frac{\partial n}{\partial \lambda_0} + n_0 \frac{\partial \nu}{\partial \lambda_0};$$

but, if $$(l, \lambda) = K(l_0, \lambda_0),$$

we have $$\Omega = K(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)^2,$$

and it follows that

$$\lambda_0 \frac{\partial l}{\partial \lambda_0} + l_0 \frac{\partial \lambda}{\partial \lambda_0} + \mu_0 \frac{\partial m}{\partial \lambda_0} + m_0 \frac{\partial \mu}{\partial \lambda_0} + \nu_0 \frac{\partial n}{\partial \lambda_0} + n_0 \frac{\partial \nu}{\partial \lambda_0} = \tfrac{1}{2} \frac{\partial \Omega}{\partial \lambda_0};$$

hence $$\frac{\partial \Omega}{\partial \lambda_0} = 2l,$$

and so $$\left(\frac{\partial \Omega}{\partial \lambda_0}, \frac{\partial \Omega}{\partial \mu_0}, \frac{\partial \Omega}{\partial \nu_0}, \frac{\partial \Omega}{\partial l_0}, \frac{\partial \Omega}{\partial m_0}, \frac{\partial \Omega}{\partial n_0}\right)$$

are proportional to the six coordinates of a line. This shows that (x, y, z) is a singular point of the complex Ω, and so the two complexes Ω and Φ have the same singular surface.

It is known that the singular line which is the intersection of the planes of the two pencils associated with a singular point (x, y, z) of the

complex Φ is a tangent to the Kummer's surface which is the locus of the singular points of Φ;* hence it follows that $(l, m, n, \lambda, \mu, \nu)$, $(l_0, m_0, n_0, \lambda_0, \mu_0, \nu_0)$ are both tangents to this surface.

3. Let B, E, H, D be four vectors whose components satisfy a system of linear partial differential equations of the type

$$\left.\begin{aligned} \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}&=\frac{\partial D_x}{\partial t}, & \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=-\frac{\partial B_x}{\partial t}\\ \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}&=\frac{\partial D_y}{\partial t}, & \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=-\frac{\partial B_y}{\partial t}\\ \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}&=\frac{\partial D_z}{\partial t}, & \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=-\frac{\partial B_z}{\partial t}\\ \frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}&=0, & \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}&=0. \end{aligned}\right\} \qquad \text{(I)}$$

These equations are invariant for any transformation of coordinates from (x, y, z, t) to (x', y', z', t') provided we define the corresponding vectors B', E', H', D' by means of the equations

$$\begin{aligned} &B_x dy\,dz+B_y dz\,dx+B_z dx\,dy+E_x dx\,dt+E_y dy\,dt+E_z dz\,dt\\ &\quad=B'_x dy'dz'+B'_y dz'dx'+B'_z dx'dy'+E'_x dx'dt'+E'_y dy'dt'+E'_z dz'dt',\\ &D_x dy\,dz+D_y dz\,dx+D_z dx\,dy-H_x dx\,dt-H_y dy\,dt-H_z dz\,dt\\ &\quad=D'_x dy'dz'+D'_y dz'dx'+D'_z dx'dy'-H'_x dx'dt'-H'_y dy'dt'-H'_z dz'dt', \end{aligned}$$

where a term such as $dy\,dz$ is interpreted to mean

$$\frac{\partial(y, x)}{\partial(\alpha, \beta)}\,d\alpha\,d\beta,$$

when x, y, z, t are expressed in an arbitrary manner as functions of two parameters α, β.

The invariance of the fundamental equations (I) is then a direct consequence of the equations of type†

$$\begin{aligned} &\iint (B_x dy\,dz+B_y dz\,dx+B_z dx\,dy+E_x dx\,dt+E_y dy\,dt+E_z dz\,dt)\\ &\quad=\iiint\left[\left(\frac{\partial B_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)dy\,dz\,dt+\left(\frac{\partial B_y}{\partial t}+\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)dz\,dx\,dt\right.\\ &\qquad\left.+\left(\frac{\partial B_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)dx\,dy\,dt+\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)dx\,dy\,dz\right]. \end{aligned}$$

* Jessop's *Treatise on the Line Complex*, p. 91.

† H. F. Baker, *Camb. Phil. Trans.*, Vol. XVIII, p. 442.

We shall now study the system of differential equations (I) in connection with a set of six constitutive relations connecting the components of the different vectors. These relations may be regarded as defining the properties of a medium through which disturbances can be propagated.

The form which these relations take after a transformation of coordinates depends on the nature of the transformation; this is only natural because, in general, a transformation of coordinates alters the properties of a medium.

The constitutive relations which will be studied here are of the type

$$\left.\begin{aligned} -B_x &= \kappa_{11}H_x+\kappa_{12}H_y+\kappa_{13}H_z+\kappa_{14}D_x+\kappa_{15}D_y+\kappa_{16}D_z \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ E_x &= \kappa_{41}H_x+\kappa_{42}H_y+\kappa_{43}H_z+\kappa_{44}D_x+\kappa_{45}D_y+\kappa_{46}D_z \end{aligned}\right\}, \qquad \text{(II)}$$

where the coefficients κ_{rs} are arbitrary constants satisfying the relations

$$\kappa_{rs} = \kappa_{sr} \quad (r = 1, 2, 3, 4, 5, 6;\ s = 1, 2, 3, 4, 5, 6).$$

To obtain the form of an elementary wave issuing from a point of the medium specified by these equations, we write $H+\frac{\partial H}{\partial\theta}\delta\theta$, instead of H, $\frac{\partial H}{\partial x}+\frac{\partial H}{\partial\theta}\frac{\partial\theta}{\partial x}$, instead of $\frac{\partial H}{\partial x}$, and similarly for the other quantities.

Putting

$$(l, m, n, \lambda, \mu, \nu) \equiv \left(\frac{\partial H_x}{\partial\theta}, \frac{\partial H_y}{\partial\theta}, \frac{\partial H_z}{\partial\theta}, \frac{\partial D_x}{\partial\theta}, \frac{\partial D_y}{\partial\theta}, \frac{\partial D_z}{\partial\theta}\right),$$

$$(l', m', n', \lambda', \mu', \nu') \equiv \left(\frac{\partial E_x}{\partial\theta}, \frac{\partial E_y}{\partial\theta}, \frac{\partial E_z}{\partial\theta}, -\frac{\partial B_x}{\partial\theta}, -\frac{\partial B_y}{\partial\theta}, -\frac{\partial B_z}{\partial\theta}\right),$$

$$\xi = \frac{\partial\theta}{\partial x}, \quad \eta = \frac{\partial\theta}{\partial y}, \quad \zeta = \frac{\partial\theta}{\partial z}, \quad \tau = \frac{\partial\theta}{\partial\tau},$$

after the other substitutions have been made in (I) and (II), we obtain the system of equations

$$\lambda+mz-ny = 0, \qquad \lambda'+m'z-n'y = 0,$$

$$\lambda x+\mu y+\nu z = 0, \qquad \lambda' x+\mu' y+\nu' z = 0,$$

where $$x = \frac{\xi}{\tau}, \quad y = \frac{\eta}{\tau}, \quad z = \frac{\zeta}{\tau},$$

and $$(\lambda', \mu', \nu', l', m', n') = \kappa(l, m, n, \lambda, \mu, \nu), \qquad \text{(III)}$$

or $$l' = \frac{\partial\Phi}{\partial\lambda}, \quad \lambda' = \frac{\partial\Phi}{\partial l}.$$

Now, as was shown in § 2, these equations imply that the point (x, y, z) is a singular point of the complex Φ, and accordingly that $\left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial t}\right)$ satisfy a homogeneous equation of degree 4, which is the equation of Kummer's surface.

This equation represents the differential equation of the characteristics of the system of equations which determine the components of the vectors H and D. For, if $\theta(x, y, z, t) = \text{const.}$ represents a system of characteristics, and ϕ, ψ, χ are three other functions of x, y, z, t, which are chosen so that (x, y, z, t) can be expressed as functions of the four coordinates θ, ϕ, ψ, χ, then the problem of determining the values of the vectors H and D, when their values are given on the characteristic, must be indeterminate. In other words, when the equations are expressed in terms of the variables $(\theta, \phi, \psi, \chi)$ in place of (x, y, z, t), it must not be possible to solve the equations and obtain the values of the derivatives

$$\frac{\partial H_x}{\partial\theta}, \quad \frac{\partial D_x}{\partial\theta}, \quad \ldots.$$

Now the condition that this should be the case is that the determinant of the coefficients of these quantities in the transformed equations should vanish.

Putting $$\frac{\partial H_x}{\partial\theta} = h_x, \quad \frac{\partial D_x}{\partial\theta} = d_x, \quad \ldots,$$

and substituting in equations (I) and (II), we obtain

$$h_z\frac{\partial\theta}{\partial y} - h_y\frac{\partial\theta}{\partial z} + \ldots = d_x\frac{\partial\theta}{\partial t} + \ldots,$$

$$(\kappa_{61}h_x + \kappa_{62}h_y + \kappa_{63}h_z + \kappa_{64}d_x + \kappa_{65}d_y + \kappa_{66}d_z)\frac{\partial\theta}{\partial y}$$

$$-(\kappa_{51}h_x + \kappa_{52}h_y + \kappa_{53}h_z + \kappa_{54}d_x + \kappa_{55}d_y + \kappa_{56}d_z)\frac{\partial\theta}{\partial z} + \ldots$$

$$= (\kappa_{11}h_x + \kappa_{12}h_y + \kappa_{13}h_z + \kappa_{14}d_x + \kappa_{15}d_y + \kappa_{16}d_z)\frac{\partial\theta}{\partial t} + \ldots.$$

The determinant of the coefficient of h_x, h_y, h_z, d_x, d_y, d_z vanishes, if quantities $(l, m, n, \lambda, \mu, \nu)$, $(l', m', n', \lambda', \mu', \nu')$ can be found, so that equations (III) are satisfied. The reasoning is then the same as before.

4. Having obtained the differential equation for the characteristics, we must now proceed to integrate it. Regarding (x, y, z, t) for the moment

as the coordinates of a point in a space of four dimensions, the equation

$$\theta(x, y, z, t) = \text{const.}$$

represents a variety, and

$$(X-x)\frac{\partial\theta}{\partial x}+(Y-y)\frac{\partial\theta}{\partial y}+(Z-z)\frac{\partial\theta}{\partial z}+(T-t)\frac{\partial\theta}{\partial t}=0,$$

the equation of the tangent threefold at the point (x, y, z, t). The coefficients $\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial t}$, however, are connected by a *homogeneous* equation of the fourth degree; hence the equation $\theta = \text{const.}$ represents a cone, and since the reciprocal of a Kummer's surface is a Kummer's surface, it follows that the sections of this cone by a system of parallel three folds given by $t = \text{const.}$ are a system of similar Kummer's surfaces.

Now it is known that the series of surfaces obtained by giving different constant values to t in the equation of a characteristic

$$\theta(x, y, z, t) = \text{const.}$$

are the successive wave fronts of an elementary disturbance in the medium whose properties are specified by the given system of differential equations;* hence in the case under consideration the elementary wave surfaces are Kummer's quartics.

To make matters clear, we shall assume that at the wave front the derivatives of the vectors D, H, E, B are discontinuous. This implies that there is more than one set of possible values for the quantities $\frac{\partial D}{\partial\theta}, \frac{\partial H}{\partial\theta}, \ldots$, and this is exactly what was assumed when we formed the equations of the characteristics.

The two sets of values of $\frac{\partial H_x}{\partial\theta}, \ldots, \frac{\partial D_x}{\partial\theta}, \ldots$, differ by quantities $l, m, n, \lambda, \mu, \nu$, which are the coordinates of a line belonging to a certain quadratic complex, while the two sets of values of $\frac{\partial E_x}{\partial\theta}, \ldots, \frac{\partial B_x}{\partial\theta}, \ldots$, differ by quantities $l'', m'', n'', \lambda'', \mu'', \nu''$, which are the coordinates of a line belonging to a cosingular quadratic complex. If we are dealing with a pulse of small thickness $\delta\theta$, the quantities $(l\delta\theta, m\delta\theta, n\delta\theta)$, $(\lambda\delta\theta, \mu\delta\theta, \nu\delta\theta)$, ..., may be interpreted as the vectors belonging to the pulse. Denoting these by (H'_x, H'_y, H'_z), (D'_x, D'_y, D'_z), ..., we see that

* See a paper by T. H. Havelock, "Wave Fronts considered as the Characteristics of Partial Differential Equations," *Proc. London Math. Soc.*, Ser. 2, Vol. 2 (1904), p. 297.

they satisfy the relations

$$D'_x H'_x + D'_y H_y + D'_z H'_z = 0 ;$$

and are therefore at right angles. The vector D' also satisfies the relation

$$D'_x \frac{\partial \theta}{\partial x} + D'_y \frac{\partial \theta}{\partial y} + D'_z \frac{\partial \theta}{\partial z} = 0,$$

and so lies in the tangent plane to the wave. In the same way it can be shown that the vector B lies in the tangent plane. On account of the correspondence between the Kummer's surface forming the wave front and the Kummer's surface described by the point $\left(\frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}, \frac{\partial \theta}{\partial t}\right)$, it appears that the two vectors in the wave front correspond to singular lines of two cosingular complexes.

THE LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT BY THE TREASURER FOR THE YEAR 1908–09 (NOV. 15TH, 1908, TO NOV. 11TH, 1909).

GENERAL CASH ACCOUNT OF THE SOCIETY.

Cr.

	£.	s.	d.	£.	s.	d.
Balance from 1907–08 :—						
Against General Fund ... £1 16 2						
In Life Compositions Fund 141 15 0						
			139	18	10	
Interest on Capital—						
Lieut.-Col. Campbell's Fund 14 12 0						
Life Compositions Fund ... 65 4 7						
Invested Surplus Fund ... 19 17 4						
			99	13	11	
Income Tax returned (3 years)			15	2	9	
Interest on Lord Rayleigh's Fund			46	13	2	
7 Entrance Fees			7	7	0	
3 Life Compositions (3 old, 0 new)			31	10	0	
125 Annual Subscriptions—						
5 before 1906 ... ... 5 5 0						
16 for 1906–07 ... ... 16 16 0						
22 for 1907–08 ... ... 23 2 0						
73 for 1908–09 ... ... 76 13 0						
9 in advance 9 9 0						
			131	5	0	
Sale of *Proceedings*, &c.			116	15	6	
			£588	6	2	

Dr.

	£.	s.	d.	£.	s.	d.
Printing and Circulating *Proceedings*, &c. ...				311	11	4
Purchase of Journals, &c., for Library				8	3	9
Postage and Sundries				4	8	11
Rent				50	0	0
Attendance				8	8	0
Providing Tea at Council				2	8	6
Purchase of £105. 9*s.* 5*d.* New South Wales $3\frac{1}{2}$% Stock for Life Compositions Fund...				105	0	0
Balance at Bank—						
In General Fund... ... £30 0 8						
In Life Compositions Fund 68 5 0						
				98	5	8
				£588	6	2

TRUST FUND ACCOUNTS.

I. Lord Rayleigh's Fund.

	£.	s.	d.	£.	s.	d.
Annuity (less transfer to Sinking Fund)... ...	34	2	0	Transferred to the General Account of the Society, as above	46	13	2
Dividends on Stock	6	0	2				
Income Tax returned (3 years)	6	11	0				
	£46	13	2		£46	13	2

II. De Morgan Medal Fund.

	£.	s.	d.	£.	s.	d.
Balance from previous account	0	15	9	Balance at Bank	4	1	1
Interest during the year 1908–09	2	16	10				
Income Tax returned (3 years)	0	8	6				
	£4	1	1		£4	1	1

STATEMENT OF CAPITAL, November, 1909.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
1. Lieut.-Col. Campbell's Fund	499	7	5	385	0	0	Guaranteed 4 per Cent. London & North Western Railway Consolidated Stock.
2. Life Compositions Fund, from 1907–8 ...	1975	15	0	1970	8	4	New South Wales $3\frac{1}{2}$ per Cent. Inscribed Stock of 1918.
Amount received during 1908–9... ...	31	10	0	68	5	0	In Bank awaiting investment.
3. Invested Surplus Fund	700	0	0	700	7	1	India 3 per Cent. Stock.
Trust Funds.							
1. Lord Rayleigh's Fund	1000	0	0				Terminable Annuity of £50. 8*s.* 8*d.* (Class B, deduction for Sinking Fund) Great Indian Peninsula Railway.
				200	0	0	Guaranteed 3 per Cent. Stock of same Railway Co.
2. De Morgan Medal Fund	102	5	9	60	0	0	Great Western Railway 5 per Cent. Preference Stock.

December 9th, 1909.

Signed on behalf of the Council by { W. D. NIVEN, *President*. A. E. H. LOVE, *Secretary*.

I report to the Members that I have obtained all the information and explanation I required as Auditor, and that I have examined the above Accounts with the books, and that, to the best of my information and of the explanations given to me, such Accounts are properly drawn up so as to exhibit a true and correct view of the state of the affairs of the Society as shown by the books of the Society.

J. G. LEATHEM. *December 4th*, 1909.

The ray with the parameters u, v is therefore intersected by two consecutive ones, so that through each ray of the congruence two developables pass. The points on the ray where it is so intersected are called the focal points.

In particular congruences these developables may be coincident, but we also exclude the consideration of such congruences from this paper.

A congruence is said to be a cyclical one if, corresponding to each ray, a circle can be drawn whose centre is at the extremity of z and plane perpendicular to λ, and if this system of circles is cut orthogonally by a system of surfaces.

We shall prove that such cyclic systems exist in the field to which we limit ourselves, viz., the field where λ does not depend on one parameter only and where the developables of the congruence are distinct.

Whether congruences exist outside this field which also are cyclical we shall not consider.

We may choose the parameters so that the developables are given by

$$du\,dv = 0,$$

and therefore $\quad S z_1 \lambda_1 \lambda = 0, \qquad S z_2 \lambda_2 \lambda = 0,$

so that $\quad z_1 = a\lambda_1 + b\lambda, \qquad z_2 = c\lambda_2 + d\lambda,$

where a, b, c, d are scalar functions of the parameters u and v.

For the focal points on a ray we have

$$dz + w\,d\lambda + \lambda\,dw = 0,$$

and therefore $\quad (z_1 + w\lambda_1)du + (z_2 + w\lambda_2)dv + \lambda\,dw = 0.$

The focal lengths are therefore given by

$$S(z_1 + w\lambda_1)(z_2 + w\lambda_2)\,\lambda = 0.$$

2. Let now λ, μ, ν be three unit vectors mutually at right angles and depending on two parameters. We must have

$$\lambda_1 = a\lambda + r'\mu - q'\nu, \qquad \mu_1 = b\mu + p'\nu + c\lambda,$$

where a, b, r', q', p', c are scalars.

Since $\lambda^2 + 1 = 0$, $S\lambda\lambda_1 = 0$; similarly we see that $S\mu\mu_1 = 0$, and therefore a and b are both zero. We also have $S\lambda\mu = 0$, and therefore

$$S\lambda_1\mu + S\lambda\mu_1 = 0,$$

so that $c + r' = 0$.

As we should so often have to write $S\lambda\mu$ and $V\lambda\mu$, it will be found more convenient to denote these symbols respectively by $\lambda \underset{\smile}{\mu}$ and $\lambda \widehat{\mu}$.

We now have $\lambda_1 = \mu r' - \nu q', \qquad \mu_1 = \nu p' - \lambda r',$

and from $\nu\nu_1 = 0, \quad \nu\lambda_1 + \nu_1\lambda = 0, \quad \nu\mu_1 + \nu_1\mu = 0,$

we see that $\nu_1 = \lambda q' - \mu p'.$

We thus obtain the formulæ

$$(1) \qquad \begin{cases} \lambda_1 = \mu r' - \nu q', & \mu_1 = \nu p' - \lambda r', & \nu_1 = \lambda q' - \mu p'; \\ \lambda_2 = \mu r'' - \nu q'', & \mu_2 = \nu p'' - \lambda r'', & \nu_2 = \lambda q'' - \mu p'', \end{cases}$$

where $p', q', r', p'', q'', r''$ are scalars; and, as these differential equations must be consistent, that is, give the same values for λ_{12} and λ_{21}, say, we see that these scalars are related by the equations

$$(2) \quad p_2' - p_1'' = q'r'' - q''r', \quad q_2' - q_1'' = r'p'' - r''p', \quad r_2' - r_1'' = p'q'' - p''q'.$$

It must be noticed that, by the limitation we have imposed, we cannot have

$$q'r'' - q''r' = 0,$$

for this relation would lead to some equation of the form

$$\lambda_1 - k\lambda_2 = 0,$$

where k is a scalar: in which case, if

$$\lambda = \xi i + \eta j + \zeta k,$$

where i, j, k are fixed unit vectors at right angles, and ξ, η, and ζ are scalars, then ξ, η, ζ would each be functionally connected; and, by a change of variables, we could take them to be functions of u only, so that λ would depend on u only.

The geometrical interpretation of the scalars $p', q', r', p'', q'', r''$ is that

$$p'du + p''dv, \qquad q'du + q''dv, \qquad r'du + r''dv$$

are the angular displacements of the axes formed by the vectors λ, μ, and ν.

We easily deduce from (1) the formulæ

$$\mu\nu_1 = -\nu\mu_1 = p', \qquad \nu\lambda_1 = -\lambda\nu_1 = q', \qquad \lambda\mu_1 = -\mu\lambda_1 = r';$$

$$\mu\nu_2 = -\nu\mu_2 = p'', \qquad \nu\lambda_2 = -\lambda\nu_2 = q'', \qquad \lambda\mu_2 = -\mu\lambda_2 = r''.$$

When the vectors are given, the functions p, q, r, p', q', r' are given; we may call these the functions which define the motion, and we shall now show that when these functions are given the vectors may be found by the solution of an equation of Riccati's.

Let a be any fixed vector, and let

$$a = a\underset{\sim}{\lambda}, \qquad b = a\underset{\sim}{\mu}, \qquad c = a\underset{\sim}{\nu}.$$

We therefore have

$$a_1 = br' - cq', \qquad b_1 = cp' - ar', \qquad c_1 = aq' - bp';$$
$$a_2 = br'' - cq'', \qquad b_2 = cp'' - ar'', \qquad c_2 = aq'' - bp''.$$

Let $$\sigma = (a + \iota b)/\{\sqrt{(a^2+b^2+c^2)} - c\},$$

and we can prove that

$$2\sigma_1 = (\iota p' + q')\sigma^2 - 2\iota r'\sigma + q' - \iota p',$$
$$2\sigma_2 = (\iota p'' + q'')\sigma^2 - 2\iota r''\sigma + q'' - \iota p'',$$

where ι stands for the square root of minus unity.

These equations are consistent, and we thus see that σ depends on a total differential equation of Riccati's form.

As σ is a complex quantity when it is known, we have the ratios of a, b and c, and therefore the position of a with respect to the three vectors λ, μ, and ν.

Similarly we obtain the position of any other two fixed vectors with respect to λ, μ, and ν, and thus the position of these vectors with respect to fixed axes in space, and, as the vectors are of unit length, they are thus obtained.

3. Let R be a scalar function of the parameters. Then, if

$$z' = z + R(\mu \cos t + \nu \sin t),$$

the vector z' will trace out a circle as t varies and u and v remain fixed.

The centre of this circle is at the extremity of z, its radius is R, and its plane is perpendicular to λ. As u and v vary, we get a double infinity of circles; if we take t to be some function of u and v, the vector z' describes a surface: the condition that this surface should be orthogonal to all the circles of the system is

$$S(\nu \cos t - \mu \sin t)[z_1' du + z_2' dv + R(\nu \cos t - \mu \sin t)\, dt] = 0,*$$

that is, $$R dt = S(\nu \cos t - \mu \sin t)(z_1' du + z_2' dv).$$

Now t being a function of u and v, this equation is equivalent with the two

$$t_1 = \frac{\cos t}{R}\,\underset{\sim}{\nu z_1'} - \frac{\sin t}{R}\,\underset{\sim}{\mu z_1'}, \qquad t_2 = \frac{\cos t}{R}\,\underset{\sim}{\nu z_2'} - \frac{\sin t}{R}\,\underset{\sim}{\mu z_2'},$$

* It will be noticed that z_1' and z_2' are here partial derivatives of z' with respect to u and v as occurring explicitly and not as involved in t. If we did not take them thus the condition would be

$$S(\nu \cos t - \mu \sin t)(z_1' du + z_2' dv) = 0.$$

that is, with the two

$$t_1 = \frac{\cos t}{R}\,\underset{\sim}{\nu z}_1 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_1 + \underset{\sim}{\nu\mu}_1, \qquad t_2 = \frac{\cos t}{R}\,\underset{\sim}{\nu z}_2 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_2 + \underset{\sim}{\nu\mu}_2.$$

Since $\underset{\sim}{\nu\mu}_1 = -p'$ and $\underset{\sim}{\nu\mu}'_2 = -p''$, we therefore have the two equations

$$t_1 = \frac{\cos t}{R}\,\underset{\sim}{\nu z}_1 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_1 - p', \qquad t_2 = \frac{\cos t}{R}\,\underset{\sim}{\nu z}_2 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_2 - p'',$$

which must be consistent.

Forming the expressions for t_{12} and t_{21}, which must be equivalent, and in these expressions substituting for t_1 and t_2 the right-hand members of the above equations, we get

$$\begin{aligned} p_1'' - p_2' &+ \left(\frac{\sin t}{R}\,\underset{\sim}{\nu z}_2 + \frac{\cos t}{R}\,\underset{\sim}{\mu z}_2\right)\left(\frac{\cos t}{R}\,\underset{\sim}{\nu z}_1 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_1 - p'\right) \\ &- \left(\frac{\cos t}{R}\,\underset{\sim}{\mu z}_1 + \frac{\sin t}{R}\,\underset{\sim}{\nu z}_1\right)\left(\frac{\cos t}{R}\,\underset{\sim}{\nu z}_2 - \frac{\sin t}{R}\,\underset{\sim}{\mu z}_2 - p''\right) \\ &+ \frac{\cos t}{R}\,(\underset{\sim}{\nu_2 z}_1 - \underset{\sim}{\nu_1 z}_2) - \frac{\sin t}{R}\,(\underset{\sim}{\mu_2 z}_1 - \underset{\sim}{\mu_1 z}_2) \\ &+ \frac{R_1}{R^2}\,(\cos t\,\underset{\sim}{\nu z}_2 - \sin t\,\underset{\sim}{\mu z}_2) - \frac{R_2}{R^2}\,(\cos t\,\underset{\sim}{\nu z}_1 - \sin t\,\underset{\sim}{\mu z}_1) = 0\,; \end{aligned}$$

that is,

$$\begin{aligned} &\frac{\cos t}{R}\left(p''\underset{\sim}{\mu z}_1 - p'\underset{\sim}{\mu z}_2 + \underset{\sim}{\nu_2 z}_1 - \underset{\sim}{\nu_1 z}_2 - \underset{\sim}{\nu z}_1\frac{R_2}{R} + \underset{\sim}{\nu z}_2\frac{R_1}{R}\right) \\ &+ \frac{\sin t}{R}\left(p''\underset{\sim}{\nu z}_1 - p'\underset{\sim}{\nu z}_2 + \underset{\sim}{\mu_1 z}_2 - \underset{\sim}{\mu_2 z}_1 + \underset{\sim}{\mu z}_1\frac{R_2}{R} - \underset{\sim}{\mu z}_2\frac{R_1}{R}\right) \\ &+ \frac{1}{R^2}\,(\underset{\sim}{\nu z}_1\underset{\sim}{\mu z}_2 - \underset{\sim}{\mu z}_1\underset{\sim}{\nu z}_2) + p_1'' - p_2' = 0. \end{aligned}$$

Since

$$p''\mu + \nu_2 = \lambda q'', \quad p'\mu + \nu_1 = \lambda q', \quad p''\nu - \mu_2 = \lambda r'', \quad p'\nu - \mu_1 = \lambda r',$$

$$p_2' - p_1'' = q'r'' - q''r',$$

this equation may be written

$$\frac{\cos t}{R}\left(q''\lambda\underset{\sim}{z}_1 - q'\lambda\underset{\sim}{z}_2 + \underset{\sim}{\nu z}_2\frac{R_1}{R} - \underset{\sim}{\nu z}_1\frac{R_2}{R}\right) + \frac{\sin t}{R}\left(r''\lambda\underset{\sim}{z}_1 - r'\lambda\underset{\sim}{z}_2 + \underset{\sim}{\mu z}_1\frac{R_2}{R} - \underset{\sim}{\mu z}_2\frac{R_1}{R}\right)$$

$$= q'r'' - q''r' + \frac{1}{R^2}\,(\underset{\sim}{\mu z}_1\underset{\sim}{\nu z}_2 - \underset{\sim}{\mu z}_2\underset{\sim}{\nu z}_1).$$

Since the developables of the congruence we are investigating are given by

$$du\,dv = 0,$$

we have $z_1 = a\lambda_1 + b\lambda, \qquad z_2 = c\lambda_2 + d\lambda,$

and therefore the equation becomes

$$\frac{\cos t}{R}\left(dq' - bq'' + c\underset{\smile}{\nu\lambda_2}\frac{R_1}{R} - a\underset{\smile}{\nu\lambda_1}\frac{R_2}{R}\right) + \frac{\sin t}{R}\left(dr' - br'' + a\underset{\smile}{\mu\lambda_1}\frac{R_2}{R} - c\underset{\smile}{\mu\lambda_2}\frac{R_1}{R}\right)$$

$$= q'r'' - q''r' + \frac{1}{R^2}(ac\,\underset{\smile}{\mu\lambda_1}\underset{\smile}{\nu\lambda_2} - ca\,\underset{\smile}{\mu\lambda_2}\underset{\smile}{\nu\lambda_1}).$$

Now $\underset{\smile}{\nu\lambda_1} = q', \quad \underset{\smile}{\nu\lambda_2} = q'', \quad \underset{\smile}{\mu\lambda_1} = -r', \quad \underset{\smile}{\mu\lambda_2} = -r'',$

so that the equation is

$$\frac{\cos t}{R}\left[q'\left(d - a\frac{R_2}{R}\right) - q''\left(b - c\frac{R_1}{R}\right)\right] + \frac{\sin t}{R}\left[r'\left(d - a\frac{R_2}{R}\right) - r''\left(b - c\frac{R_1}{R}\right)\right]$$

$$= (q'r'' - q''r')\left(1 + \frac{ac}{R^2}\right).$$

We now see that, if the system of circles cut three surfaces orthogonally, they cut an infinity of surfaces orthogonally.

In order that they may cut an infinity of surfaces orthogonally, we must have, firstly,

$$(q'r'' - q''r')(ac + R^2) = 0.$$

And as, by what we have proved, we must not allow $q'r'' - q''r'$, to be zero, we have

$$ac = -R^2.$$

We then may say that

$$a = R\cot\theta, \qquad c = -R\tan\theta,$$

where θ is some angle.

Also, since the coefficients of $\cos t$ and of $\sin t$ must vanish, we have

$$q'(d - R_2\cot\theta) - q''(b + R_1\tan\theta) = 0,$$

$$r'(d - R_2\cot\theta) - r''(b + R_1\tan\theta) = 0;$$

and, as $q'r'' - r'q''$ is not zero, this gives

$$b = -R_1\tan\theta, \qquad d = R_2\cot\theta.$$

We now have as defining the cyclic congruence in our field,

(3) $\qquad z_1 = R\cot\theta\lambda_1 - R_1\tan\theta\lambda, \qquad z_2 = -R\tan\theta\lambda_2 + R_2\cot\theta\lambda.$

If we eliminate z by differentiation, we get

$$\lambda_{12}-\theta_2\cot\theta\lambda_1+\theta_1\tan\theta\lambda_2-\frac{\lambda}{R}(R_{12}+\theta_2\tan\theta R_1-\theta_1\cot\theta R_2)=0,$$

and therefore λ satisfies an equation of the form

$$\lambda_{12}-\theta_2\cot\theta\lambda_1+\theta_1\tan\theta\lambda_2+c\lambda=0, \tag{4}$$

while R satisfies the equation

$$R_{12}+\theta_2\tan\theta R_1-\theta_1\cot\theta R_2+cR=0. \tag{5}$$

Conversely, if the scalar R satisfies the equation (5) and the unit vector λ satisfies the equation (4), then a vector z can be found to satisfy the equations (3).

The distance of the foci of the congruence from the centre of the circle which corresponds to the vector z is given by

$$S(z_1+w\lambda_1)(z_2+w\lambda_2)\lambda=0,$$

and therefore by $w=-R\cot\theta$ and $w=R\tan\theta$.

We shall now find the necessary relations between the functions p', q', r', p'', q'', r'' in order that λ may satisfy an equation of the form

$$\lambda_{12}-\theta_2\cot\theta\lambda_1+\theta_1\tan\theta\lambda_2+c\lambda=0.$$

Since $$\lambda_{12}=\mu r_2'-\nu q_2'+r'(\nu p''-\lambda r'')-q'(\lambda q''-p''\mu),$$

$$\lambda_1=\mu r'-\nu q', \qquad \lambda_2=\mu r''-\nu q'',$$

we get $$r_2'+q'p''=\theta_2\cot\theta r'-\theta_1\tan\theta r'',$$

$$q_2'-r'p''=\theta_2\cot\theta q'-\theta_1\tan\theta q'',$$

$$c=r'r''+q'q''.$$

The first two conditions are, of course, equivalent with

$$r_1''+p'q''=\theta_2\cot\theta r'-\theta_1\tan\theta r'',$$

$$q_1''-p'r''=\theta_2\cot\theta q'-\theta_1\tan\theta q'',$$

since $$r_2'-r_1''=p'q''-p''q', \qquad q_2'-q_1''=r'p''-r''p'.$$

Conversely, any functions which satisfy these four equations and the equation

$$p_2'-p_1''=q'r''-q''r'$$

will generate vectors λ, μ, ν of unit length and such that

$$\lambda_{12}-\theta_2\cot\theta\lambda_1+\theta_1\tan\theta\lambda_2+c\lambda=0.$$

4. If we now take

$$(6)\qquad \begin{cases} p` = p', & q` = -\iota q' \cot\theta, & r` = -\iota r' \cot\theta\,; \\ p`` = p'', & q`` = \iota q'' \tan\theta, & r`` = \iota r'' \tan\theta, \end{cases}$$

we may verify without difficulty that

$$r`_2 + q` p`` = -r` \theta_2 \tan\theta + r`` \theta_1 \cot\theta,$$

$$q`_2 - r` p`` = -q` \theta_2 \tan\theta + q`` \theta_1 \cot\theta,$$

$$p`_2 - p``_1 = q` r`` - q`` r`, \qquad q`_2 - q``_1 = r` p`` - r`` p`, \qquad r`_2 - r``_1 = p` q`` - p`` q`\,;$$

and therefore we see that the new set of functions generate a new set of vectors λ', μ', ν' such that

$$\lambda'_{12} - \theta'_2 \cot\theta' \lambda'_1 + \theta'_1 \tan\theta' \lambda'_2 + c\lambda' = 0,$$

where the c is the same as in the first equation and $\theta' = \frac{1}{2}\pi - \theta$.

The centre of the circle perpendicular to the vector λ is the extremity of the vector z, and we may say that

$$z = \alpha\lambda + \beta\mu + \gamma\nu,$$

where α, β, γ are scalars to be determined.

We easily deduce from (1) and (3) that

$$-R_1 \tan\theta = \alpha_1 - \beta r' + \gamma q', \qquad R_2 \cot\theta = \alpha_2 - \beta r'' + \gamma q'',$$

$$R \cot\theta r' = \beta_1 - \gamma p' + \alpha r', \qquad -R \tan\theta r'' = \beta_2 - \gamma p'' + \alpha r'',$$

$$-R \cot\theta q' = \gamma_1 - \alpha q' + \beta p', \qquad R \tan\theta q'' = \gamma_2 - \alpha q'' + \beta p''.$$

As z is obtained by quadrature, so can α, β, γ, as these equations, in fact, tell us. We see that α is the length of the perpendicular from the origin on the cyclic plane, and it is given by the equation

$$\alpha + z\breve{\lambda} = 0.$$

Since $$\lambda_{12} - \theta_2 \cot\theta \lambda_1 + \theta_1 \tan\theta \lambda_2 + c\lambda = 0,$$

we see that $$c + \lambda_1 \breve{\lambda}_2 = 0.$$

And, since $$R_{12} + \theta_2 \tan\theta R_1 - \theta_1 \cot\theta R_2 + cR$$

and $$\alpha_1 + z\breve{\lambda}_1 + R_1 \tan\theta = 0, \qquad \alpha_2 + z\breve{\lambda}_2 - R_2 \cot\theta = 0,$$

we deduce that

$$(7)\qquad \alpha_{12} - \alpha_1 \theta_2 \cot\theta + \alpha_2 \theta_1 \tan\theta + c\alpha = 0.$$

If we now take $\quad \alpha' = -\iota R, \quad R' = -\iota\alpha, \quad \theta' = \frac{1}{2}\pi - \theta,$

$$\beta' = \beta, \qquad \gamma' = \gamma,$$

we easily verify that

$$-R_1' \tan\theta' = a_1' - \beta' r` + \gamma' q`, \qquad R_2' \cot\theta' = a_2' - \beta' r`` + \gamma' q``,$$

$$R' \cot\theta' r` = \beta_1' - \gamma' p` + a' r`, \qquad -R' \tan\theta' r`` = \beta_2' - \gamma' p`` + a' r``,$$

$$-R' \cot\theta' q` = \gamma_1' - a' q` + \beta' p`, \qquad R' \tan\theta' q`` = \gamma_2' - a q`` + \beta p``.$$

And therefore, if λ', μ', ν' are the unit vectors generated by the functions $p`$, $q`$, $r`$, $p``$, $q``$, $r``$, and

$$z = a'\lambda' + \beta'\mu' + \gamma'\nu',$$

we see that the extremity of the vector z' traces out the surface formed by the centres of a cyclic system of which λ' is the axis.

There is clearly a reciprocal relation between these two cyclic congruences which we may call complementary. It is important to note that the complementary congruence is derived from the original cyclic congruence by the solution of an equation of Riccati's form and by quadrature.

If l is a vector parallel to λ such that

$$l\underset{\smile}{z} = 1,$$

then l will be the vector of the surface reciprocal to the envelope of the cyclic planes with respect to a sphere whose centre is the origin and radius the square root of minus unity.

From the relation $$l a + \lambda = 0$$

we easily deduce, by aid of (4) and (7), that

$$a l_{12} + l_1(a_2 - a\theta_2 \cot\theta) + l_2(a_1 + a\theta_1 \tan\theta) = 0,$$

and therefore the parametric lines are conjugate on the surface traced out by l. They are therefore also conjugate on the envelope of the cyclic planes.

5. We shall now prove that the envelopes of the cyclic planes in two complementary cyclic systems are applicable on one another.

Let the envelope of the first be traced out by the extremity of the vector ξ, where $$\xi = a\lambda + m\mu + n\nu,$$

where m and n are scalars to be determined. We have, as before,

$$\xi_1 = (a_1 - mr' + nq')\lambda + (m_1 - p'n + r'a)\mu + (n_1 - aq' + mp')\nu,$$

$$\xi_2 = (a_2 - mr'' + nq'')\lambda + (m_2 - p''n + r''a)\mu + (n_2 - aq'' + mp'')\nu,$$

and, as $$\lambda\underset{\smile}{\xi_1} = \lambda\underset{\smile}{\xi_2} = 0,$$

we see that $$a_1 = mr' - nq', \qquad a_2 = mr'' - nq''.$$

To find the envelope of the corresponding complementary cyclic plane we may take

$$\xi' = z' + b\mu' + c\nu',$$

where b and c are scalars, since $\lambda'\underset{\smile}{\xi}' = \lambda'\underset{\smile}{z}'$. We now deduce that

$$\xi_1' = (-R_1' \cot\theta - br` + cq`)\lambda' + (b_1 - cp` + R' \tan\theta r`)\mu' + (c_1 - R' \tan\theta q` + bp`)\nu',$$

$$\xi_2' = (R_2' \tan\theta - br`` + cq``)\lambda' + (b_2 - cp`` - R' \cot\theta r``)\mu' + (c_2 + R' \cot\theta q`` + bq``)\nu',$$

and therefore, as $\lambda'\underset{\smile}{\xi}_1' = 0 = \lambda'\underset{\smile}{\xi}_2'$,

$$-R_1' \cot\theta - br` + cq` = 0, \qquad R_2' \tan\theta - br`` + cq`` = 0,$$

which equations are equivalent with

$$a_1 = -br' + cq', \qquad a_1 = -br'' + cq''.$$

It follows that $b = -m$, and $c = -n$,

and

$$-\xi_1' = (m_1 - np' + \alpha r')\mu' + (n_1 - \alpha q' + mp')\nu',$$

$$-\xi_2' = (m_2 - np'' + r''\alpha)\mu' + (n_2 - \alpha q'' + mp'')\nu'.$$

It now follows that

$$\xi_1^2 = \xi_1'^2, \quad \xi_1\underset{\smile}{\xi}_2 = \xi_1'\underset{\smile}{\xi}_2', \quad \text{and} \quad \xi_2^2 = \xi_2'^2,$$

and therefore the surfaces are applicable on one another, as we have the same expression for the element of arc on each.

It may also be shown that if we have any two surfaces which are applicable on one another and we take, as parametric lines, lines which are conjugate on both surfaces then expressed in terms of these parameters, the expressions for the elements of arc will still be the same on both surfaces. From two such surfaces we can then deduce by solutions of equations of Riccati's form two complementary cyclic congruences.

I do not here give the proof of this converse theorem, which is closely related to the theorem on cyclic congruences given by Darboux in the third volume of his treatise on surfaces, § 761, p. 354.

THE HARMONIC FUNCTIONS ASSOCIATED WITH THE PARABOLIC CYLINDER

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1. It has been shewn by Weber* that the harmonic functions associated with the parabolic cylinder satisfy a differential equation of the form

$$\frac{d^2y}{dz^2}+Zy=0,$$

where Z is a quadratic function of z; we may, without loss of generality, take

$$Z=n+\tfrac{1}{2}-\tfrac{1}{4}z^2.$$

Some series which satisfy the equation are given by Baer† and Haentzschel.‡ Various properties of a particular solution of the equation, $D_n(z)$, have been obtained by Whittaker.§

More recently, Adamoff‖ has obtained the asymptotic expansion of $D_n(z)$, when z is real and n is a large positive integer; his result is

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* Weber, *Mathematische Annalen*, Bd. I (1869), pp. 1–36.

† Baer, *Dissertation*, Cüstrin (1883).

‡ Haentzschel, *Zeitschrift für Mathematik*, Bd. XXXIII (1888), pp. 22–30.

§ Whittaker, *Proc. London Math. Soc.*, Vol. XXXV (1903), pp. 417–427.

‖ A. Adamoff, "Sur les expansions des polynômes $U_n(x)=e^{\frac{1}{2}ax^2}\frac{d^n}{dx^n}\{e^{-\frac{1}{2}ax^2}\}$ pour les grandes valeurs de n," *Annales de l'Institut Polytechnique de St. Pétersbourg*, T. V (1906), pp. 127–143 (Russian).

obtained by a somewhat elaborate transformation of a definite integral, and his work involves some extremely elegant analysis.

The object of the present paper is to amplify and generalise some of Whittaker's results. For these purposes, contour integrals of Barnes's type (involving gamma functions) will be employed to obtain the asymptotic expansion of $D_n(z)$, when $|z|$ is large, for all values of arg (z); and it will be shewn that from these results it is possible to evaluate various types of integrals involving the functions $D_n(z)$. I then propose to extend Adamoff's work by obtaining the asymptotic expansion of $D_n(z)$ when n is large and real and z is complex. Unfortunately, it seems impossible to obtain this asymptotic expansion by the use of contour integrals of Barnes's type, and it does not seem to be possible to adapt Adamoff's analysis to the case when n is complex. By means of this asymptotic expansion, it will be possible to investigate the range of validity of Whittaker's expansion of an arbitrary function in a series of the functions D_n.

It should be mentioned that the harmonic functions associated with the parabolic cylinder can be expressed in terms of a particular form of the generalised hypergeometric function usually denoted by

$$ {}_1F_1(\alpha_1;\ \rho_1;\ x), $$

and it has also been shewn that Bessel's functions can be expressed in terms of a (different) particular form of this function.

2. Following Whittaker, we shall take as the standard form of the equation

$$ \frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)\, y = 0, \tag{1} $$

n being any constant, real or complex.

Introducing a new dependent variable by the relation

$$ y = e^{-\frac{1}{4}z^2} z^n u, $$

the equation (1) is transformed into

$$ z^2 \frac{d^2u}{dz^2} + (2nz - z^3)\frac{du}{dz} + n(n-1)\, u = 0. \tag{1a} $$

Then this equation is satisfied by the integral

$$ I_1 = -\frac{1}{2\pi i}\int_{C'} \frac{\Gamma(s)\,\Gamma(-s-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-s-\frac{1}{2}n)}{2^s\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}\, z^{2s} ds $$

round a contour C' embracing the ray on which the sequence of poles of $\Gamma(-s-\frac{1}{2}n)$ lies (provided that n is not a positive integer).

This statement may readily be verified by substituting in the equation (1a). The integral I_1 exists for all finite values of arg z.

Consequently,

$$D_n(z) = -\frac{1}{2\pi i} e^{-\frac{1}{4}z^2} z^n \int_C \frac{\Gamma(s)\,\Gamma(-s-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-s-\frac{1}{2}n)}{2^s\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)} z^{2s}\,ds$$

is a solution of the equation (1).

We write this in the modified form

$$D_n(z) = -\frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_C \frac{\Gamma(\frac{1}{2}t-\frac{1}{2}n)\,\Gamma(-t)}{\Gamma(-n)} (\sqrt{2})^{t-n-2} z^t\,dt \qquad (2)$$

round a contour C embracing the positive half of the real axis, but not containing the poles of $\Gamma(\frac{1}{2}t-\frac{1}{2}n)$.

Evaluating this integral by Cauchy's theorem, we obtain the expansion

$$D_n(z) = e^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n)}{m!\,\Gamma(-n)} (\sqrt{2})^{m-n-2} (-z)^m, \qquad (2a)$$

agreeing with Whittaker's definition of $D_n(z)$, and shewing that $D_n(z)$ is a function of z which is uniform over the finite part of the z-plane.

When n is a positive integer, we may use the limiting form of the series (2a), shewing that $D_n(z)\,e^{\frac{1}{4}z^2}$ then reduces to a polynomial in z.

Whittaker points out that when n is an integer

$$D_n(z) = (-)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n} \{e^{-\frac{1}{2}z^2}\}.$$

We observe that $D_n(-z)$ is also a solution of (1), reducing, when n is an integer, to a constant multiple of $D_n(z)$; and since the equation (1) is unaltered if we write $\pm iz$ for z and $-n-1$ for n, the functions $D_{-n-1}(iz)$, $D_{-n-1}(-iz)$ are also solutions of (1).

By the theory of linear differential equations, a relation of the form

$$D_n(z) = AD_{-n-1}(iz) + BD_{-n-1}(-iz)$$

must hold, A and B being constants.

Making use of the coefficients of the first two terms in the series (2a), and the corresponding series for $D_{-n-1}(\pm iz)$, we find that this relation is

$$D_n(z) = \frac{\Gamma(n+1)}{\sqrt{(2\pi)}} [e^{\frac{1}{2}n\pi i} D_{-n-1}(iz) + e^{-\frac{1}{2}n\pi i} D_{-n-1}(-iz)]. \qquad (3)$$

3. We are now in a position to find the asymptotic expansion of $D_n(z)$ for all values of arg z, when $|z|$ is large.

Consider the integral

$$-\frac{1}{2\pi i}\int \frac{\Gamma(\frac{1}{2}t-\frac{1}{2}n)\,\Gamma(-t)}{\Gamma(-n)}(\sqrt{2})^{t-n-2}z^t\,dt$$

taken round that part of the circle at infinity for which* $R(t) > -k$, and the line $t = -k$.

The integral round the circle will vanish if $|\arg z| < \frac{3}{4}\pi$; the integral along the line is of order $|z|^{-k}$.

Consequently the sum of the residues of the integrand at the poles within the contour is of order $|z|^{-k}$, by Cauchy's theorem; but the poles of the integrand within the contour are the points 0, 1, 2, ...; n, $n-2$, ..., $n-2l$, where l is the greatest integer such that

$$R(n)-2l > -k.$$

Evaluating these residues, we get

$$D_n(z) = e^{-\frac{1}{4}z^2}\left[z^n \sum_{m=0}^{l} \frac{\Gamma(2m-n)}{m!\,\Gamma(-n)}(-2z^2)^{-m}+J_k\right], \tag{4}$$

where $R(2l-n) < k < R(2l-n+2)$ and $|J_k|$ is, when $|z|$ is large, of order $|z|^{-k}$. This result holds when

$$|\arg z| < \tfrac{3}{4}\pi.$$

We now require the asymptotic expansion of $D_n(z)$ for large values of $|z|$ when $\frac{3}{4}\pi < \arg z < \frac{5}{4}\pi$.

If $\frac{3}{4}\pi < \arg z < \frac{5}{4}\pi$, and we interpret iz and $-iz$ as meaning $z\exp(-\frac{3}{2}\pi i)$, $z\exp(-\frac{1}{2}\pi i)$ respectively, then

$$|\arg(iz)| < \tfrac{3}{4}\pi, \quad |\arg(-iz)| < \tfrac{3}{4}\pi;$$

so that when $\frac{3}{4}\pi < \arg z < \frac{5}{4}\pi$,

the asymptotic expansions of $D_{-n-1}(iz)$, $D_{-n-1}(-iz)$ are

$$D_{-n-1}(iz) = e^{\frac{1}{4}z^2}\exp\{\tfrac{3}{2}(n+1)\pi i\}\left[z^{-n-1}\sum_{m=0}^{l}\frac{\Gamma(2m+n+1)}{m!\,\Gamma(n+1)}(2z^2)^{-m}+J_k'\right],$$

$$D_{-n-1}(-iz) = e^{\frac{1}{4}z^2}\exp\{\tfrac{1}{2}(n+1)\pi i\}\left[z^{-n-1}\sum_{m=0}^{l}\frac{\Gamma(2m+n+1)}{m!\,\Gamma(n+1)}(2z^2)^{-m}+J_k''\right],$$

where $R(2l+n+1) < k < R(2l+n+3)$ and $|J_k'|$, $|J_k''|$ are of order $|z|^{-k}$, when $|z|$ is large.

Consequently, using (3), the asymptotic expansion of $D_n(z)$ when

* Throughout the paper, "R" denotes "the real part of."

$\frac{3}{4}\pi < \arg z < \frac{5}{4}\pi$ is given by the formula

$$D_n(z) = \frac{2\sin n\pi}{\sqrt{(2\pi)}}\, e^{n\pi i}\, e^{\frac{1}{4}z^2} z^{-n-1}\left[\sum_{m=0}^{l} \frac{\Gamma(2m+n+1)}{m!}\,(2z^2)^{-m} + J_k\right], \quad (4a)$$

where $2l < k < 2l+2$, and J_k is of order $|z|^{-k}$ when $|z|$ is large.

This formula shews that when n is a positive integer, $D_n(z)$ contains no terms of order $e^{\frac{1}{4}z^2}$; this is only to be expected, since, when n is a positive integer, $D_n(z)$ is equal to $e^{-\frac{1}{4}z^2}$ multiplied by a polynomial in z; *i.e.*, $D_n(z)$ is of order $e^{-\frac{1}{4}z^2}$.

To obtain the asymptotic expansion of $D_n(z)$ when arg (z) is nearly equal to $\frac{3}{4}\pi$, we put $-n-1$ for n and $-iz$ for z in (3); we then get

$$D_n(z) = e^{n\pi i} D_n(-z) + \frac{\sqrt{(2\pi)}}{\Gamma(-n)}\, e^{\frac{1}{2}(n+1)\pi i}\, D_{-n-1}(-iz),$$

which may be written in the form

$$D_n(z) = e^{n\pi i} D_n(e^{-\pi i}z) + \frac{\sqrt{(2\pi)}}{\Gamma(-n)}\, e^{\frac{1}{2}(n+1)\pi i} D_{-n-1}(e^{-\frac{1}{2}\pi i}z),$$

and $$|\arg(e^{-\pi i}z)| < \tfrac{3}{4}\pi, \qquad |\arg(e^{-\frac{1}{2}\pi i}z)| < \tfrac{3}{4}\pi.$$

Using (4), we get

$$D_n(z) = e^{-\frac{1}{4}z^2} z^n \left[\sum_{m=0}^{l} \frac{\Gamma(2m-n)}{m!\,\Gamma(-n)}\,(-2z^2)^{-m} + J_l\right]$$

$$- \frac{\sqrt{(2\pi)}}{\Gamma(-n)}\, e^{n\pi i} e^{\frac{1}{4}z^2} z^{-n-1}\left[\sum_{m=0}^{l'} \frac{\Gamma(2m+n+1)}{m!\,\Gamma(n+1)}\,(2z^2)^{-m} + J_l'\right], \quad (4b)$$

where $|z^l J_l|$, $|z^{l'} J_l'|$ tend to zero as z tends to infinity.

The corresponding formula when arg (z) is nearly equal to $-\frac{3}{4}\pi$ is obtained in a similar manner from the formula

$$D_n(z) = e^{-n\pi i} D_n(-z) + \frac{\sqrt{(2\pi)}}{\Gamma(-n)}\, e^{-\frac{1}{2}(n+1)\pi i} D_{-n-1}(iz).$$

It differs from (4*b*) only in containing $e^{-n\pi i}$ in place of $e^{n\pi i}$.

4. We shall now indicate briefly how integrals of the type

$$\int_L e^{(\frac{1}{4}-\alpha)z^2} z^m D_n(z)\, dz = I_2$$

may be evaluated; the contour of integration L starts at positive infinity, encircles the origin in a positive (counter-clockwise) direction, and returns to positive infinity; it is assumed that m is not a positive integer (in which case the integral evidently vanishes), and that

$$0 < \arg z < 2\pi$$

throughout the path ; the integral converges if

$$R(a) > 0.$$

We have

$$\int_L e^{(\frac{1}{4}-a)z^2} z^m D_n(z)\,dz$$
$$= -\frac{1}{2\pi i}\int_L\int_C e^{-az^2}(\sqrt{2})^{t-n-2}\frac{\Gamma(\frac{1}{2}t-\frac{1}{2}n)\,\Gamma(-t)}{\Gamma(-n)}\,z^{t+m}\,dt\,dz.$$

It may be shewn by an easy but rather lengthy investigation that we may invert the order of integration when

$$|\arg a| < \tfrac{1}{2}\pi, \qquad |a| > \tfrac{1}{2}.$$

Consequently, in these circumstances,

$$I_2 = -\frac{1}{2\pi i}\int_C (\sqrt{2})^{t-n-2}\frac{\Gamma(\frac{1}{2}t-\frac{1}{2}n)\,\Gamma(-t)}{\Gamma(-n)}\int_L e^{-az^2} z^{t+m}\,dz\,dt.$$

To evaluate the integral

$$I_2' = \int_L e^{-az^2} z^{2p-1}\,dz,$$

we put

$$z = ua^{-\frac{1}{2}},$$

so that

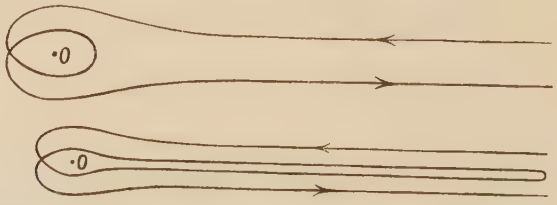
$$I_2' = a^{-p}\int e^{-u^2} u^{2p-1}\,du$$

along a contour which (by Cauchy's theorem) can be deformed into the contour L.

Putting $v = u^2$, we get

$$\int_L e^{-u^2} u^{2p-1}\,du = \tfrac{1}{2}\int e^{-v} v^{p-1}\,dv,$$

taken along a contour starting at $+\infty$, encircling the origin twice and returning to $+\infty$; and initially $\arg(v) = 0$, finally $\arg(v) = 4\pi$; we may stretch out the circuit round the origin so that it extends to $+\infty$ (*i.e.*, we may change the contour of the first figure to that of the second), so that the new contour is effectively the contour L taken twice.



We thus get

$$I_2' = \tfrac{1}{2}a^{-p}\left[\int_L e^{-v} v^{p-1}\,dv + e^{2p\pi i}\int_L e^{-v} v^{p-1}\,dv\right],$$

where the initial value of $\arg(v)$, in both integrals, is zero. But (Whittaker, *Modern Analysis*, p. 182)

$$\int_L e^{-v} v^{p-1}\, dv = (e^{2\pi p i} - 1)\,\Gamma(p)\ ;$$

so that $$\int_L e^{-az^2} z^{2p-1}\, dz = i a^{-p} e^{2p\pi i} \sin(2p\pi)\,\Gamma(p).$$

Using this result, we get

$$I_2 = -\frac{1}{2\pi i}\int_C (\sqrt{2})^{t-n-2}\,\frac{\Gamma(\frac{1}{2}t-\frac{1}{2}n)\,\Gamma(-t)}{\Gamma(-n)}\, i a^{-\frac{1}{2}(t+m+1)} e^{(t+m+1)\pi i} \times \{-\sin(t+m)\,\pi\}\,\Gamma\{\tfrac{1}{2}(t+m+1)\}\, dt$$

$$= -\frac{1}{2\pi i}\int_C (\sqrt{2})^{-2v-n-2}\,\frac{\Gamma(v-\frac{1}{2}n)\,\Gamma(-v)\,\Gamma(\frac{1}{2}-v)}{\Gamma(-n)\sqrt{\pi}}\, i a^{-v-\frac{1}{2}m-\frac{1}{2}} e^{2\pi i v+(m+1)\pi i} \times \{-\sin(2v+m)\,\pi\}\,\Gamma\{v+\tfrac{1}{2}(m+1)\}\, dv,$$

on using the duplication formula for the gamma function.

This integral may be evaluated by Cauchy's theorem; it is equal to the sum of the residues of the integrand at the points $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, when $|2a| > 1$.

But when v is an integer or half an odd integer*

$$e^{2\pi i v} \sin(2v+m)\,\pi = \sin m\pi.$$

Consequently the integral I_1 is equal to

$$-\frac{1}{2\pi i}\int_C (\sqrt{2})^{-2v-n-2}\,\frac{\Gamma(v-\frac{1}{2}n)\,\Gamma(-v)\,\Gamma(\frac{1}{2}-v)}{\Gamma(-n)\sqrt{\pi}}\, a^{-v-\frac{1}{2}m-\frac{1}{2}} \times i e^{m\pi i} \sin m\pi\,\Gamma\{v+\tfrac{1}{2}(m+1)\}\, dv \text{ (this integral being convergent)}$$

$$= -\frac{1}{2\pi i}\,\frac{(\sqrt{2})^{-n-2} a^{-\frac{1}{2}m-\frac{1}{2}} i e^{m\pi i} \sin m\pi}{\Gamma(-n)\sqrt{\pi}} \times \int_C \Gamma(v-\tfrac{1}{2}n)\,\Gamma\{v+\tfrac{1}{2}(m+1)\}\,\Gamma(-v)\,\Gamma(\tfrac{1}{2}-v)(2a)^{-v}\, dv.$$

But by a known theorem,† due to Barnes,

$$-\frac{1}{2\pi i}\int_C \Gamma(-s)\,\Gamma(\rho-a_1-a_2-s)\,\Gamma(a_1+s)\,\Gamma(a_2+s)(1-x)^s\, ds$$

$$= \frac{\Gamma(\rho-a_1)\,\Gamma(\rho-a_2)\,\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(\rho)}\, F(a_1, a_2;\ \rho;\ x),$$

---

* Dr. Barnes suggested this device for abbreviating the labour of evaluating the integral.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 154, 155.

when $$|\arg(1-x)| < \pi.$$

Hence we get, after a little algebra,

$$I_2 = \frac{\pi^{\frac{3}{2}} 2^{\frac{1}{2}n-m} e^{\pi i(m-\frac{1}{2})}}{\Gamma(-m)\,\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)\,a^{\frac{1}{2}(m+1)}} F\left\{-\frac{n}{2},\ \frac{m+1}{2};\ \frac{m-n}{2}+1;\ 1-\frac{1}{2a}\right\}.$$

But both I_2 and this function are uniform analytic functions of a when $R(a) > 0$ and $|\arg(a)| < \frac{1}{2}\pi$; consequently, over this extended range of values of a,

$$\int_L e^{(\frac{1}{4}-a)z^2} z^m D_n(z)\,dz$$
$$= \frac{\pi^{\frac{3}{2}} 2^{\frac{1}{2}n-m} e^{\pi i(m-\frac{1}{2})}}{\Gamma(-m)\,\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)\,a^{\frac{1}{2}(m+1)}} F\left\{-\frac{n}{2},\ \frac{m+1}{2};\ \frac{m-n}{2}+1;\ 1-\frac{1}{2a}\right\}. \tag{5}$$

We may also now remove the restriction that n is not a positive integer.

When $R(m) > -1$, we can deform the contour L into a contour consisting of the positive half of the real axis taken twice and an infinitesimal circle round the origin.

We thus obtain the result

$$\int_0^\infty e^{(\frac{1}{4}-a)z^2} z^m D_n(z)\,dz$$
$$= \frac{\pi^{\frac{1}{2}} 2^{\frac{1}{2}n-m-1}\,\Gamma(m+1)}{a^{\frac{1}{2}(m+1)}\,\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)} F\left\{-\frac{n}{2},\ \frac{m+1}{2};\ \frac{m-n}{2}+1;\ 1-\frac{1}{2a}\right\}. \tag{5a}$$

A particular case of these results is of interest; by a formula due to Kummer, we have

$$F\{a, \beta;\ a+\beta+\tfrac{1}{2};\ 4x(1-x)\} = F\{2a, 2\beta;\ a+\beta+\tfrac{1}{2};\ x\}.$$

Putting $x = \frac{1}{2}$, we have

$$F\{2a, 2\beta;\ a+\beta+\tfrac{1}{2};\ \tfrac{1}{2}\} = \frac{\Gamma(a+\beta+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+a)\,\Gamma(\frac{1}{2}+\beta)}.$$

Consequently, on taking $a = 1$, $n = m+1$ in the above integrals, we get

$$\int_L e^{-\frac{3}{4}z^2} z^m D_{m+1}(z)\,dz = (\sqrt{2})^{1-m}\, e^{(m+\frac{1}{2})\pi i}\,\Gamma(m+1)\sin(\tfrac{1}{4}-\tfrac{1}{4}m)\pi \,.\, \sin m\pi,$$

$$\int_0^\infty e^{-\frac{3}{4}z^2} z^m D_{m+1}(z)\,dz = (\sqrt{2})^{-1-m}\,\Gamma(m+1)\sin(\tfrac{1}{4}-\tfrac{1}{4}m)\pi.$$

We shall, subsequently, need the value of the integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}z^2} z^m D_n(z)\, dz,$$

where n is an integer, and the origin is avoided by a small semi-circle either above or below it; and the value of $\arg z$ on the positive part of the axis of real quantities is zero. The integral is a uniform analytic function of m for all values of m.

Suppose that $R(m) > -1$; then the path of integration may be deformed by taking the semicircle indefinitely small; and the integral is equal to

$$\int_0^{\infty} e^{-\frac{1}{4}z^2} z^m D_n(z)\, dz + \int_0^{\infty} e^{-\frac{1}{4}z^2} e^{\pm m\pi i} z^m D_n(-z)\, dz,$$

where the upper or lower sign is taken according as the original contour passed above or below the origin.

This is equal to

$$(1 + e^{(n \pm m)\pi i}) \int_0^{\infty} e^{-\frac{1}{4}z^2} z^m D_n(z)\, dz$$

(n being an integer), and this, when we use $(5a)$ with α put equal to $\frac{1}{2}$, becomes, on reduction,

$$e^{\pm\frac{1}{2}(m+n)\pi i} \frac{\Gamma(\frac{1}{2}n - \frac{1}{2}m)}{\Gamma(-m)} \pi^{\frac{1}{2}} 2^{\frac{1}{2}(n-m-1)}.$$

This is, for all values of m, a uniform analytic function of m. Consequently, we have, for all values of m,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}z^2} z^m D_n(z)\, dz = e^{\pm\frac{1}{2}(m+n)\pi i} \frac{\Gamma(\frac{1}{2}n - \frac{1}{2}m)}{\Gamma(-m)} \pi^{\frac{1}{2}} 2^{\frac{1}{2}(n-m-1)}, \qquad (6)$$

the plus or minus sign being taken according as the contour avoids the origin by passing above or below it.

From the results which have been obtained, we can evaluate the integral

$$I_3 = \int_L z^p D_m(z)\, D_n(z)\, dz$$

in terms of hypergeometric functions of the third order. When p is a negative integer and $p + n - m = -2$, the result can be expressed in terms of gamma functions.

We have

$$\int_L z^p D_m(z)\, D_n(z)\, dz$$
$$= -\frac{1}{2\pi i} \int_L \int_C e^{-\frac{1}{4}z^2} (\sqrt{2})^{t-m-2} \frac{\Gamma(\frac{1}{2}t - \frac{1}{2}m)\,\Gamma(-t)}{\Gamma(-m)} D_n(z)\, z^{t+p}\, dt\, dz.$$

When $R(m-n-p) > 0$, we may change the order of integration; using (5), we get

$$I_3 = -\frac{1}{2\pi i}\int_C \pi^{\frac{3}{2}}(\sqrt{2})^{n-m-p-1}e^{\pi i(t+p-\frac{1}{2})} \times \frac{\Gamma(\frac{1}{2}t-\frac{1}{2}m)\,\Gamma(-t)}{\Gamma(-m)\,\Gamma(-t-p)\,\Gamma(\frac{1}{2}t+\frac{1}{2}p-\frac{1}{2}n+1)}\,dt.$$

On evaluating this by Cauchy's theorem, we find that

$$I_3 = \sum_{t=0}^{\infty} \pi^{\frac{3}{2}}(\sqrt{2})^{n-m-p-1}e^{\pi i(p-\frac{1}{2})}\frac{\Gamma(\frac{1}{2}t-\frac{1}{2}m)}{\Gamma(-m)\Gamma(t+1)\Gamma(\frac{1}{2}t+\frac{1}{2}p-\frac{1}{2}n+1)\,\Gamma(-t-p)}.$$

Taking the even terms of this series apart from the odd terms, we get

$$I_3 = \pi^{\frac{3}{2}}(\sqrt{2})^{n-m-p-1}e^{\pi i(p-\frac{1}{2})}$$
$$\times\Big[\frac{\Gamma(-\frac{1}{2}m)}{\Gamma(-m)\Gamma(-p)\,\Gamma(\frac{1}{2}p-\frac{1}{2}n+1)}\,{}_3F_2\{-\tfrac{1}{2}m, \tfrac{1}{2}p+\tfrac{1}{2}, \tfrac{1}{2}p+1; \tfrac{1}{2}, 1+\tfrac{1}{2}p-\tfrac{1}{2}n; 1\}$$
$$+\frac{\Gamma(\frac{1}{2}-\frac{1}{2}m)}{\Gamma(-m)\,\Gamma(-p-1)\,\Gamma(\frac{1}{2}p-\frac{1}{2}n+\frac{3}{2})} \times {}_3F\{\tfrac{1}{2}-\tfrac{1}{2}m,\ \tfrac{1}{2}p+1,\ \tfrac{1}{2}p+\tfrac{3}{2};\ \tfrac{3}{2},\ \tfrac{3}{2}+\tfrac{1}{2}p-\tfrac{1}{2}n;\ 1\}\Big].$$

We thus have the result stated.

Now when $\alpha+\beta+\gamma-\delta-\epsilon = -1$ and α or β is a negative integer*

$${}_3F_2(\alpha, \beta, \gamma; \delta, \epsilon; 1) = \frac{\Gamma(\alpha-\delta+1)\,\Gamma(\beta-\delta+1)\,\Gamma(\alpha-\epsilon+1)\,\Gamma(\beta-\epsilon+1)}{\Gamma(\epsilon-\gamma)\,\Gamma(\delta-\gamma)\,\Gamma(1-\delta)\,\Gamma(1-\epsilon)};$$

applying this theorem, after observing that

$$\left.\begin{aligned}(\tfrac{1}{2}p+\tfrac{1}{2})+(\tfrac{1}{2}p+1)+(\eta-\tfrac{1}{2}m)-(\eta+\tfrac{1}{2})-(1+\tfrac{1}{2}p-\tfrac{1}{2}n) &= -1\\ (\tfrac{1}{2}p+1)+(\tfrac{1}{2}p+\tfrac{3}{2})+(\eta+\tfrac{1}{2}-\tfrac{1}{2}m)-(\eta+\tfrac{3}{2})-(\tfrac{3}{2}+\tfrac{1}{2}p-\tfrac{1}{2}n) &= -1\end{aligned}\right\}$$

(since $p+n-m = -2$), we get, when $p+1$ is a negative integer,

$${}_3F_2(\tfrac{1}{2}p+\tfrac{1}{2}, \tfrac{1}{2}p+1, -\tfrac{1}{2}m;\ \tfrac{1}{2}, 1+\tfrac{1}{2}p-\tfrac{1}{2}n;\ 1)$$
$$= \operatorname*{Lt}_{\eta=0}\,{}_3F_2(\tfrac{1}{2}p+\tfrac{1}{2}, \tfrac{1}{2}p+1, \eta-\tfrac{1}{2}m;\ \tfrac{1}{2}+\eta, 1+\tfrac{1}{2}p-\tfrac{1}{2}n;\ 1)$$
$$= \operatorname*{Lt}_{\eta=0}\frac{\Gamma(\frac{1}{2}p-\eta+1)\,\Gamma(\frac{1}{2}p-\eta+\frac{3}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})\,\Gamma(\frac{1}{2}n+1)}{\Gamma(1+\frac{1}{2}p-\frac{1}{2}n+\frac{1}{2}m-\eta)\Gamma(\frac{1}{2}+\frac{1}{2}m)\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n-\frac{1}{2}p)}$$
$$= \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})\,\Gamma(\frac{1}{2}n+1)}{2^{p+2}\,\Gamma(\frac{1}{2}+\frac{1}{2}m)\,\Gamma(\frac{1}{2}n-\frac{1}{2}p)}.$$

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* A theorem equivalent to this is given by Whittaker, *Modern Analysis*, p. 203, Ex. 54. It is due to Saalschütz.

Similarly

$$ {}_3F_2(\tfrac{1}{2}p+1,\ \tfrac{1}{2}p+\tfrac{3}{2},\ \tfrac{1}{2}-\tfrac{1}{2}m\ ;\ \tfrac{3}{2},\ 1+\tfrac{1}{2}p-\tfrac{1}{2}n\ ;\ 1) $$

$$ = \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})\,\Gamma(\frac{1}{2}n+1)\,\Gamma(-p-1)}{2^{p+2}\,\Gamma(-p)\,\Gamma(1+\frac{1}{2}m)\,\Gamma(\frac{1}{2}n-\frac{1}{2}p-\frac{1}{2})}. $$

So that we get, after some algebra,

$$ I_3 = \frac{\pi\sqrt{(2\pi)}\,e^{\pi i(p-\frac{1}{2})}}{2^{p+1}\,\Gamma(-n)\,\Gamma(-p)}. \tag{7} $$

When $p=-1$, the coefficient of the second hypergeometric function vanishes, and the hypergeometric function itself is finite; so when $p=-1$, we get, simply

$$ I_3 = \frac{\pi\sqrt{(2\pi)}\,e^{\frac{1}{2}\pi i}}{\Gamma(-n)}. \tag{7a} $$

These will be sufficient applications of the theory to the method of evaluating contour integrals.

5. We shall now investigate the asymptotic expansions of $D_n(z)$ and and $D_{-n-1}(iz)$ when n is large and real and z may be complex. The methods which will be employed are based on Adamoff's* investigation for the case when z is real and n is an integer.

[His result is that, if

$$ U_n(x) = e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}\{e^{-\frac{1}{2}x^2}\}, $$

then

$$ U_n(x) = (-)^{\mu}\sqrt{2}\,(\sqrt{n})^n e^{-\frac{1}{2}n}e^{\frac{1}{2}x^2}\left[\begin{matrix}\cos\\ \sin\end{matrix}(x\sqrt{n})+\frac{w_n(x)}{\sqrt{n}}\right], $$

where μ is the integer equal to $\frac{1}{2}n$ or $\frac{1}{2}(1+n)$, and the cosine or sine is taken as n is even or odd; while

$$ w_n(x) = \frac{1}{x\sqrt{\pi}}e^{\frac{1}{4}x^2}\vartheta_n(x), $$

and $|\vartheta_n(x)|<3{\cdot}35\ldots$, when $n>2$; when x is zero, $w_n=0$, or $\frac{1}{6}\vartheta n^{-\frac{1}{2}}$ (where $-1<\vartheta<1$), according as n is odd or even.]

For the purposes of our investigation, we take the integral

$$ e^{\frac{1}{4}z^2}\int_0^{\infty} e^{-2u^2}e^{2izu}u^n\,du, $$

where

$$ \arg u = 0. $$

* See the paper referred to on p. 393.

The integral converges if $n > -1$, and it satisfies the equation (1); on expanding in powers of z and integrating,* we find that it is equal to $2^{-n-1}\Gamma(n+1)D_{-n-1}(-iz)$.

Put $2u = v\sqrt{n}$ (it being assumed that $n > 0$ and $\sqrt{n}$ is positive), and we get

$$D_{-n-1}(-iz) = \frac{(\sqrt{n})^{n+1}}{\Gamma(n+1)} e^{\frac{1}{4}z^2} \int_0^\infty e^{-\frac{1}{2}nv^2} \exp(izv\sqrt{n})\, v^n\, dv$$

$$= \frac{(\sqrt{n})^{n+1} e^{-\frac{1}{2}n}}{\Gamma(n+1)} e^{\frac{1}{4}z^2} \left[\int_{-\infty}^{\infty} e^{-n(v-1)^2} \exp(izv\sqrt{n})\, dv \right.$$

$$+ \int_0^\infty \left\{ e^{\frac{1}{2}n(1-v^2)} v^n - e^{-n(v-1)^2} \right\} \exp(izv\sqrt{n})\, dv$$

$$\left. - \int_{-\infty}^0 e^{-n(v-1)^2} \exp(izv\sqrt{n})\, dv \right].$$

In the first of these three integrals, put

$$v = 1 + wn^{-\frac{1}{2}};$$

it becomes

$$\frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-w^2 + izw} \exp(iz\sqrt{n})\, dw,$$

and this, by a known formula, is equal to $(\pi/n)^{\frac{1}{2}} e^{-\frac{1}{4}z^2} \exp(iz\sqrt{n})$.

Consequently,

$$D_{-n-1}(-iz) = \frac{\sqrt{\pi}\,(\sqrt{n})^n\, e^{-\frac{1}{2}n}}{\Gamma(n+1)} \left[\exp(iz\sqrt{n}) + \left(\frac{n}{\pi}\right)^{\frac{1}{2}} e^{\frac{1}{4}z^2} \{J_1 - J_2\} \right], \qquad (8)$$

where

$$J_1 = \int_0^\infty \left\{ e^{\frac{1}{2}n(1-v^2)} v^n - e^{-n(v-1)^2} \right\} \exp(izv\sqrt{n})\, dv,$$

$$J_2 = \int_0^\infty e^{-n(v+1)^2} \exp(-izv\sqrt{n})\, dv.$$

We shall now find upper limits for $|J_1|$ and $|J_2|$; put $z = x + iy$, where x and y are real, and suppose that†

$$|y| < \tfrac{1}{2}\sqrt{n}.$$

Consider the integrals

$$J_3, J_4 = \int_0^\infty e^{-n(v+1)^2 + yv\sqrt{n}} \begin{matrix}\cos \\ \sin\end{matrix} (xv\sqrt{n})\, dv.$$

* This procedure is legitimate.

† In equation (9) we shall place a further restriction on $|y|$.

The function $\exp\{-n(v+1)^2+yv\sqrt{n}\}$ is monotone and positive throughout the range of integration.

Now, by the second mean-value theorem, if $\phi(x)$ is positive and decreasing throughout the interval a to b,

$$\int_a^b \phi(x) f(x)\,dx = \phi(a)\int_a^\xi f(x)\,dx,$$

where $$a \leqslant \xi < b.$$

Consequently

$$|J_3, J_4| = e^{-n}\left|\int_0^\xi \begin{matrix}\cos\\ \sin\end{matrix}(xv\sqrt{n})\,dv\right| < \frac{2e^{-n}}{|x|\sqrt{n}},$$

so that $$|J_2| < \frac{2\sqrt{2}\,e^{-n}}{|x|\sqrt{n}},$$

and, for all values of $|x|$ (no matter how small)

$$\begin{aligned}|J_3, J_4| &< \int_0^\infty \exp\left[-n(v+1)^2+yv\sqrt{n}\right]dv\\ &< e^{-n}\int_0^\infty \exp\left[-nv^2-v(2n-y\sqrt{n})\right]dv\\ &< e^{-n}\int_0^\infty \exp(-nv^2)\,dv,\\ &< \tfrac{1}{2}e^{-n}\left(\frac{\pi}{n}\right)^{\frac{1}{2}},\end{aligned}$$

so that $$|J_2| < \left(\frac{\pi}{2n}\right)^{\frac{1}{2}} e^{-n}.$$

We now have to find an upper limit to the value of J_1; we take the two integrals

$$J_5, J_6 = \int_0^\infty \left\{e^{\frac{1}{2}n(1-v^2)}v^n - e^{-n(v-1)^2}\right\} e^{-yv\sqrt{n}} \begin{matrix}\cos\\ \sin\end{matrix}(xv\sqrt{n})\,dv.$$

The function $e^{\frac{1}{2}n(1-v^2)}v^n - e^{-n(v-1)^2}$ is termed $\sigma(v)$ by Adamoff.* He shews that it possesses the following properties, when $n > 2$ (his proofs do not depend on n being an integer): $|\sigma(v)|$ has a maximum when $v = 1-h_1$, and when $v = 1+h_2$; and $\sigma'(v)$ vanishes, but does not change

* See p. 393.

sign when $v = 1$; also

$$\frac{1}{2}\sqrt{\frac{3}{2n}} < h_1 < \sqrt{\frac{3}{2n}}, \qquad \frac{1}{2}\sqrt{\frac{3}{2n}} < h_2 < \sqrt{\frac{3}{2n}};$$

when $0 < v < 1$, $\sigma(v)$ is negative; when $v > 1$, $\sigma(v)$ is positive; and $|\sigma(1-h_1)|$, $\sigma(1+h_2)$ are each less than $An^{-\frac{1}{2}}$, where $A = 0{\cdot}742\ldots$.

We shall shew that it is possible to impose such a limitation on y that a quantity k may be determined, such that

Case I: $\sigma(v)e^{-yv\sqrt{n}}$ may decrease, throughout the interval $1+k$ to $+\infty$, as v increases, y being negative.

Case II: $|\sigma(v)|e^{-yv\sqrt{n}}$ may increase, throughout the interval 0 to $1-k$, as v increases, y being positive.

The limitation which we shall impose is

$$|y| < \tfrac{1}{3}n^{a}, \tag{9}$$

and we put

$$k = 2n^{a-\frac{1}{2}}, \tag{9a}$$

a being any quantity such that $0 \leqslant a < \frac{1}{2}$; and we shall suppose that

$$n > \exp\left\{\frac{2}{1-2a}\log 2\right\},$$

so that $n > 4$ and $k < 1$.

We have

$$\frac{1}{n}\frac{d}{dv}\left\{\sigma(v)\,e^{-yv\sqrt{n}}\right\} = (1-v)\,e^{\frac{1}{2}n(1-v^2)-yv\sqrt{n}} \times\left[v^{n-1}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\} - 2\left\{1-\frac{y}{2(1-v)\sqrt{n}}\right\}e^{-n(\frac{1}{3}-2v+\frac{1}{2}v^2)}\right].$$

Consequently the roots of $\dfrac{d}{dv}\left\{\sigma(v)\,e^{-yv\sqrt{n}}\right\}$ are those of

$$\phi(v) = \log\left[v^{n-1}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\} \Big/ 2\left\{1-\frac{y}{2(1-v)\sqrt{n}}\right\}e^{-n(\frac{1}{3}-2v+\frac{1}{2}v^2)}\right] = 0; \tag{10}$$

($v = 1$ is a root of this equation), and there is not more than one root of $\phi(v)$ in the interval between each consecutive pair of zeros or infinities of $\phi'(v) = 0$; now $\phi'(v) = \psi(v)/v$, where

$$\psi(v) = n(1-v)^2 - \frac{(1-v)\left\{2(1-v)-(1+v)\,yn^{-\frac{1}{2}}\right\}}{\left\{1-v^2-vyn^{-\frac{1}{2}}\right\}\left\{2(1-v)-yn^{-\frac{1}{2}}\right\}}. \tag{10a}$$

The equation $\psi(v) = 0$ is an equation of the fifth degree. We may also write

$$\psi(v) = n(1-v)^2 - \frac{1}{1+v} - \frac{vy}{\sqrt{n}} \frac{\{(1-v)^2 - yn^{-\frac{1}{2}}\}}{\{(1-v^2) - yvn^{-\frac{1}{2}}\}\{2(1-v) - yn^{-\frac{1}{2}}\}(1+v)}. \tag{10b}$$

Put
$$\frac{1}{1+v}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\} = f_1(v),$$

$$1-\frac{y}{2(1-v)\sqrt{n}} = f_2(v).$$

Then $$\frac{d}{dv}\left\{\frac{f_1(v)}{f_2(v)}\right\} = -\tfrac{1}{2}\frac{y}{\sqrt{n}}\left\{(1-v)^2 - \frac{y}{\sqrt{n}}\right\}(1-v^2)^{-2}\{f_2(v)\}^{-2}.$$

We now have to treat Cases I and II separately.

Case I :—Let $yn^{-\frac{1}{2}}$ be negative, so that $f_1(v)/f_2(v)$ continually increases with v; $f_2(v)$ vanishes when $v = 1 - \frac{1}{2}yn^{-\frac{1}{2}} = v_2$ (say), and $f_1(v)$ vanishes when $v = v_1$ or v_3, where $v_1 > v_2$, and $-1 < v_3 < 0$.

We now draw up the following scheme of values of $\psi(v)$:—

$v = -\infty$	v_3	0	1	v_2	v_1	$+\infty$
$\psi(v) = +\infty$	$+\infty, -\infty$	$+$	$-0, +0$	$+\infty, -\infty$	$-\infty, +\infty$	$+\infty$

Consequently $\psi(v)$ has a zero in each of the intervals v_3 to 0, 0 to 1; it has another zero at $v = 1$; and it may have two more zeros both being in one of the intervals $-\infty$ to 0, 0 to 1, 1 to v_2, v_2 to v_1, v_1 to $+\infty$.

We can now discuss the values of

$$v^{n-1}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\} - 2\left\{1-\frac{y}{2(1-v)\sqrt{n}}\right\} e^{-n(\frac{2}{3}-2v+\frac{1}{2}v^2)} = \phi_1(v).$$

As v increases through the value 1, $\phi_1(v)$ changes from $+$ to $-$; $\phi_1(0)$ is negative; there is consequently one zero of $\phi_1(v)$ in the interval 0 to 1 corresponding to the zero of $\psi(v)$ in the interval. When $v_2 < v < v_1$, $\phi_1(v)$ is obviously negative; since $\phi_1(+\infty) = +\infty$, there is one zero of $\phi_1(v)$ in the interval v_1 to $+\infty$; and, corresponding to the two possible zeros of $\psi(v)$ in the intervals 0 to 1, 1 to v_2, v_1 to $+\infty$, there may be two additional zeros (but no more), both in one of these three intervals.

Now, let that root of $n(v-1)^2(v+1) - 1 = 0$, which is greater than 1, be v_4. Then, if $v > v_1$ and $v > v_4$, $\psi(v)$ is positive, by (10*b*).

Accordingly, if we can shew that $1+k > v_1$, $1+k > v_4$, $\phi_1(1+k) > 0$, then $\phi_1(v) > 0$ when $v > 1+k$.

And consequently $\frac{d}{dv}\{\sigma(v)e^{-yv\sqrt{n}}\}$ will be negative when $v > 1+k$.

It will then be possible to find quantities $k_1, \ldots, k_6$ in accordance with the following table :—

	$\sigma(v)e^{-yv\sqrt{n}}$	$\lvert\sigma(v)e^{-yv\sqrt{n}}\rvert$
$0 < v < 1-k_1$	negative	increasing
$1-k_1 < v < 1-k_2$	,,	decreasing
$1-k_2 < v < 1-k_3$	,,	increasing
$1-k_3 < v < 1$	,,	decreasing
$1 < v < 1+k_4$	positive	increasing
$1+k_4 < v < 1+k_5$	,,	decreasing
$1+k_5 < v < 1+k_6$	,,	increasing
$1+k_6 < v < +\infty$	,,	decreasing

and $$k_1 = k_2 = k_3,$$

or else $$k_4 = k_5 = k_6,$$

or else $$k_1 = k_2 = k_3 \quad \text{and} \quad k_4 = k_5 = k_6.$$

Also $$k_1 \leqslant h_1, \quad h_2 \leqslant k_4, \quad k_6 \leqslant k.$$

We now prove what has just been assumed, namely that, with the values of y and k given by formulæ (9) and (9a), $f_1(1+k)$ and $\phi_1(1+k)$ are positive, and $1+k > v_4$.

For $f_1(1+k) > 0$, if

$$(1+k)^2 - 1 > -(1+k)yn^{-\frac{1}{2}};$$

i.e., if $$-yn^{-\frac{1}{2}} < k+1-\frac{1}{1+k},$$

and this is the case since

$$-y < \tfrac{1}{6}kn^{\frac{1}{2}}, \quad 0 < k < 1.$$

Also $(n-1)\log(1+k)+\log(2+k)+\frac{1}{2}nk(2-k)-\log 2$

$$> (n-1)(k-\tfrac{1}{2}k^2+\tfrac{1}{3}k^3-\tfrac{1}{4}k^4)+\tfrac{1}{2}k-\tfrac{1}{8}k^2-n(k-\tfrac{1}{2}k^2)$$

$$> k[\tfrac{1}{12}(n-1)k^2-\tfrac{1}{8}][4-3k]$$

$$> k[\tfrac{1}{16}nk^2-\tfrac{1}{8}] \quad \left(\text{since } \frac{n-1}{n} > \tfrac{3}{4} \text{ and } 4-3k>1\right)$$

$$> k[\tfrac{1}{4}n^{2a}-\tfrac{1}{8}]$$

$$> \tfrac{1}{8}k.$$

Consequently, when $v = 1+k$,

$$\tfrac{1}{2}v^{n-1}(1+v)\,e^{n(\frac{3}{2}-2v+\frac{1}{2}v^2)}\left\{1+\frac{yv}{(v^2-1)\sqrt{n}}\right\}\Big/\left\{1+\frac{y}{2(v-1)\sqrt{n}}\right\}$$

$$> e^{\frac{1}{8}k}\left\{1+\frac{y(1+k)}{(2k+k^2)\sqrt{n}}\right\}\Big/\left\{1+\frac{y}{2k\sqrt{n}}\right\}$$

$$> (1+\tfrac{1}{8}k)\left\{1+\frac{y(1+k)}{(2k+k^2)\sqrt{n}}\right\}\Big/\left\{1+\frac{y}{2k\sqrt{n}}\right\}.$$

This is greater than unity, if

$$(1+\tfrac{1}{8}k)\left\{1+\frac{y(1+k)}{(2k+k^2)\sqrt{n}}\right\} > 1+\frac{y}{2k\sqrt{n}},$$

i.e., if

$$\tfrac{1}{8}k > -\frac{y}{\sqrt{n}}\left\{\frac{5+k}{8(2+k)}\right\},$$

which is true since

$$-y < \tfrac{1}{6}kn^{\frac{1}{2}}.$$

That is to say, we have proved that $\phi_1(1+k)$ is positive. Also $1+k > v_4$, if $n(v-1)^2(v+1)-1>0$ when $v = 1+k$, which is the case.

The maximum value of $|\sigma(v)e^{-vy\sqrt{n}}|$ in the range 1 to ∞ is consequently its maximum value in the range 1 to $1+k$, which is less than $\sigma(1+h_2)e^{-y(1+k)\sqrt{n}}$; the maximum value of the same function in the range 0 to 1 is less than $|\sigma(1-h_1)|\,e^{-y\sqrt{n}}$.

We now employ the mean-value theorem, that, if in the interval a to b, $f(x)$ has one sign, and $|f(x)|$ $\begin{matrix}\text{decreases}\\ \text{increases}\end{matrix}$ as x increases, then

$$\int_a^b f(x)\,\phi(x)\,dx = \begin{cases} f(a)\displaystyle\int_a^{\xi}\phi(x)\,dx & (\text{where } a \leqslant \xi < b),\\ f(b)\displaystyle\int_{\xi'}^{b}\phi(x)\,dx & (\text{where } a < \xi' \leqslant b).\end{cases}$$

Noting that $\left|\int_\alpha^\beta \begin{matrix}\cos\\ \sin\end{matrix} (vx\sqrt{n})\,dv\right| < \frac{2}{|x|\sqrt{n}}$, where α, β are any real quantities, we apply these theorems to $\sigma(v)\,e^{-yv\sqrt{n}}\,[=\tau(v)]$ for each of the intervals 0 to $1-k_1$, $1-k_1$ to $1-k_2$, $1-k_2$ to $1-k_3$, $1-k_3$ to 1, 1 to $1+k_4$, $1+k_4$ to $1+k_5$, $1+k_5$ to $1+k_6$, $1+k_6$ to $+\infty$, and we get

$$\left|\int_0^\infty \tau(v) \begin{matrix}\cos\\ \sin\end{matrix} (vx\sqrt{n})\,dv\right| < \left|\frac{2}{x\sqrt{n}}\right|\Big[2\,|\tau(1-k_1)|+2\,|\tau(1-k_3)| +2\tau(1+k_4)+2\tau(1+k_6)\Big]$$

$$< \left|\frac{8}{x\sqrt{n}}\right|\big[|\sigma(1-h_1)|\,e^{-y\sqrt{n}}+|\sigma(1+h_2)|\,e^{-y(1+k)\sqrt{n}}\big].$$

Consequently

$$\int_0^\infty \tau(v) \begin{matrix}\cos\\ \sin\end{matrix} (vx\sqrt{n})\,dv = \frac{8A\Im e^{-y\sqrt{n}}[1+e^{-yk\sqrt{n}}]}{nx},$$

where $$-1 < \Im < 1.$$

So that $$\int_0^\infty \sigma(v)\exp(ivz\sqrt{n})\,dv = \frac{8A\Im' \sqrt{2}\,e^{iz\sqrt{n}}[1+e^{-yk\sqrt{n}}]}{nx}, \qquad (11)$$

where $\Im'$ is a complex quantity such that $|\Im'| \leqslant 1$.

We now require an upper limit for

$$\left|\int_0^\infty \sigma(v)\,e^{-yv\sqrt{n}} \begin{matrix}\cos\\ \sin\end{matrix} (xv\sqrt{n})\,dv\right|,$$

when $|x|$ is small. We have

$$\left|\int_0^\infty \sigma(v)e^{-yv\sqrt{n}} \begin{matrix}\cos\\ \sin\end{matrix} (xv\sqrt{n})dv\right| < \int_0^1 |\sigma(v)|\,e^{-yv\sqrt{n}}\,dv+\int_1^{1+k}\sigma(v)e^{-yv\sqrt{n}}dv +\int_{1+k}^\infty \sigma(v)e^{-yv\sqrt{n}}dv$$

$$< J_7+J_8+J_9, \text{ say.}$$

Now $$J_7 < \int_0^{1-k} e^{-n(1-v)^2}e^{-yv\sqrt{n}}\,dv+\int_{1-k}^1 |\sigma(v)|\,e^{-yv\sqrt{n}}\,dv$$

$$< e^{-y\sqrt{n}}\left[\int_0^{1-k} e^{-n(1-v)^2}\,dv+\int_{1-k}^1 |\sigma(v)|\,dv\right],$$

and $|\sigma(v)| < An^{-\frac{1}{2}}$ through the range of the second integral. Therefore

$$J_7 < e^{-y\sqrt{n}}\left[\int_0^{1-k} e^{-n(k+w)^2}\,dw+Akn^{-\frac{1}{2}}\right]$$

$$< e^{-y\sqrt{n}}\left[e^{-nk^2}\int_0^\infty e^{-nw^2}\,dw+Akn^{-\frac{1}{2}}\right]$$

$$< e^{-y\sqrt{n}}\left[\tfrac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}e^{-nk^2}+\frac{Ak}{\sqrt{n}}\right].$$

Also $J_8 < k \times$ the maximum value of the integrand in the range

$$< kAn^{-\frac{1}{2}} e^{-y(1+k)\sqrt{n}}.$$

And
$$J_9 < \int_{1+k}^{\infty} v^n e^{\frac{1}{2}n(1-v^2)} e^{-yv\sqrt{n}}\, dv.$$

Throughout the range of integration v^n increases less rapidly than

$$\exp\{nv/(1+k)\}.$$

Therefore

$$J_9 < (1+k)^n e^{-n} \int_{1+k}^{\infty} \exp\{nv(1+k)^{-1} + \tfrac{1}{2}n(1-v^2) - yv\sqrt{n}\}\, dv.$$

Put $v = 1+k+w$, and we get

$$J_9 < (1+k)^n \exp\{\tfrac{1}{2}n - \tfrac{1}{2}n(1+k)^2 - y\sqrt{n}(1+k)\} \int_0^{\infty} \exp\{-\tfrac{1}{2}nw^2 - \lambda w\}\, dw,$$

where
$$\lambda = y\sqrt{n} - n\{1+k-(1+k)^{-1}\}$$
$$> y\sqrt{n} - nk > 0.$$

Hence $J_9 < (1+k)^n \exp \tfrac{1}{2}n\{1-(1+k)^2\}\, e^{-y(1+k)\sqrt{n}} \int_0^{\infty} \exp(-\tfrac{1}{2}nw^2)\, dw$

$$< \left(\frac{\pi}{2n}\right)^{\frac{1}{2}} \mu\, e^{-y(1+k)\sqrt{n}},$$

where
$$\log \mu = n \log(1+k) - \tfrac{1}{2}n(2k+k^2) < -\tfrac{1}{2}nk^2.$$

Collecting the values of J_7, J_8, J_9, we see that

$$\left| \int_0^{\infty} \sigma(v)\, e^{-yv\sqrt{n}} \begin{matrix}\cos\\ \sin\end{matrix} (vx\sqrt{n})\, dv \right|$$

is less than
$$\frac{e^{-y\sqrt{n}}}{\sqrt{n}} [kA + \tfrac{1}{2}\sqrt{(2\pi)}\, e^{-\frac{1}{2}nk^2}][1+e^{-yk\sqrt{n}}].$$

Consequently

$$\left| \int_0^{\infty} \sigma(v) \exp(izv\sqrt{n})\, dv \right| = \frac{\Im' \sqrt{2}\, e^{iz\sqrt{n}}}{\sqrt{n}} \{kA + \tfrac{1}{2}e^{-\frac{1}{2}nk^2}\sqrt{(2\pi)}\}\{1+e^{-yk\sqrt{n}}\}, \tag{11a}$$

where $\Im'$ is a complex quantity such that $|\Im'| < 1$. This result has to be used in conjunction with (11). The investigation when y is negative is complete.

Case II.—We have now to discuss the behaviour of the function $\sigma(v)e^{-yv\sqrt{n}}$ when y is positive.

The function $f_2(v)$ vanishes at $v = v_2$ as before, but now $0 < v_2 < 1$; $f_1(v)$ vanishes where $v = v_1$, v_3 where $v_2 < v_1 < 1$, and $v_3 < -1$.

We now draw up the following scheme of values of $\psi(v)$:

$v = -\infty$	v_3	0	v_2	v_1	1	$+\infty$
$\psi(v) = +\infty$	$+\infty, -\infty$	$+$	$+\infty, -\infty$	$-\infty, +\infty$	$+0, -0$	$+\infty$

Consequently $\psi(v)$ has one zero in the interval v_3 to 0, another at $v = 1$, and another in the interval 1 to $+\infty$; it may have two more, both being in any one of the intervals $-\infty$ to v_3, v_3 to 0, 0 to v_2, v_2 to v_1, v_1 to 1, 1 to $+\infty$.

We can now discuss the values of

$$v^{n-1}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\}-2\left\{1-\frac{y}{2(1-v)\sqrt{n}}\right\}e^{-n(\frac{2}{3}-2v+\frac{1}{2}v^2)} = \phi_1(v).$$

When v increases through the value 1, $\phi_1(v)$ changes from $+$ to $-$, $\phi_1(+\infty) = +\infty$; consequently $\phi_1(v)$ has one zero greater than 1, corresponding to the zero of $\psi(v)$ which is greater than 1; also $\phi_1(v)$ is positive throughout the interval v_2 to v_1 and $\phi_1(0)$ is negative; there is therefore one zero of $\phi_1(v)$ in the interval 0 to v_2; corresponding to the two possible zeros of $\psi(v)$ in the intervals 0 to v_2, v_1 to 1, 1 to $+\infty$, there may be two (but not more) zeros of $\phi_1(v)$ in one of these intervals.

We can consequently draw up the following table:—

	$\sigma(v)\,e^{-yv\sqrt{n}}$	$\lvert\sigma(v)\,e^{-yv\sqrt{n}}\rvert$
$0 < v < 1-k_1$	negative	increasing
$1-k_1 < v < 1-k_2$	,,	decreasing
$1-k_2 < v < 1-k_3$	,,	increasing
$1-k_3 < v < 1$	,,	decreasing
$1 < v < 1+k_4$	positive	increasing
$1+k_4 < v < 1+k_5$	,,	decreasing
$1+k_5 < v < 1+k_6$	,,	increasing
$1+k_6 < v < +\infty$	,,	decreasing

and $$k_1 = k_2 = k_3,$$

or else $$k_4 = k_5 = k_6,$$

or else $$k_1 = k_2 = k_3 \quad \text{and} \quad k_4 = k_5 = k_6.$$

Also $$k_1 \leqslant k, \quad h_1 \leqslant k_3, \quad k_6 \leqslant h_2.$$

We now shew that with the values of y and k given by formulæ (9) and (9a), $\phi_1(v)$ actually is negative throughout the interval 0 to $1-k$; this will certainly be the case if $\psi(v)$ is finite and positive throughout the interval and $\phi_1(1-k)$ is negative.

Now $2(1-v)-(1+v)yn^{-\frac{1}{2}}$, $1-v^2-vyn^{-\frac{1}{2}}$, $2(1-v)-yn^{-\frac{1}{2}}$ are positive throughout the interval 0 to $1-k$, and also $yn^{-\frac{1}{2}} < \frac{1}{6}k < \frac{1}{6}(1-v)$.

Consequently

$$\frac{(1-v)\{2(1-v)-(1+v)yn^{-\frac{1}{2}}\}}{\{1-v^2-vyn^{-\frac{1}{2}}\}\{2(1-v)-yn^{-\frac{1}{2}}\}} < \frac{(1-v)2(1-v)}{\{1-v^2-\frac{1}{6}v(1-v)\}\{2(1-v)-\frac{1}{6}(1-v)\}}$$

$$< \frac{12}{11(1+\frac{5}{6}v)} < \frac{12}{11}$$

$$< nk^2 < n(1-v)^2.$$

And therefore $\psi(v)$ is positive throughout the interval, by (10a). Also, when $v = 1-k$,

$$\log 2 - n(\tfrac{3}{2}-2v+\tfrac{1}{2}v^2) - (n-1)\log v - \log(v+1)$$

$$> -n(k+\tfrac{1}{2}k^2)+(n-1)\{k+\tfrac{1}{2}k^2+\tfrac{1}{3}k^3+\tfrac{1}{4}k^4\}+\tfrac{1}{2}k+\tfrac{1}{8}k^2$$

$$> k\{\tfrac{1}{12}(n-1)k^2-\tfrac{1}{8}\}(4+3k)$$

$$> 4k\{\tfrac{1}{12}n.\tfrac{3}{4}k^2-\tfrac{1}{8}\} \quad (\text{since } n > 4)$$

$$> \tfrac{1}{2}k.$$

Therefore, when $v = 1-k$,

$$2\left\{1-\frac{y}{2(1-v)\sqrt{n}}\right\} e^{-n(\frac{3}{2}-2v+\frac{1}{2}v^2)} \Big/ v^{n-1}\left\{1+v-\frac{yv}{(1-v)\sqrt{n}}\right\}$$

$$> e^{\frac{1}{2}k}\left\{1-\frac{y}{2k\sqrt{n}}\right\} \Big/ \left\{1-\frac{y(1-k)}{k(2-k)\sqrt{n}}\right\}$$

$$> (1+\tfrac{1}{2}k)\left\{1-\frac{y}{2k\sqrt{n}}\right\} \Big/ \left\{1-\frac{y(1-k)}{k(2-k)\sqrt{n}}\right\}.$$

This is greater than unity, if

$$(1-\tfrac{1}{2}k)\left(1-\frac{y}{2k\sqrt{n}}\right) > 1-\frac{y(1-k)}{k(2-k)\sqrt{n}},$$

i.e., if

$$\tfrac{1}{2}k > \frac{y}{\sqrt{n}}\,\frac{4-k}{2-k},$$

which is the case, since $0<k<1$ and $y<\frac{1}{6}k\sqrt{n}$.

We have thus proved that $\phi_1(v)$ is negative when

$$0<v<1-k.$$

The maximum value of $|\sigma(v)\,e^{-yv\sqrt{n}}|$ in the range 0 to 1 is its maximum value in the range $1-k$ to 1, and is therefore less than

$$e^{-y(1-k)\sqrt{n}}\,|\sigma(1-h_1)|;$$

the maximum value of the same function in the range 1 to $+\infty$ is less than $\sigma(1+h_2)e^{-y\sqrt{n}}$.

We now apply the mean-value theorem as in Case I, and we get

$$\left|\int_0^\infty \tau(v)\begin{matrix}\cos\\ \sin\end{matrix}(vx\sqrt{n})\,dv\right|$$

$$<\left|\frac{2}{x\sqrt{n}}\right|[2\,|\tau(1-k_1)|+2\,|\tau(1-k_3)|+2\tau(1+k_4)+2\tau(1+k_6)]$$

$$<\left|\frac{8}{x\sqrt{n}}\right|e^{-y\sqrt{n}}[\,|\sigma(1-h_1)|\,e^{yk\sqrt{n}}+\sigma(1+h_2)].$$

We deduce that

$$\int_0^\infty \sigma(v)\exp(ivz\sqrt{n})\,dv = \frac{8A\vartheta'\sqrt{2}\,e^{iz\sqrt{n}}[1+e^{yk\sqrt{n}}]}{nx}, \qquad (12)$$

where ϑ' is a complex quantity such that $|\vartheta'|\leqslant 1$.

We now want an upper limit for

$$\left|\int_0^\infty \sigma(v)\,e^{-yv\sqrt{n}}\begin{matrix}\cos\\ \sin\end{matrix}(xv\sqrt{n})\,dv\right|$$

when $|x|$ is small. We have

$$\left|\int_0^\infty \sigma(v)e^{-yv\sqrt{n}}\begin{matrix}\cos\\ \sin\end{matrix}(xv\sqrt{n})\,dv\right|$$

$$<\int_0^{1-k}|\sigma(v)|\,e^{-yv\sqrt{n}}\,dv+\int_{1-k}^1|\sigma(v)|\,e^{-yv\sqrt{n}}\,dv+\int_1^\infty|\sigma(v)|\,e^{-yv\sqrt{n}}\,dv$$

$$<J_{10}+J_{11}+J_{12},\ \text{say}.$$

Now $\int_0^{1-k} |\sigma(v)|\, e^{-yv\sqrt{n}}\, dv < \int_0^{1-k} \exp\{-n(1-v)^2 - yv\sqrt{n}\}\, dv$

$$< \int_0^{1-k} \exp\{-n(w+k)^2 - y\sqrt{n}(1-w-k)\}\, dw.$$

But $2nk > y\sqrt{n}$, so this is less than

$$\exp\{-nk^2 - y\sqrt{n}(1-k)\} \int_0^\infty \exp(-nw^2)\, dv,$$

so that $J_{10} < \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}} \exp(-nk^2) \exp\{-y\sqrt{n}(1-k)\}$.

Also $J_{11} < k \times$ the maximum value of $|\sigma(v)|\, e^{-yv\sqrt{n}}$ in the range of integration

$$< \frac{kA}{\sqrt{n}}\, e^{-y(1-k)\sqrt{n}}.$$

And $J_{12} < e^{-y\sqrt{n}} \int_1^\infty \sigma(v)\, dv$

$$< e^{-y\sqrt{n}} \left[\int_1^{1+k} \sigma(v)\, dv + \int_{1+k}^\infty \sigma(v)\, dv\right]$$

$$< e^{-y\sqrt{n}} \left[k\, \frac{A}{\sqrt{n}} + \int_{1+k}^\infty v^n e^{\frac{1}{2}n(1-v^2)} dv\right].$$

Now v^n increases less rapidly than $\exp\left(\frac{n}{1+k}\, v\right)$ throughout the range of integration. Therefore

$$\int_{1+k}^\infty v^n e^{\frac{1}{2}n(1-v^2)} dv < \int_{1+k}^\infty \exp\left\{\tfrac{1}{2}n(1-v^2) + \frac{nv}{1+k}\right\} dv\, e^{-n}(1+k)^n$$

$$< e^{-n}(1+k)^n \int_0^\infty \exp\left\{\tfrac{1}{2}n - \tfrac{1}{2}n(1+k+w)^2 + n + \frac{nw}{1+k}\right\} dw$$

$$< (1+k)^n\, e^{-\frac{1}{2}nk(2+k)} \int_0^\infty \exp\{-\tfrac{1}{2}nw^2 + \lambda w\}\, dw$$

(where λ is negative)

$$< \left(\frac{\pi}{2n}\right)^{\frac{1}{2}} \mu,$$

where $\log \mu = n \log(1+k) - nk - \frac{1}{2}nk^2 < -\frac{1}{2}nk^2$.

Collecting our values for J_{10}, J_{11}, J_{12}, we get

$$\left|\int_0^\infty \sigma(v)e^{-yv\sqrt{n}}\begin{matrix}\cos\\ \sin\end{matrix}(xv\sqrt{n})\,dv\right|$$

$$< \frac{e^{-y\sqrt{n}}}{\sqrt{n}}\left[kA+e^{-\frac{1}{2}nk^2}\tfrac{1}{2}\sqrt{(2\pi)}+e^{yk\sqrt{n}}(kA+\tfrac{1}{2}e^{-nk^2}\sqrt{\pi})\right].$$

Consequently

$$\left|\int_0^\infty \sigma(v)\exp(izv\sqrt{n})\,dv\right| = \frac{\vartheta'\sqrt{2}\,e^{iz\sqrt{n}}}{\sqrt{n}}(kA+e^{-\frac{1}{2}nk^2}\sqrt{\pi})[1+e^{yk\sqrt{n}}], \quad (12a)$$

where ϑ' is a complex quantity such that $|\vartheta'|<1$. This formula has to be used in conjunction with (12).

We can now write down the asymptotic expansion of $D_{-n-1}(-iz)$. Noting that $-\frac{1}{3}n < \pm y\sqrt{n}$, we see that the inequalities concerning J_2 may be written in the form

$$|J_2|<\frac{2\sqrt{2}\,e^{-y\sqrt{n}}\,e^{-\frac{2}{3}n}}{|x|\sqrt{n}}, \qquad |J_2|<\left(\frac{\pi}{2n}\right)^{\frac{1}{2}}e^{-y\sqrt{n}}e^{-\frac{2}{3}n},$$

i.e.,

$$|J_2|<\frac{2\sqrt{2}\,|e^{iz\sqrt{n}}|\,e^{-\frac{2}{3}n}}{|x|\sqrt{n}}, \qquad |J_2|<\left(\frac{\pi}{2n}\right)^{\frac{1}{2}}|e^{iz\sqrt{n}}|\,e^{-\frac{2}{3}n}.$$

We consequently get, from (8),

$$D_{-n-1}(-iz)=\frac{\sqrt{\pi}(\sqrt{n})^n\,e^{-\frac{1}{2}n}}{\Gamma(n+1)}\,e^{\frac{1}{4}z^2}e^{iz\sqrt{n}}[e^{-\frac{1}{4}z^2}+\Theta], \quad (13)$$

where

$$|\Theta|<\frac{1}{|x|}\left[2e^{-\frac{2}{3}n}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}+8A\left(\frac{2}{\pi n}\right)^{\frac{1}{2}}(1+e^{|y|k\sqrt{n}})\right], \quad (13a)$$

and

$$|\Theta|<\frac{1}{\sqrt{2}}\,e^{-\frac{2}{3}n}+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\{kA+e^{-\frac{1}{2}nk^2}\sqrt{\pi}\}\{1+e^{|y|k\sqrt{n}}\}, \quad (13b)$$

where

$$z=x+iy, \quad k=2n^{\alpha-\frac{1}{2}}, \quad n>\exp\left\{\frac{2}{1-2\alpha}\log 2\right\}, \quad A=0{\cdot}742\ldots,$$

and y is so limited that $|y|<\frac{1}{3}n^\alpha$,

α being *any* quantity such that $0\leqslant\alpha<\frac{1}{2}$.

We can, if necessary, obtain the asymptotic expansion of $D_n(z)$ when n is large, by making use of formula (3) in conjunction with these results.

6. We are now in a position to investigate the range of validity of the expansion of an arbitrary function $f(x)$ in a series of D_n functions.

Whittaker has shewn that if the expansion exists, so that

$$f(x) = \sum_{n=0}^{\infty} a_n D_n(x),$$

then

$$a_n = \frac{1}{(2\pi)^{\frac{1}{2}}\, n!} \int_{-\infty}^{\infty} D_n(t) f(t).$$

This result had previously been given in a slightly different form by Hermite.*

The method which we shall employ is based on Neumann's† corresponding work for the functions of Legendre.

We first investigate the function

$$E_n(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{4}z^2} (z-x)^{-1} D_n(z)\, dz,$$

where x is not a real quantity and n is a positive integer. Suppose that $I(x)$ is positive.‡

Deform the path of integration near the origin into a semicircle, of radius a, whose centre is at the origin and which is below the real axis, where $a > |x|$. Then at all points of the path of integration, $|z| > |x|$.

We may consequently expand $(z-x)^{-1}$ in powers of x, and get

$$E_n(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} x^m z^{-m-1} D_n(z)\, dz,$$

where the path of integration passes below the origin.

Using formula (6), we get

$$E_n(x) = \sum_{m=0}^{\infty} e^{\frac{1}{2}(m+1-n)\pi i} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(m+1)} \pi^{\frac{1}{2}} 2^{\frac{1}{2}(n+m)} x^m$$

$$= e^{-\frac{1}{2}n\pi i} i \sqrt{(2\pi)}\, \Gamma(n+1)\, D_{-n-1}(-ix)\, e^{-\frac{1}{4}x^2}. \qquad (14)$$

Similarly, when $I(x)$ is negative, we get

$$E_n(x) = -e^{\frac{1}{2}n\pi i} i \sqrt{(2\pi)}\, \Gamma(n+1)\, D_{-n-1}(ix) e^{-\frac{1}{4}x^2}. \qquad (14a)$$

The function $E_n(x)$ thus has a cut along the axis of real quantities.

* Hermite, *Comptes Rendus*, t. LVIII, pp. 266–273.

† See Whittaker, *Modern Analysis*, § 128.

‡ We use $I(x)$ to denote $R(-ix)$; x is no longer used to denote $R(z)$.

Now take Whittaker's recurrence formulæ

$$\left.\begin{aligned} D_{n+1}(z)-zD_n(z)+nD_{n-1}(z) &= 0 \\ D_{n+1}(x)-xD_n(x)+nD_{n-1}(x) &= 0 \end{aligned}\right\},$$

and, after multiplying them by $D_n(x)$, $D_n(z)$ respectively, subtract. We get

$$D_{n+1}(z)\,D_n(x)-D_n(z)D_{n+1}(x)-n\{D_n(z)D_{n-1}(x)-D_{n-1}(z)\,D_n(x)\}$$
$$=(z-x)\,D_n(z)\,D_n(x),$$

so that
$$\sum_{n=0}^{m}\frac{1}{n!}D_n(z)\,D_n(x)=\frac{D_{m+1}(z)\,D_m(x)-D_m(z)\,D_{m+1}(x)}{m!\,(z-x)}. \tag{15}$$

Multiply by $e^{-\frac{1}{4}z^2}(z-t)^{-1}$, t not being real, and integrate; we get

$$\sum_{n=0}^{m}\frac{1}{n!}D_n(x)\int_{-\infty}^{\infty}e^{-\frac{1}{4}z^2}D_n(z)(z-t)^{-1}\,dz$$
$$=\frac{1}{x-t}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{4}z^2}}{m!\,(z-x)}\left[D_{m+1}(z)\,D_m(x)-D_m(z)\,D_{m+1}(x)\right]dz$$
$$-\frac{1}{x-t}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{4}z^2}}{m!\,(z-t)}\left[D_{m+1}(z)\,D_m(x)-D_m(z)\,D_{m+1}(x)\right]dz.$$

In the first of these integrals replace $D_{m+1}(z)\,D_m(x)-D_m(z)\,D_{m+1}(x)$ by the series given in (15); noticing that $e^{-\frac{1}{4}z^2}=D_0(z)$, and using Whittaker's integral

$$\int_{-\infty}^{\infty}D_m(z)\,D_n(z)\,dz=0 \quad (m\neq n)$$
$$=(2\pi)^{\frac{1}{2}}m! \quad (m=n),$$

we get

$$\sum_{n=0}^{m}\frac{1}{n!}D_n(x)\,E_n(t)=\frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{4}x^2}}{x-t}-\frac{E_{m+1}(t)\,D_m(x)-E_m(t)\,D_{m+1}(x)}{m!\,(x-t)}$$
$$=\frac{(2\pi)^{\frac{1}{2}}e^{-\frac{1}{4}x^2}}{x-t}-F(x,\,t;\,m), \tag{16}$$

where

$$F(x,\,t;\,m)=\frac{e^{-\frac{1}{4}t^2}\Gamma(m+2)}{x-t}$$
$$\times\left[e^{\mp\frac{1}{2}m\pi i}D_{-m-2}(\mp it)\left\{e^{\frac{1}{2}m\pi i}D_{-m-1}(ix)+e^{-\frac{1}{2}m\pi i}D_{-m-1}(-ix)\right\}\right.$$
$$\left.-e^{\mp\frac{1}{2}(m-1)\pi i}D_{-m-1}(\mp it)\left\{e^{\frac{1}{2}(m+1)\pi i}D_{-m-2}(ix)+e^{-\frac{1}{2}(m-1)\pi i}D_{-m-2}(-ix)\right\}\right], \tag{17}$$

the upper or lower signs being taken according as $I(t)$ is $+$ or $-$.

Now suppose that $f(x)$ is an arbitrary function of x with no singularities in the strip of the plane defined by

$$|I(x)| \leqslant a,$$

and that $f(x)$ satisfies the two following conditions :—

(i) $|f(x)\, e^{\frac{1}{4}x^2} x^{-1}| \to 0$ as $|x| \to \infty$, when $|I(x)| \leqslant a$.

(ii) Each of the four integrals

$$\int^{\pm\infty \pm ai} f(x)\, e^{\frac{1}{4}x^2} \frac{dx}{x}$$

is absolutely convergent (the paths of integration being parallel to the real axis).

Let γ be a contour formed by the lines

$$I(t) = -a \quad \text{(going from left to right)},$$

$$I(t) = a \quad \text{(going from right to left)}.$$

Then
$$\frac{1}{2\pi i}\int_\gamma f(t)\, e^{\frac{1}{4}t^2}(t-x)^{-1}dt = f(x)\, e^{\frac{1}{4}x^2},$$

by Cauchy's theorem.

Consequently, by (16),

$$\frac{1}{2\pi i}\sum_{n=0}^{m}\frac{1}{n!}D_n(x)\int_\gamma E_n(t)\, f(t)\, e^{\frac{1}{4}t^2}dt$$

$$= -(2\pi)^{\frac{1}{2}} f(x) - \frac{1}{2\pi i}\int_\gamma f(t)\, e^{\frac{1}{4}t^2} F(x, t; m)\, dt. \quad (18)$$

(Each of the integrals on the left-hand side converges.)

We now return to the asymptotic expansion of $D_{-n-1}(iz)$. Suppose that m starts by being equal to m_0, where m_0 is greater than $16a^2(1+\epsilon)$, where ϵ is a positive quantity; and let $\alpha = (\log 4a)/(\log m)$, or zero, if this expression be negative. Then $\exp\left(\frac{2}{1-2\alpha}\log 2\right)$ is less than some finite quantity K, when $m \geqslant m_0$.

Now take $m > K$, and put

$$k = 2m^{\alpha-\frac{1}{2}}.$$

Then, if z lies in the strip $|I(z)| \leqslant a$, we have

$$|I(z)\, k\sqrt{m}| \leqslant a \,.\, 2m^{\alpha} \leqslant 8a^2$$

(or $2a$ if $3a < 1$, so that $\alpha = 0$).

Consequently, using (13), (13a), and (13b),

$$D_{-m-1}(-it) = \frac{\sqrt{\pi}(\sqrt{m})^m e^{-\frac{1}{2}m}}{\Gamma(m+1)} e^{\frac{1}{4}t^2} e^{it\sqrt{m}} \{e^{-\frac{1}{4}t^2} + \Theta'\},$$

where Θ' certainly does not exceed the smaller of the two quantities

$$\frac{1}{|R(t)|}\left[2e^{-\frac{2}{3}K}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} + 8A\left(\frac{2}{\pi K}\right)^{\frac{1}{2}}(1+e^{8a^2})\right]$$

and

$$\frac{1}{\sqrt{2}} e^{-\frac{2}{3}K} + \frac{2}{\pi}\{A+\sqrt{\pi}\}\{1+e^{8a^2}\}$$

(if $3a < 1$, we replace $8a^2$ by $2a$ in these expressions).

Consequently, when $m >$ a finite quantity K,

$$D_{-m-1}(-it) = \frac{\sqrt{\pi}(\sqrt{m})^m e^{-\frac{1}{2}m}}{\Gamma(m+1)} e^{\frac{1}{4}t^2} e^{it\sqrt{n}} \{e^{-\frac{1}{4}t^2} + \Theta_1\},$$

where $|\Theta_1|$ does not exceed a finite quantity independent of t and m; we have a similar formula for $D_{-m-1}(ix)$.

We consequently get, from (17),

$$F(x, t; m) = \frac{e^{\frac{1}{4}x^2}\pi m^{\frac{1}{2}m}(m+1)^{\frac{1}{2}(m+1)} e^{-m-\frac{1}{2}}}{(x-t)\Gamma(m+1)}$$

$$\times[\lambda_1 \exp\{ix\sqrt{m} \pm it\sqrt{(m+1)}\} + \lambda_2 \exp\{-ix\sqrt{m} \pm it\sqrt{(m+1)}\}$$

$$+\lambda_3 \exp\{ix\sqrt{(m+1)} \pm it\sqrt{m}\} + \lambda_4 \exp\{-ix\sqrt{(m+1)} \pm it\sqrt{m}\}],$$

where the upper or lower signs are to be taken according as $I(t)$ is positive or negative and $|\lambda_1|$, $|\lambda_2|$, $|\lambda_3|$, $|\lambda_4|$ do not exceed finite quantities independent of m.

It is easily seen that the term in square brackets tends uniformly to zero as m tends to infinity, if we can find a finite quantity m_1, such that

$$|I(x)|\sqrt{(m_1+1)} < |I(t)|\sqrt{m_1};$$

that is, effectively, if $|I(x)| < a$, since $|I(t)| = a$.

Also, by the asymptotic expansion of the gamma function,

$$m^{\frac{1}{2}m}(m+1)^{\frac{1}{2}(m+1)} e^{-m-\frac{1}{2}} / \Gamma(m+1)$$

tends uniformly to $(2\pi)^{-\frac{1}{2}}$ as m tends to infinity. Therefore

$$\left|\int_\gamma f(t) e^{\frac{1}{4}t^2} F(x, t; m)\, dt\right| < \epsilon_m \times \int_\gamma |f(t)\, e^{\frac{1}{4}t^2}(x-t)^{-1}|\, d|t|,$$

where ϵ_m tends uniformly to zero as m tends to infinity.

That is to say,

$$\left| \int_\gamma f(t)\, e^{\frac{1}{4}t^2} F(x, t; m)\, dt \right|$$

tends uniformly to zero as m tends to infinity.

It follows from a consideration of equation (18) that we have now proved the following theorem :—

Let $f(x)$ be a function of x which has no singularities in the strip of the plane $|I(x)| \leqslant a$, where a is a finite quantity; and let $f(x)$ satisfy the two following conditions :—

(i) $|f(x)\, e^{\frac{1}{4}x^2} x^{-1}| \to 0$ as $|x| \to \infty$, when $|I(x)| \leqslant a$.

(ii) Each of the four integrals

$$\int^{\pm\infty\, \pm ai} f(x)\, e^{\frac{1}{4}x^2} \frac{dx}{x}$$

is absolutely convergent (the paths of integration being parallel to the real axis).

Then $f(x)$ can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} b_n D_n(x),$$

where

$$b_n = -\frac{1}{2\pi i} \frac{1}{(2\pi)^{\frac{1}{2}} n!} \int_\gamma E_n(t)\, f(t)\, e^{\frac{1}{4}t^2} dt,$$

and this expansion is valid for all points in the finite part of the x plane which are in the interior of the strip $|I(x)| \leqslant a$.

ON THE DIFFRACTION OF A SOLITARY WAVE

By HORACE LAMB, F.R.S.

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IN a former communication* a simplified proof was given of Sommerfeld's results relating to the diffraction of an infinite train of plane waves of simple-harmonic type by the straight edge of a plane perfectly reflecting screen. Since a plane wave of arbitrary character can always be resolved, by Fourier's theorem, into simple-harmonic trains, the formulæ appropriate to the case of a solitary wave can theoretically be deduced. It was in this way that the particular cases to be hereafter considered were in the first instance worked out; but in the paper as presented I have somewhat varied and simplified the procedure. I find that, in the case of one or two special types which present themselves quite naturally for consideration, the results assume a comparatively simple form, so that conclusions as to the whole history of the disturbance in various parts of the field can be drawn without much difficulty. The more interesting points are illustrated by means of figures drawn to scale.

It should be added that the diffraction of a solitary wave has also been discussed by Sommerfeld,† but chiefly in relation to the theory of Röntgen rays, where the boundary condition is different; the method employed is moreover quite distinct.

1. The screen, whose thickness is neglected, is supposed to occupy that half of the xz-plane for which x is positive. For simplicity the case of normal incidence is alone considered, although there is no difficulty in extending the results in this respect, in the manner explained in the former paper. The primary wave is accordingly taken to be of the type

$$\phi = F(ct+y). \tag{1}$$

* "On Sommerfeld's Diffraction Problem," *Proceedings*, Vol. 4, p. 190 (1906).

† "Theoretisches über die Beugung der Röntgenstrahlen," *Zeitschrift für Math. u. Physik*, Bd. 46, p. 11 (1901).

In the complete solution, representing both the primary and the diffracted waves, we have to satisfy the equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right), \tag{2}$$

subject to the condition either that the value of ϕ, or that the value of its normal derivative, shall vanish over both faces of the screen. We begin with the latter form of the problem, the boundary condition being

$$\frac{\partial\phi}{\partial y} = 0. \tag{3}$$

This corresponds to the case of sound waves, or of electric waves polarized in the plane xy. In the latter case, the components of magnetic and electric force are connected with ϕ by the relations

$$\left.\begin{aligned} &\alpha = 0, \qquad \beta = 0, \qquad \gamma = \phi, \\ &\dot{X} = c\,\partial\phi/\partial y, \quad \dot{Y} = -c\,\partial\phi/\partial x, \quad \dot{Z} = 0. \end{aligned}\right\} \tag{4}$$

It is easily seen that the function

$$\chi = \frac{\partial\phi}{\partial x}, \tag{5}$$

which is also a solution of (2), must vanish over the negative half of the plane xz, whilst its normal derivative $\partial\chi/\partial y$ must vanish at both faces of the screen. If we write

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where θ is supposed to range from 0 to 2π, these conditions are satisfied by

$$\chi = f(ct-r).r^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta, \tag{6}$$

which is an obvious generalization of the assumption (7) of the former paper.

To deduce a value of ϕ, we introduce the parabolic coordinates

$$\xi = r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \qquad \eta = r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \tag{7}$$

in terms of which we have

$$x = \xi^2-\eta^2, \quad y = 2\xi\eta, \quad r = \xi^2+\eta^2. \tag{8}$$

"The curves $\xi = \text{const.}$, $\eta = \text{const.}$ form a system of confocal para-

bolas. The coordinate η is everywhere positive, and the line $\eta = 0$

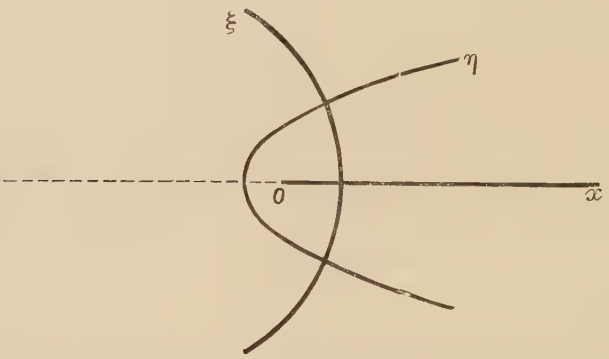

FIG. 1.

represents the section of the screen. The coordinate ξ has opposite signs on the two sides of the axis of x, and the line $\xi = 0$ represents the free portion of this axis."

In terms of these variables we have

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{1}{2r}\left(\xi \frac{\partial \phi}{\partial \xi} - \eta \frac{\partial \phi}{\partial \eta}\right), \\ \frac{\partial \phi}{\partial y} &= \frac{1}{2r}\left(\eta \frac{\partial \phi}{\partial \xi} + \xi \frac{\partial \phi}{\partial \eta}\right). \end{aligned}\right\} \tag{9}$$

Hence, from (5) and (6),

$$\xi \frac{\partial \phi}{\partial \xi} - \eta \frac{\partial \phi}{\partial \eta} = 2\xi f(ct - \xi^2 - \eta^2). \tag{10}$$

The integration of this equation follows exactly the same lines as in the special case treated in the former paper (Vol. 4, p. 193). If we adjust the complementary function so as to make $\partial\phi/\partial y = 0$, or $\partial\phi/\partial\eta = 0$, at the screen, where $\eta = 0$, we obtain

$$\phi = \int_0^{\xi+\eta} f(ct + y - \zeta^2)\, d\zeta + \int_0^{\xi-\eta} f(ct - y - \zeta^2)\, d\zeta + \tfrac{1}{2}F(ct+y) + \tfrac{1}{2}F(ct-y). \tag{11}$$

The functions f and F are so far arbitrary and independent. The relation between them in the present problem is supplied by the consideration that for large negative values of x the result must tend to the form (1). Now when θ is nearly equal to π, and r is very great, the upper limits of the integrals in (11) are $+\infty$ and $-\infty$, respectively, and the

limiting form is accordingly

$$\phi = \int_0^\infty f(ct+y-\xi^2)\,d\xi - \int_0^\infty f(ct-y-\xi^2)\,d\xi + \tfrac{1}{2}F(ct+y) + \tfrac{1}{2}F(ct-y). \quad (12)$$

This becomes identical with (1), provided

$$\int_0^\infty f(y-\xi^2)\,d\xi = \tfrac{1}{2}F(y). \quad (13)$$

The complete solution of the problem of diffraction of a plane wave of arbitrary type is thus reduced to that of the integral equation (13).

By way of verification we may put

$$F(y) = e^{iky}. \quad (14)$$

Since

$$\int_0^\infty e^{-ik\xi^2}\,d\xi = \tfrac{1}{2}\frac{\sqrt{\pi}}{\sqrt{k}}e^{-\frac{1}{4}i\pi}, \quad (15)$$

the solution is

$$f(y) = \frac{\sqrt{k}}{\sqrt{\pi}}\,e^{\frac{1}{4}i\pi}e^{iky}. \quad (16)$$

This agrees with the known result (Vol. 4, p. 194), if regard be had to the slightly altered definition of ξ, η.

This case leads at once to a general solution of (13). Writing Fourier's theorem in the form

$$F(y) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty F(a)\,e^{ik(y-a)}\,da, \quad (17)$$

with the convention that the real part alone is to be retained, we have

$$f(y) = \frac{1}{\pi^{\frac{3}{2}}}\int_0^\infty k^{\frac{1}{2}}\,dk \int_{-\infty}^\infty F(a)\,e^{i\,[k(y-a)+\frac{1}{4}\pi]}\,da, \quad (18)$$

subject to the same convention.*

2. There is one type of function expressing a more or less concentrated disturbance which has been especially affected by mathematical physicists from the time of Cauchy downwards, on account of the facility of analytical treatment, viz.,

$$F(y) = \frac{b}{b^2+y^2}. \quad (19)$$

* Mr. H. Bateman, to whom I submitted the question, has obtained a simpler solution in the form

$$f(y) = \frac{1}{\pi}\int_{-y}^\infty \frac{F'(-z)\,dz}{\sqrt{(z+x)}}.$$

This makes
$$\int_{-\infty}^{\infty} F(y)\,dy = \pi, \tag{20}$$

independently of the value of b; but the disturbance is the more concentrated the smaller this value. The familiar graphical representation of this function is given, for purposes of subsequent comparison, in Fig. 2.

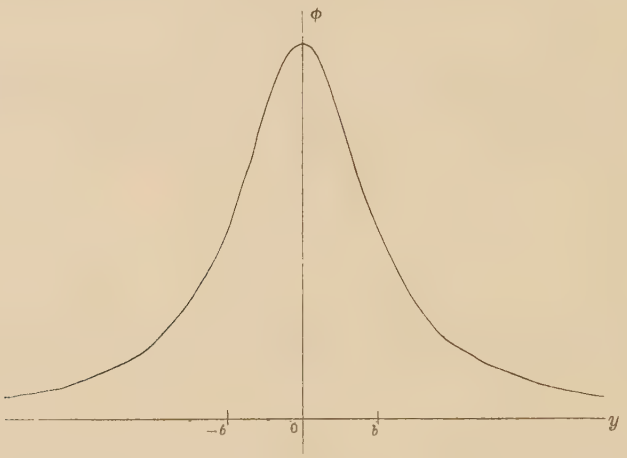

FIG. 2.

We have now
$$\int_{-\infty}^{\infty} F(a)e^{-ik\alpha}\,da = \pi e^{-kb}, \tag{21}$$

and therefore
$$\begin{aligned} f(y) &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-kb+i(ky+\frac{1}{4}\pi)}\,k^{\frac{1}{2}}\,dk \\ &= \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}(b-iy)^{\frac{3}{2}}}\,\Gamma(\tfrac{3}{2}) \\ &= \frac{e^{\frac{1}{4}i\pi}}{2\,(b-iy)^{\frac{3}{2}}}, \end{aligned} \tag{22}$$

provided that in the denominator that value of the square root of $b-iy$ be taken which is real and positive when $y = 0$. It is understood, as before, that the real part of the expression is alone to be preserved.

If we include the imaginary part, the formula (22) corresponds to

$$F(y) = \frac{1}{b+iy}. \tag{23}$$

This may immediately be verified. Since

$$\frac{d}{d\xi}\,\frac{\xi}{\sqrt{(b+iy-i\xi^2)}} = \frac{b+iy}{(b+iy-i\xi^2)^{\frac{3}{2}}}, \tag{24}$$

the formula (22) makes

$$\int f(y-\xi^2)\,d\xi = \frac{e^{-\frac{1}{4}i\pi}}{2\,(b+iy)}\,\frac{\xi}{\sqrt{(b+iy-i\xi^2)}}. \tag{25}$$

Taking this between the limits 0 and ∞, we find

$$\int_0^\infty f(y-\xi^2)\,d\xi = \frac{1}{2(b+iy)}, \tag{26}$$

in accordance with (13).

If we substitute from (23) and (25) in (11), making use of (8), we obtain

$$\phi = \frac{\frac{1}{2}}{b+i(ct+y)} + \frac{\frac{1}{2}}{b+i(ct-y)} + \frac{\frac{1}{2}e^{-\frac{1}{4}i\pi}}{\sqrt{\{b+i(ct-r)\}}}\left\{\frac{\xi+\eta}{b+i(ct+y)} + \frac{\xi-\eta}{b+i(ct-y)}\right\}, \tag{27}$$

where the square root is to be interpreted as before.

In order to isolate the real part of this expression we write

$$ct+y = b\tan\alpha, \quad ct-y = b\tan\beta, \quad ct-r = b\tan\omega, \tag{28}$$

where the angles α, β, ω may range from $-\frac{1}{2}\pi$ through 0 to $+\frac{1}{2}\pi$. We find

$$\phi = \frac{1}{2b}\left\{\cos^2\alpha+\cos^2\beta+\frac{\xi+\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\alpha\cos(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi) + \frac{\xi-\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\cos(\tfrac{1}{2}\omega+\beta+\tfrac{1}{4}\pi)\right\}, \tag{29}$$

corresponding to a primary wave

$$\phi = \frac{b}{b^2+(ct+y)^2} = \frac{\cos^2\alpha}{b}. \tag{30}$$

The origin of t is the instant at which the centre of the primary wave would coincide with the plane $y=0$ if the screen were absent. It may also be noted that the values of $\xi+\eta$, $\xi-\eta$ in the four quadrants of the plane xy are respectively as follows :—

$$\left.\begin{aligned}\xi+\eta &= +\sqrt{(r+y)}, \; +\sqrt{(r+y)}, \; +\sqrt{(r+y)}, \; -\sqrt{(r+y)},\\ \xi-\eta &= +\sqrt{(r-y)}, \; -\sqrt{(r-y)}, \; -\sqrt{(r-y)}, \; -\sqrt{(r-y)},\end{aligned}\right\} \tag{31}$$

as appears at once from (7) and (8).

3. To interpret this solution we notice in the first place that when the ratio ct/b is large and negative we have, in the region for which $y > 0$,

$$\beta = -\tfrac{1}{2}\pi, \qquad \omega = -\tfrac{1}{2}\pi,$$

approximately. The formula (29) therefore reduces to

$$\phi = \frac{\cos^2 \alpha}{2b} \left\{ 1 + \sqrt{\left(\frac{r+y}{b}\right)} \cos^{\frac{1}{2}} \omega \right\}, \tag{32}$$

i.e., since

$$(r+y)/b = \tan \alpha - \tan \omega, \tag{33}$$

practically to the form (30), the primary wave being as yet unaffected by the presence of the screen. In the region where $y < 0$ we have

$$\alpha = -\tfrac{1}{2}\pi, \qquad \omega = -\tfrac{1}{2}\pi,$$

nearly, whence, having regard to (31),

$$\phi = \frac{\cos^2 \beta}{2b} \left\{ 1 - \sqrt{\left(\frac{r-y}{b}\right)} \cos^{\frac{1}{2}} \omega \right\}. \tag{34}$$

Since

$$(r-y)/b = \tan \beta - \tan \omega, \tag{35}$$

this is practically insensible. That the formula should indicate any disturbance at all on the far side of the plane $y = 0$, at this stage, is due to the fact that our assumed primary wave has no clearly defined front.

When, on the other hand, ct/b is large and positive, we have, in the region for which $y > 0$, $\alpha = \frac{1}{2}\pi$, and therefore $\cos \alpha = 0$, nearly. The value of $\cos \beta$ will also be small except in the neighbourhood of the plane $y = ct$. Excluding for the present the critical region near the plane $x = 0$, we will further suppose that $r - ct$ is large compared with b, so that $\omega = -\frac{1}{2}\pi$, nearly. The formula thus reduces to

$$\phi = \frac{\cos^2 \beta}{2b} \left\{ 1 \pm \sqrt{\left(\frac{r-y}{b}\right)} \cos^{\frac{1}{2}} \omega \right\}, \tag{36}$$

where the upper or lower sign is to be taken according as $x \gtrless 0$. In virtue of (35) this is equivalent to

$$\phi = \frac{\cos^2 \beta}{b}, \quad \text{or} \quad 0. \tag{37}$$

Hence in the first quadrant of the plane xy we have a regularly reflected wave

$$\phi = \frac{b}{b^2 + (ct-y)^2}, \tag{38}$$

whilst in the second quadrant the disturbance is insensible. Again, in the region for which $y < 0$, we have $\beta = \frac{1}{2}\pi$, nearly, and the more important part of the disturbance is given by

$$\phi = \frac{\cos^2 \alpha}{2b} \left\{ 1 \mp \sqrt{\left(\frac{r+y}{b}\right)} \cos^{\frac{1}{2}} \omega \right\}, \tag{39}$$

where the rule of signs is as before. In virtue of (33) this reduces to

$$\phi = 0, \quad \text{or} \quad \frac{\cos^2 \alpha}{b}. \tag{40}$$

The disturbance is accordingly negligible in the region behind the screen, in the fourth quadrant, whilst in the third quadrant the primary wave proceeds undisturbed.

That the main part of the disturbance at any instant is to be sought in the neighbourhood of the planes $y = \pm ct$ is indicated already by the formula (27). The same equation suggests, in addition, an examination of the circumstances in the neighbourhood of the cylindrical surface $r = ct$. If we exclude the neighbourhood of the points where this touches the planes $y = \pm ct$, we have $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, nearly; and therefore

$$\cos \alpha = b/(r+y), \qquad \cos \beta = b/(r-y), \tag{41}$$

approximately. The formula (29) then gives

$$\phi = \frac{1}{2\sqrt{b}} \left\{ \frac{\pm 1}{\sqrt{(r+y)}} + \frac{\pm 1}{\sqrt{(r-y)}} \right\} \cos^{\frac{1}{2}} \omega \cos(\tfrac{1}{2}\omega + \tfrac{3}{4}\pi), \tag{42}$$

where the combinations of signs to be taken in the several quadrants are $+\,+$, $+\,-$, $+\,-$, $-\,-$ respectively. This may be interpreted as representing a cylindrical wave diverging from the edge of the screen; but the amplitude at a distance r from the edge, in any assigned direction, is of the order $\sqrt{(b/r)}$ as compared with that of the primary wave, and therefore usually negligible.

4. It remains to examine the conditions near the edges of the reflected and transmitted waves. Since the circumstances are exactly similar in these two regions it will be sufficient to take the latter case. At the medial plane of the transmitted wave we have $y = -ct$, and we will suppose as before that the ratio ct/b is large. We have then $\alpha = 0$, whilst β is nearly $= \frac{1}{2}\pi$. Also

$$\xi + \eta = \mp \sqrt{(r+y)} = \mp \sqrt{(r-ct)} = \mp \sqrt{(-b \tan \omega)}. \tag{43}$$

Thus (29) becomes

$$\phi = \frac{1}{2b}\left\{1 \mp \sqrt{(-\sin\omega)}\cos(\tfrac{1}{2}\omega + \tfrac{1}{4}\pi)\right\}, \tag{44}$$

where ω ranges from 0 (at the edge of the geometrical shadow) to $-\frac{1}{2}\pi$.

Again, over the plane in question, we have $r^2 - c^2t^2 = x^2$, and therefore, if x be small compared with ct,

$$-\tan\omega = x^2/2bct. \tag{45}$$

The following table gives a series of corresponding values of $x/\sqrt{(2bct)}$ and $b\phi$, calculated from (45) and (44); and the results are exhibited graphically in Fig. 3. It will be seen that the value of ϕ shades off gradually, without fluctuation, in the neighbourhood of the edge of the geometrical shadow. The transition is the more rapid the smaller the value of b, *i.e.*, the smaller the breadth of the primary wave, the horizontal scale being proportional to $\sqrt{b}$.

$\pm \frac{x}{\sqrt{(2bct)}}$	$b\phi$	$\pm \frac{x}{\sqrt{(2bct)}}$	$b\phi$
0	·500 ·500	·852	·158 ·842
·281	·397 ·603	1·000	·112 ·888
·398	·350 ·650	1·173	·072 ·928
·490	·310 ·690	1·401	·041 ·959
·570	·275 ·725	1·754	·018 ·982
·644	·243 ·757	2·513	·005 ·995
·714	·213 ·787	∞	0 1·000

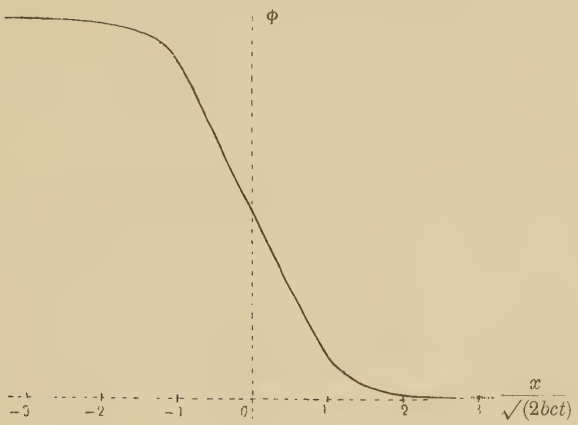

FIG. 3.

The figure gives an instantaneous view of the distribution of ϕ in the medial plane of the transmitted wave, but since the epochs of maximum amplitude are different for different points of any fixed plane $y = \text{const.}$, it is desirable to supplement it by an investigation of the whole history of the disturbance at various points, especially at such as are near the edge of the geometrical shadow.

This may be conducted as follows. Fixing our attention on any one point we may conveniently take the angle α as our independent variable, this being connected with t by the relation

$$ct = -y + b \tan \alpha. \tag{46}$$

The corresponding value of ω is given by

$$\tan \omega = \tan \alpha - (r+y)/b, \tag{47}$$

whilst β can be found, if necessary, from the relation

$$\tan \beta = \tan \alpha - 2y/b. \tag{48}$$

It appears, however, from (28) that when ct and $-y$ are both large compared with b, the angle β is very nearly equal to $\frac{1}{2}\pi$, so that the terms in (29) which contain $\cos \beta$ have little influence during the more important part of the disturbance. We have then, approximately,

$$\phi = \frac{1}{2b} \left\{ \cos^2 \alpha \pm \sqrt{\left(\frac{r+y}{b}\right)} \cos^{\frac{1}{2}} \omega \cos \alpha \cos \left(\tfrac{1}{2}\omega + \alpha + \tfrac{1}{4}\pi\right) \right\}. \tag{49}$$

This function can be tabulated without very much labour. I have chosen for computation the points for which

$$r+y=b, \quad \text{or} \quad x^2=-2by. \tag{50}$$

At any given distance $(-y)$ from the plane of the screen there are two of these, one lying just outside, and the other just inside the geometrical shadow. The resulting curves, shewing the variation of ϕ with t, are given in Fig. 4. The scale corresponds to that of Fig. 2, which may now be taken to represent the history of the disturbance which would be produced at any point by the primary wave alone.

In the later stages of the disturbance, the terms neglected in (49) may of course become comparable with those which are retained.

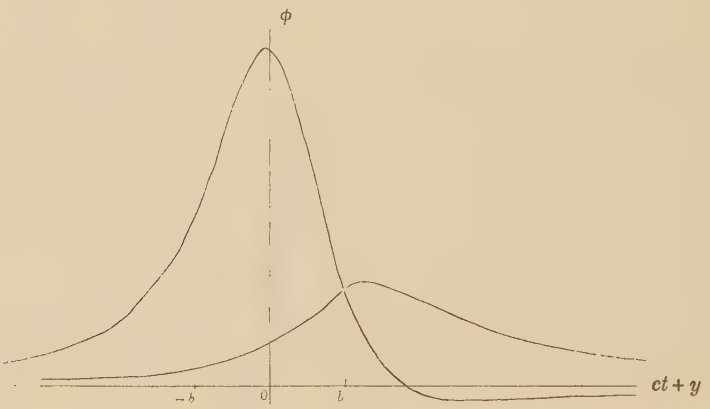

FIG. 4.

5. The second form of the diffraction problem, where the condition (3) which was to be satisfied at both faces of the screen is replaced by

$$\phi=0, \tag{51}$$

need not be dwelt upon at length. The general solution corresponding to a primary wave of the type (1) is

$$\phi=\int_0^{\xi+\eta} f(ct+y-\zeta^2)\,d\zeta-\int_0^{\xi-\eta} f(ct-y-\zeta^2)\,d\zeta+\tfrac{1}{2}F(ct+y)-\tfrac{1}{2}F(ct-y), \tag{52}$$

where the function $f(y)$ is determined by (15).

In the case of

$$F(y)=\frac{1}{b+iy}, \tag{53}$$

we should find

$$\phi = \frac{\frac{1}{2}}{b+i(ct+y)} - \frac{\frac{1}{2}}{b+i(ct-y)} + \frac{\frac{1}{2}e^{-\frac{1}{4}i\pi}}{\sqrt{\{b+i(ct-r)\}}}\left\{\frac{\xi+\eta}{b+i(ct+y)} - \frac{\xi-\eta}{b+i(ct-y)}\right\}, \quad (54)$$

the real part of which is

$$\phi = \frac{1}{2b}\left\{\cos^2\alpha - \cos^2\beta + \frac{\xi+\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\alpha\cos(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi) - \frac{\xi-\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\cos(\tfrac{1}{2}\omega+\beta+\tfrac{1}{4}\pi)\right\}, \quad (55)$$

in the notation of (28).

The interpretation would take exactly the same course as before, the chief difference being in the sign of the reflected wave. The main part of the disturbance on the far side of the plane $y = 0$ will be given as before by the formula (49); and the curves in Fig. 4 will still serve to represent the course of events near the edge of the shadow.

6. The preceding results admit of a great variety of applications. Their relation to the theory of electric waves has been indicated in § 1; but even if we confine ourselves to acoustics we find that there is still a considerable freedom of interpretation. Thus in the form of the problem treated in §§ 1–4 the function ϕ may be taken to represent either the velocity-potential or the condensation. In § 5, again, ϕ may be identified with the component (v) of the velocity normal to the plane xz, the positive half of this plane being occupied as before by a rigid screen.

If in the particular case to which Fig. 2 refers we identify ϕ with the *condensation* (usually denoted by s), we have a primary wave in which s has one sign only. It has been pointed out, however, by Stokes that a plane wave which is merely the ultimate form of a wave diverging from a source must necessarily contain both condensed and rarefied portions. A wave of this character, such as may be supposed to have originated in a local condensation, is obtained if in equation (30) we take ϕ to represent the *velocity-potential*, from which s is derived by the relation

$$c^2 s = \frac{\partial\phi}{\partial t}. \quad (56)$$

The consequent distribution of s is shewn in Fig. 5.

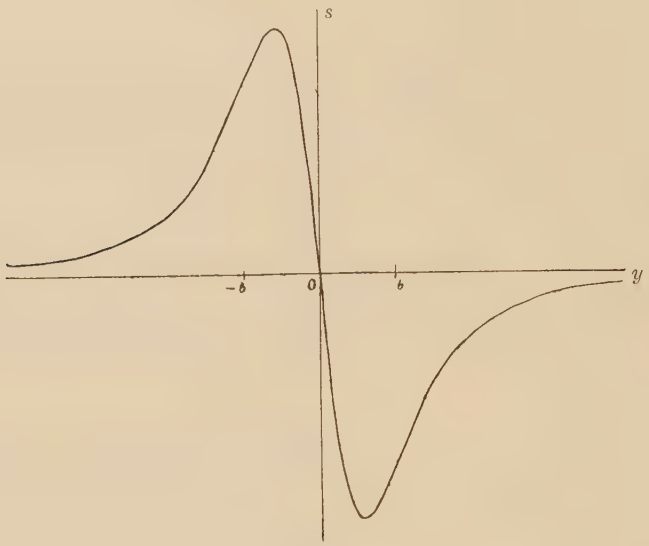

FIG. 5.

If we deal with the complex form (27), we deduce

$$cs = -\frac{\frac{1}{2}i}{\{b+i(ct+y)\}^2} - \frac{\frac{1}{2}i}{\{b+i(ct-y)\}^2}$$

$$-\frac{\frac{1}{4}ie^{-\frac{1}{4}i\pi}}{\{b+i(ct-r)\}^{\frac{3}{2}}}\left\{\frac{\xi+\eta}{b+i(ct+y)}+\frac{\xi-\eta}{b+i(ct-y)}\right\}$$

$$-\frac{\frac{1}{2}ie^{-\frac{1}{4}i\pi}}{\{b+i(ct-r)\}^{\frac{1}{2}}}\left\{\frac{\xi+\eta}{\{b+i(ct+y)\}^2}+\frac{\xi-\eta}{\{b+i(ct-y)\}^2}\right\}. \qquad (57)$$

Introducing the notations of (28), and rejecting the imaginary part, we obtain

$$cs = -\frac{1}{2b^2}\left\{2\sin\alpha\cos^3\alpha+2\sin\beta\cos^3\beta\right.$$

$$+\frac{\xi+\eta}{2\sqrt{b}}\cos^{\frac{3}{2}}\omega\cos\alpha\sin(\tfrac{3}{2}\omega+\alpha+\tfrac{1}{4}\pi)+\frac{\xi-\eta}{2\sqrt{b}}\cos^{\frac{3}{2}}\omega\cos\beta\sin(\tfrac{3}{2}\omega+\beta+\tfrac{1}{4}\pi)$$

$$\left.+\frac{\xi+\eta}{2\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos^2\alpha\sin(\tfrac{1}{2}\omega+2\alpha+\tfrac{1}{4}\pi)+\frac{\xi-\eta}{2\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\sin(\tfrac{1}{2}\omega+2\beta+\tfrac{1}{4}\pi)\right\}, \qquad (58)$$

corresponding to $cs = -\frac{2b(ct+y)}{\{b^2+(ct+y)^2\}^2} = -\frac{2\sin\alpha\cos^3\alpha}{b^2}$, (59)

in the primary wave.

It seems unnecessary to go through the interpretation in detail. The extreme values of s in the primary wave correspond to $\alpha = \pm\frac{1}{6}\pi$, and it is possible without much trouble to obtain an instantaneous picture of the distribution of s in the planes

$$y = -ct \pm \frac{1}{\sqrt{3}}b,$$

but the circumstance that the various particles are caught, so to speak, in different phases of their evolutions detracts, even more than in the case of Fig. 4, from the significance of the results. The main interest consists in tracing the history of the disturbance near the edge of the geometrical shadow, at a great distance from the screen. We have then, with sufficient accuracy,

$$cs = -\frac{1}{2b^2}\left\{2\sin\alpha\cos^3\alpha \mp \tfrac{1}{2}\sqrt{\left(\frac{r+y}{b}\right)}\cos^{\frac{3}{2}}\omega\cos\alpha\sin(\tfrac{3}{2}\omega+\alpha+\tfrac{1}{4}\pi)\right.$$
$$\left. \mp \sqrt{\left(\frac{r+y}{b}\right)}\cos^{\frac{1}{2}}\omega\cos^2\alpha\sin(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi)\right\}. \quad (60)$$

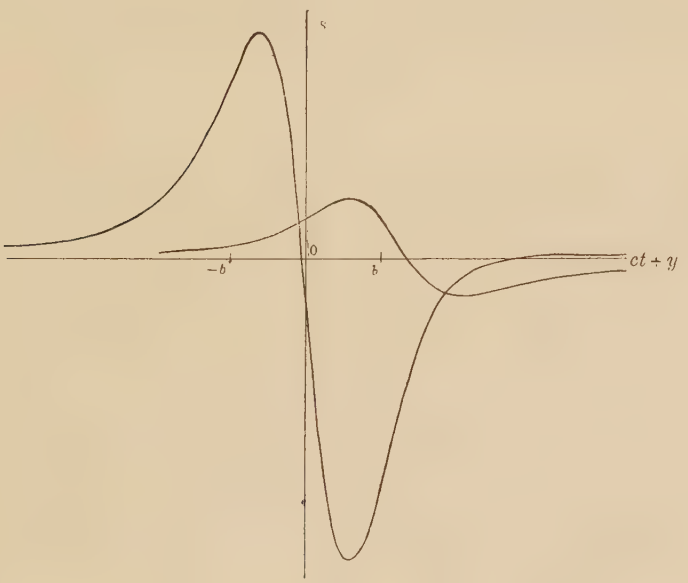

FIG. 6.

I have computed from this formula the variation of s with t at the

points, just outside and just inside the geometrical shadow, for which $r+y = b$. The results are shewn in Fig. 6, which corresponds in scale with Fig. 5. The terms neglected in (60) would of course be comparable with those retained, in the later stages of the disturbance.

7. One curious point remains to be noticed. In the case of the formula (27) the time-integral

$$\int_{-\infty}^{\infty} \phi\, dt \tag{61}$$

has the same value, viz. π/c, in all parts of the field. This easily follows by the method of contour integration, for the integral of each term in the second line of the formula referred to, taken along the real axis of t and an infinite semicircle on the negative side of this axis, must vanish, since the singular points $t = (r+ib)/c$, $t = (\pm y+ib)/c$ lie outside the region thus bounded. The fact that the integral (61) has the same value at points well within the shadow as in the primary wave shews that the smaller amplitude at such points must be compensated by a longer duration. This is already indicated to some extent by the form of the second curve in Fig. 4.

It appears in the same way that in the case of (54) the integral (61) everywhere vanishes. It has already been remarked that in the form of the problem to which (54) relates ϕ may be taken to represent the y-component of the velocity in a wave incident on a rigid semi-infinite screen. It follows that on this view the total displacement

$$\int_{-\infty}^{\infty} v\, dt$$

of a particle in the direction normal to the screen is zero at *all* points, whereas, if the primary wave (30) had not been interrupted, the displacement would have had the uniform value π/c, on the scale of our formulæ. The conclusion may be unexpected, especially as regards particles which would at first sight appear to be far removed from the influence of the screen, but the obvious explanation is that a considerable positive displacement is followed sooner or later by a negative one of much smaller extent, but correspondingly greater duration. The first curve in Fig. 4, which is valid also as an approximation for a certain range of t in the present form of the question, indicates how this may come about. To follow out the matter completely, it would be necessary to take account of the terms in (55) which we have neglected, but which ultimately become comparable with those retained.

It will be surmised that the uniformity of the value of the integral (61) throughout the medium is a result not confined to the particular types of

wave here considered, or even to the case of plane waves. A more general proof may be sought in the following manner. If we integrate both sides of the equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \tag{62}$$

with respect to t between the limits $-\infty$ and $+\infty$, and write

$$\psi = \int_{-\infty}^{\infty} \phi\, dt, \tag{63}$$

we find

$$\nabla^2\psi = 0, \tag{64}$$

on the hypothesis that $\partial\phi/\partial t$ vanishes at both limits. The theorem

$$\iiint \left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} dx\, dy\, dz = -\iint \psi \frac{\partial\psi}{\partial n}\, dS, \tag{65}$$

applied to an infinite region with fixed internal boundaries, then shews that ψ is constant, provided we can assert that the part of the surface integral which relates to an infinite enclosing boundary vanishes. Its precise value may then be found by a consideration of the circumstances at some particular point. Thus in our present case of a plane screen, where $v = 0$, the value must be zero if ϕ denotes the velocity-component v.

PERPETUANT SYZYGIES OF THE n-TH KIND

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A GREAT part of modern invariant theory deals with the question of what are known as "complete systems." Hilbert, in particular, has proved a series of theorems* of vast generality and power dealing with the formation and structure of such systems.

The present paper treats of "perpetuant types,"† and obtains for them a "syzygy chain" of the Hilbertian model. To the best of my belief, no previous example of a syzygy chain in invariant theory‡ has extended beyond the second link. The chain given here is of indefinite extent.

* The most important of these are to be found in *Math. Annalen*, Bd. XXXVI. In what follows, whenever Hilbert is mentioned the reference is to this paper.

† Defined further on.

‡ Examples for a rational integral function of n variables have been given by Hilbert himself and by Schönflies (*Gött. Nach.*, 1891).

A brief sketch of the history of the subject may not be out of place.

George Boole is generally recognised as having laid the first foundations. In 1841 he showed that if the variables of any given form were subjected to any linear transformation, then the discriminant of the transformed expression would be the same as that of the original one, save for a factor containing only the constants of the substitution. This is the property of invariance.

The next step was to find other functions of the coefficients having this property. When Cayley first turned his attention to the subject, the discovery of each single invariant or covariant* was a distinct acquisition. All this was changed by his invention of the calculus of hyperdeterminants. By this he could evolve, from a given form, an infinite number of concomitants.

To Cayley also fell the further success of showing that in the case of the binary quantics of the four lowest orders, any of these concomitants can (if rational and integral) be expressed as a rational integral function of a certain finite number of them. This is what is meant by the statement that the concomitants of the forms in question possess a *complete system.*

As the result of his investigations on the binary quintic, Cayley concluded that for it there is no such complete system.

However, Gordan, using the symbolic notation† of Aronhold and Clebsch, was able to show that not only the binary quintic, but also the general binary quantic of any order, possesses a complete system. It is interesting to note that the erroneous conclusion as to the quintic was due to the non-recognition of the existence of a syzygy of the second kind.

In the memoir to which we have already referred and in later papers,‡ Hilbert greatly extended our knowledge of complete systems. His method is founded upon the following theorem :—

Let a homogeneous function of any number of variables be formed according to any definite laws ; then, although there may be an infinite number of functions F satisfying the conditions laid down, nevertheless a finite number $F_1F_2 \ldots F_r$ can always be chosen so that any other F can be written in the form

$$F = A_1F_1+A_2F_2+\ldots+A_rF_r,$$

* A covariant only differs from an invariant in that it contains the variables as well as the coefficients. We shall use the term *concomitant* to include these as well as other functions (*e.g.*, "mixed concomitants") having the invariantive property.

† Which is really equivalent to Cayley's hyperdeterminants.

‡ *Gött. Nach.*, 1891 and 1892.

where the A's are homogeneous integral functions of the variables, but do not necessarily satisfy the conditions for the F's.

By means of this very general theorem he extended Gordan's theorem to quantics (and sets of quantics) of any order, in any number of variables, or even in several sets of variables. But this was not all. Having proved that a complete system can always be obtained, he next proceeded to study its internal structure. To follow his conclusions it will be convenient to recall a few definitions:—

The members of a complete system are called the *irreducible* concomitants. In terms of these all other concomitants can be expressed as rational integral functions.

A *syzygant* (of the *first* kind) is a rational integral function of the irreducible concomitants which, *when these last are replaced by their expressions as rational integral functions of the coefficients of the quantics*, is identically zero.* The equation expressing this fact is called a *syzygy*.

If a syzygant is obtainable from one of lower degree by multiplication throughout with a concomitant, it is called *compound*. A *reducible* syzygant is one which is compound or equivalent to the sum of two or more compound ones.

Hilbert has proved that, starting from a given form, the number of *irreducible* syzygants is finite, or, in other words, that *syzygants have a complete system.*

From syzygants of the first kind, we pass to those of higher kinds.

If $S_1 S_2 \dots S_r$ be irreducible syzygants of the first kind, and $P_1 P_2 \dots P_r$ certain concomitants or products of concomitants, it may happen that the expression

$$P_1 S_1 + P_2 S_2 + \dots + P_r S_r \equiv 0,$$

the expressions P, S being regarded as functions of the concomitants (which for the moment are treated as independent variables). If so, $P_1 S_1 + P_2 S_2 + \dots + P_r S_r$ is called a syzygant of the *second kind*. It will be seen that it is an identically vanishing sum of compound syzygants of the first kind. Syzygants of the second kind may, like those of the first, be reducible or irreducible. Hilbert has shown that the number of irreducible ones is finite.

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* The simplest example (due to Cayley) is that for the binary cubic. See Elliott's *Algebra of Quantics*, p. 110.

We go on to syzygants of the third kind. Let $S_1^{(2)}S_2^{(2)}\dots S_r^{(2)}$ be irreducible syzygants of the second kind, and $P_1P_2\dots P_r$ be certain concomitants (or products of them) such that $P_1S_1^{(2)}+P_2S_2^{(2)}+\dots+P_rS_r^{(2)}$, when the $S^{(2)}$'s are replaced by their equivalent sums of compound syzygants of the first kind, is, *solely in virtue of this replacement*,* identically zero; then $P_1S_1^{(2)}+P_2S_2^{(2)}+\dots+P_rS_r^{(2)}$ is called a syzygant of the third kind.

In the same way a syzygant of the n-th kind is defined in the terms of those of the $(n-1)$-th kind and concomitant products.†

Hilbert has succeeded in proving that the syzygants of each species possess a complete system.

The object of this paper is to illustrate these conclusions. The particular syzygies selected for consideration are those connecting "perpetuant types."‡

In a paper by Young and Wood,§ the syzygies of the first kind connecting perpetuant types have been considered as far as degree 9. These authors differ fundamentally from all previous writers‖ on the same subject in that their methods are based upon the symbolical notation, and in particular upon Grace's perpetuant type theorem.¶ This theorem of Grace's, when taken in conjunction with a paper of Wood's,** gives in a neat form the symbolical expressions for the perpetuant types. Young and Wood's paper deals with the syzygies of the first kind connecting these, and in the present paper I deal with the syzygies of the second, third, and finally n-th kind.

A general method†† for the formation of these is developed and a some-

* *I.e.*, the syzygants of the first kind are *not* to be replaced, in their turn, by their equivalent sums of concomitant products.

† It will be seen that the definition of the third and higher kinds is slightly different from that of the second, which again differs from that of the first.

‡ A *perpetuant* is an *irreducible* seminvariant of a set of binary quantics of infinite order, and a perpetuant *type* is one linear in the coefficients of each quantic concerned. In case it should be thought that the problem proposed for consideration is an unduly specialised one, it may be remarked that theorems concerning perpetuants seem to be generally capable of extension to ordinary covariants (*i.e.*, those of forms of finite order). For instance, Young has so extended Grace's perpetuant type theorem and also the results of the paper (referred to below) on perpetuant syzygies of the first kind. See *Proc. London Math. Soc.*, Ser. 2, Vols. 1 and 3.

§ *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 221–265.

‖ A full list of previous writings is given in Young and Wood's paper. They all deal exclusively with the *first* kind.

¶ *Proc. London Math. Soc.*, Vol. xxxv.

** *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

†† Explained in Section VI.

what curious theorem is arrived at, differentiating those of kind n, and of degrees less than $2n+2$, from those of the same kind and higher degrees.

This general method, which we may call *symbolical multiplication*, is an extension of the well known method of deriving syzygies of the first kind from the Jacobian identity.

The results obtained are as follows:—

No perpetuant syzygy of the n-th kind can exist for degrees $< (n+2)$.

For degree $(n+2)$ one, and only one, perpetuant syzygy of the n-th kind exists. From this, by *symbolical multiplication*, many other perpetuant syzygies of the n-th kind of all degrees $> (n+2)$ may be derived.

Let these, together with the syzygy from which they spring, be called *primary perpetuant syzygies of the n-th kind.*

Then the principal theorem obtained is that *all perpetuant syzygies of the n-th kind of degree* $< (2n+2)$ *are primary.* We have thus obtained those members of the complete system that belong to the first n degrees for which perpetuant syzygies of the n-th kind exist at all. For the first kind this merely gives the fact that the Jacobian identity is the only syzygy of degree 3.

It is somewhat unexpected that the range of simplicity should increase as the "kind" of the syzygy increases.

After an examination in detail of these primary syzygies, introducing a principle of arrangement of symbolical letters analogous to that in Grace's perpetuant type theorem, and after finding the generating function corresponding to each case, a proof is given that *for degree* $(2n+2)$ *new syzygies that are not primary do actually exist.*

Among the minor results, it may be of interest to notice that perpetuant syzygies of the first kind, which, in the notation of Young and Wood, *reduce* perpetuant products having a factor C_1, are always compound.* This agrees with their results as far as they go.

A corresponding theorem as to perpetuant syzygies of the n-th kind is also proved.†

Beside the above, I have obtained the linearly independent set of perpetuant syzygies of the second kind of degree 6, and some results for perpetuant syzygies of the second kind of degree 7, and perpetuant syzygies of the third kind of degree 8. As these are somewhat lengthy, I have not included them in this present paper.

* See Lemma I.

† See Lemma III.

I. *General Methods of Finding Canonical Sets of Perpetuant Syzygies of the n-th Kind.*

The general procedure for finding the *canonical system* of perpetuant syzygies of the n-th kind of degree δ (*i.e.*, the set of linearly independent ones in terms of which all other perpetuant syzygies of the n-th kind and degree δ may be linearly expressed), when we know those for the perpetuant syzygies of the $(n-1)$-th kind of degrees less than δ, divides itself into three parts :

(1) Deducing, as explained below, from our knowledge of perpetuant syzygies of the $(n-1)$-th kind of degree $< \delta$, what must be the generating function for linearly independent* perpetuant syzygies of the n-th kind of degree δ.

(2) Actually finding perpetuant syzygies of the n-th kind. In most cases the irreducible ones can all be found by *symbolical multiplication.*

(3) Finding how many of the perpetuant syzygies of the n-th kind (both irreducible and compound) that we now have got are linearly independent. If the generating function for these is equal to the L.G.F. formed by (1), we know that we have obtained all the linearly independent perpetuant syzygies of the n-th kind.

The method used for proving a set of perpetuant syzygies of the n-th kind to be linearly independent is as follows :—

All compound perpetuant syzygies of the $(n-1)$-th kind of the degree in question are arranged in accordance with a fixed sequence.† Now a perpetuant syzygy of the n-th kind can be looked upon as a linear relation between compound perpetuant syzygies of the $(n-1)$-th kind. In any particular perpetuant syzygy of the n-th kind, take the compound perpetuant syzygy of the $(n-1)$-th kind that precedes, according to our arrangement, all the others involved in the perpetuant syzygy of the n-th kind under consideration.

We shall say that this compound perpetuant syzygy of the $(n-1)$-th kind is *resolved* by the perpetuant syzygy of the n-th kind.

If we have m perpetuant syzygies of the n-th kind, say A, B, C, ... of the same degree, and each resolves a different compound perpetuant syzygy of the $(n-1)$-th kind, say a, b, c, ..., respectively, *they must all be*

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* We shall, in future, use L.G.F. to denote these five words.

† For the details of which see Section II.

linearly independent; for if, of $a, b, c, \ldots, a$ stand first of all the m, b second, c third, and so on, according to the fixed arrangement, then a cannot occur in any of $B, C, D, \ldots$; therefore there is certainly no linear relation connecting A with $B, C, D, \ldots$.

Similarly there is none connecting B with $C, D, E, \ldots$, none connecting C with $D, E, F, \ldots$, and so on; therefore $A, B, C, \ldots$ are all linearly independent.

We speak of a compound perpetuant syzygy of the $(n-1)$-th kind for which a perpetuant syzygy of the n-th kind can be found, to resolve it, as *resolvable.* If no such perpetuant syzygy of the n-th kind can be found, the perpetuant syzygy of the $(n-1)$-th kind is called *irresolvable.**

To explain (1),

the L.G.F. for perpetuant syzygies of the n-th kind

$=$ generating function for *all* compound perpetuant syzygies of the $(n-1)$-th kind

$-$generating function for canonical perpetuant syzygies of the $(n-1)$-th kind.

In consequence of our device for proving linear independence, there is a one-to-one correspondence between a *doubly compound* (*i.e.*, with at least two perpetuant factors) perpetuant syzygy of the $(n-2)$-th kind that is resolvable at all by a *compound* perpetuant syzygy of the $(n-1)$-th kind, and the *canonical* compound perpetuant syzygy of the $(n-1)$-th kind that resolves it. The other compound perpetuant syzygies of the $(n-1)$-th kind that resolve it are not canonical. By taking all doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ, we bring in, once and once only, all compound perpetuant syzygies of the $(n-1)$-th kind of that degree.

Therefore, if from the generating function for all *compound*† perpetuant syzygies of the $(n-1)$-th kind resolving a certain form of doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ, we subtract the generating function for all doubly compound perpetuant syzygies of the $(n-2)$-th kind of that form that *are resolvable* by *any* compound

* In the notation of Young and Wood for perpetuant products, *irreducible* is the word used. I was forced to use another word, as an *irreducible perpetuant syzygy of the* $(n-1)$*-th kind* has another meaning (see introduction).

† It is because we are only dealing with *compound* perpetuant syzygies of the $(n-1)$-th kind of degree δ that we need only know the canonical set of perpetuant syzygies of the $(n-1)$-th kind for degrees *less than* δ.

perpetuant syzygies of the $(n-1)$-th kind, we get the L.G.F. for the associated perpetuant syzygies of the n-th kind.*

Taking the sum, for all possible forms of doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree δ of these L.G.F.'s, we get the total L.G.F. for perpetuant syzygies of the n-th kind of that degree.†

We shall use C_κ to denote a perpetuant of degree κ, $S_\delta^{(n)}$ to denote an irreducible syzygant of kind n and degree δ, and $[C_\kappa S_\delta^{(n)}]$ to denote a syzygant of the $(n+1)$-th kind, resolving the compound perpetuant syzygies of the n-th kind $C_\kappa S_\delta^{(n)}$.

II. *Arrangement of Compound Perpetuant Syzygies of the n-th Kind.*

The sequence is according to the following rules :—

(i) A non-primary‡ syzygy always precedes a primary one.

(ii) Of two syzygies, both non-primary or both primary, with C_1^m and C_1^n as factors respectively, where $m < n$, that with C_1^m as a factor stands first.

(iii) Of two syzygies, whose precedence is not determined by (i) or (ii), that which has the fewest perpetuant factors stands first.

(iv) If $A = C_{\kappa_1} C_{\kappa_2} \dots C_{\kappa_p} S^{(n)} \quad (\kappa_1 \leqslant \kappa_2 \leqslant \kappa_3 \dots \leqslant \kappa_p)$,

and $B = C_{\kappa'_1} C_{\kappa'_2} \dots C_{\kappa'_p} S'^{(n)} \quad (\kappa'_1 \leqslant \kappa'_2 \leqslant \kappa'_3 \dots \leqslant \kappa'_p)$,

have not their precedence determined by (i), (ii) or (iii), then A precedes B if the first of the differences $(\kappa_1-\kappa'_1)$, $(\kappa_2-\kappa'_2)$, ..., $(\kappa_p-\kappa'_p)$ that does not vanish is *positive*.

(v) If $A = C_{\kappa_1} C_{\kappa_2} \dots C_{\kappa_p} S^{(n)}$

and $B = C'_{\kappa_1} C'_{\kappa_2} \dots C'_{\kappa_p} S'^{(n)}$ $\quad (\kappa_1 \leqslant \kappa_2 \leqslant \kappa_3 \dots \leqslant \kappa_p)$,

have not their precedence determined by (i), (ii), (iii), or (iv), and if, of the

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* For an example, see Section III or Section XI.

† For the result is, as before,

generating function for all compound perpetuant syzygies of the $(n-1)$-th kind

= ,, ,, canonical ,, ,, ,,

‡ A primary perpetuant syzygy of the n-th kind is one derived by symbolical multiplication from the perpetuant syzygies of the n-th kind of lowest possible degree. See Section VI.

symbols $a_1, a_2, \ldots, a_\delta$ involved,

C_{κ_1} (or $C_{\kappa_1} \ldots C_{\kappa_s}$, if $\kappa_1 = \kappa_2 = \kappa_3 = \ldots = \kappa_s < \kappa_{s+1}$) contain $a_{r_1}, a_{r_2}, \ldots, a_{r_t}$,

and

C'_{κ_1} (or $C'_{\kappa_1} \ldots C'_{\kappa_s}$, ,, ,, ,,) ,, $a_{r'_1}, a_{r'_2}, \ldots, a_{r'_t}$,

where $r_1 < r_2 \ldots < r_t$ and $r'_1 < r'_2 \ldots < r'_t$,

then A precedes B if the first of the differences

$$(r_1 - r'_1),\ (r_2 - r'_2),\ \ldots,\ (r_t - r'_t),$$

that does not vanish is *negative*.

If C_{κ_1} and C'_{κ_1} contain exactly the same letters, consider C_{κ_2} and C'_{κ_2}, ..., and so on.

It will not be necessary to go into further details of this kind, when all the above principles fail, to establish the results of the present paper. It is sufficient to know that of two syzygies we *can** always determine which stands first.

For perpetuant products, replace $S^{(n)}$ in the above by unity. Principle (i) does not apply.

III. *Perpetuant Syzygies of the Second Kind of Degree* 4.

We consider a perpetuant syzygy of the second kind as connecting a canonical and an uncanonical compound perpetuant syzygy of the first kind that resolve the same perpetuant product. This product must contain at least three factors, or it cannot be resolved in more than one way by a compound perpetuant syzygy of the first kind. Its degree, therefore, cannot be less than three.

Degree 3.—The only product of at least three factors is C_1^3, which is irresolvable, therefore *there are no perpetuant syzygies of the second kind of degree* 3.

Degree 4.—The only products to be considered are

(i) $C_1^2 C_2$, and (ii) C_1^4.

* By following out, for example, part of the classification of perpetuant products given in a paper on "Perpetuant Syzygies (of the First Kind)," by A. Young and P. W. Wood, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 230–233. We use the classification given in Section III, (ii), (iii) and (iv) [but *not* (i)], and apply it to the perpetuant factors of compound perpetuant syzygies of the n-th kind.

(i) $C_1^2 C_2$.—This is only resolvable* if one of the two C_1's is A_1, and if C_2 be of unit weight, *i.e.*, the generating function for resolvables is $3x$.

The number of compound perpetuant syzygies of the first kind of form $C_1[C_1 C_2]$ that resolve a form $C_1^2 C_2$ is,* however, given by $4x$; therefore the generating function for linearly independent perpetuant syzygies of the second kind is the difference

$$4x - 3x = x,$$

i.e., there is only one perpetuant syzygy of the second kind of degree 4. It is

$$\begin{aligned}(a_1a_2a_3a_4) \equiv\ & A_1(a_2a_3a_4) - A_2(a_1a_3a_4) + A_3(a_1a_2a_4) - A_4(a_1a_2a_3)\\ \equiv\ & A_1\{A_2(a_3a_4) - A_3(a_2a_4) + A_4(a_2a_3)\}\\ & - A_2\{A_1(a_3a_4) - A_3(a_1a_4) + A_4(a_1a_3)\}\\ & + A_3\{A_1(a_2a_4) - A_2(a_1a_4) + A_4(a_1a_2)\}\\ & - A_4\{A_1(a_2a_3) - A_2(a_1a_3) + A_3(a_1a_2)\}\\ \equiv\ & 0.\end{aligned}$$

A_r is here written instead of C_1 when the symbolical letter involved is a_r.

IV. *Perpetuant Syzygies of the Third Kind of Degree* 5.

Since the lowest degree of a perpetuant syzygy of the first kind is 3, therefore the lowest degree of a doubly compound perpetuant syzygy of the first kind is 5; therefore *the lowest possible degree of a perpetuant syzygy of the third kind is* 5.

The only doubly compound perpetuant syzygy of the first kind of this degree is of form $C_1^2 S_3^{(1)}$.

This is only resolvable† in more than one way if the C_1's are A_1 and A_2, when we get two resolutions.

Therefore *there is only one perpetuant syzygy of the third kind of degree* 5, *i.e.*,

$$\begin{aligned}(a_1a_2a_3a_4a_5) \equiv\ & A_1(a_2a_3a_4a_5) - A_2(a_1a_3a_4a_5) + A_3(a_1a_2a_4a_5)\\ & - A_4(a_1a_2a_3a_5) + A_5(a_1a_2a_3a_4).\end{aligned}$$

* These results are quoted from Young and Wood's paper. They can easily be obtained independently.

† Since $(a_1a_2a_3a_4)$ resolves $A_1(a_2a_3a_4)$.

V. *Perpetuant Syzygies of the n-th Kind of Degree* $(n+2)$.

Consider the zero determinant of $(n+2)$ rows and columns :—

$$\begin{vmatrix} A_1 & A_1 & \ldots\ldots & A_1 & a_1 \\ A_2 & A_2 & \ldots\ldots & A_2 & a_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n+2} & A_{n+2} & \ldots\ldots & A_{n+2} & a_{n+2} \end{vmatrix}.$$

Denote this by $(a_1 a_2 \ldots a_{n+2})$.

Use a similar notation for the first minors of the first column. Then expanding

$$(a_1 a_2 \ldots a_{n+2}) = A_1(a_2 a_3 \ldots a_{n+2}) - A_2(a_1 a_3 \ldots a_{n+2}) + A_3(a_1 a_2 a_4 \ldots a_{n+2}) \ldots .$$

If we expand the first minors in terms of the constituents of *their* first columns and first minors, we shall have the original determinant expanded in terms of its first two columns. As these are the same, the result is identically zero.

Losing sight of the determinants, and quoting only this last identity, it follows that, if $(a_{r_1} a_{r_2} \ldots a_{r_n})$ be a perpetuant syzygy of the $(n-2)$-th kind, and $(a_{r_1} a_{r_2} \ldots a_{r_n} a_{r_{n+1}})$ be a perpetuant syzygy of the $(n-1)$-th kind, and $A_1 A_2 \ldots A_{n+2}$ denote, as usual, quantics of infinite order, then $(a_1 a_2 \ldots a_{n+2})$ is a perpetuant syzygy of the n-th kind.

But we have seen that $(a_1 a_2 a_3 a_4)$ is a perpetuant syzygy of the second kind, and $(a_1 a_2 a_3 a_4 a_5)$ is a perpetuant syzygy of the third kind. Therefore $(a_1 a_2 \ldots a_{n+2})$ *is a perpetuant syzygy of the n-th kind of degree* $(n+2)$.

Generating Function for Perpetuant Syzygies of the n-th Kind of Degree $(n+2)$.

To find the generating function for perpetuant syzygies of the n-th kind of degree δ, we take all doubly compound perpetuant syzygies of the $(n-2)$-th kind of that degree. The excess of the generating function for compound perpetuant syzygies of the $(n-1)$-th kind (some of them uncanonical) that resolve these, over the generating function for resolvable perpetuant syzygies of the $(n-2)$-th kind of degree δ, will be the generating function for the perpetuant syzygies of the n-th kind of degree δ. If $(a_1 \ldots a_{n+1})$ is the only perpetuant syzygy of the $(n-1)$-th kind of degree $(n+1)$, and if there are no perpetuant syzygies of the $(n-1)$-th kind of lower degrees; if, also $(a_1 \ldots a_n)$ is the only perpetuant syzygy of the $(n-2)$-th kind of degree n, and if there are no perpetuant syzygies of the

$(n-2)$-th kind of lower degrees; then there are no doubly compound perpetuant syzygies of the $(n-2)$-th kind of degree $< (n+2)$; therefore, *there are no perpetuant syzygies of the n-th kind of degree* $< (n+2)$. This is true for kinds 2 and 3, and therefore it is true in general.

For degree $(n+2)$, we have only to consider $C_1^2 S_n^{(n-2)}$. This is resolvable in two ways only if the C_1's be A_1 and A_2; therefore, *there is only one perpetuant syzygy of the n-th kind of degree* $(n+2)$, *i.e.* $(a_1 \dots a_{n+2})$. This is true for kinds 2 and 3, and therefore in general.

VI. *Symbolical Multiplication and Primary Perpetuant Syzygies.*

$$A_1(a_2a_3) - A_2(a_1a_3) + A_3(a_1a_2) \equiv 0,$$

or, as it is often written,

$$(a_2a_3) - (a_1a_3) + (a_1a_2) \equiv 0,$$

is a perpetuant syzygy of the first kind.

If we multiply throughout by the symbols $(a_1a_4)^\omega$, we get

$$(a_1a_4)^\omega (a_2a_3) - (a_1a_4)^\omega (a_1a_3) + (a_1a_4)^\omega (a_1a_2) \equiv 0,$$

which is another perpetuant syzygy of the first kind. Denote it by $J^{(1)}[(a_1a_4)^\omega (a_1a_2a_3)]$. In fact, if the result of multiplying each term of a sum of perpetuant products (expressed symbolically) by some further symbols is to give another sum of *products** of perpetuants, and the original sum is a syzygant, so is the new one. This follows from the first principles of the symbolical notation.

In the sum of compound perpetuant syzygies of the first kind

$$(a_1a_5)^\omega (a_2a_3a_4) - A_2J^{(1)}[(a_1a_5)^\omega (a_1a_3a_4)] + A_3J^{(1)}[(a_1a_5)^\omega (a_1a_2a_4)$$
$$- A_4J^{(1)}[(a_1a_5)^\omega (a_1a_2a_3)],$$

if we replace each perpetuant syzygy of the first kind by its equivalent sum of perpetuant products, the result will be found to be identically zero. The expression is therefore a perpetuant syzygy of the second kind.

Now, if this replacement is carried out in full, and then compared with the proof† that $(a_1a_2a_3a_4)$ itself is a perpetuant syzygy of the second kind, it will be seen that one piece of work may be obtained from the

* They must be products; otherwise we get, not a syzygy, but merely a relation between reducible forms.

† See end of Section III.

other by multiplication of the symbols throughout by $(a_1 a_5)^\omega$ and omitting the capital A's.

On this account we denote the perpetuant syzygy of the second kind by $J^{(2)}[(a_1 a_5)^\omega (a_1 a_2 a_3 a_4)]$.

Similarly the sum of compound perpetuant syzygies of the second kind

$$(a_1 a_6)^\omega (a_2 a_3 a_4 a_5) - A_2 J^{(2)}[(a_1 a_6)^\omega (a_1 a_3 a_4 a_5)] + A_3 J^{(2)}[(a_1 a_6)^\omega (a_1 a_2 a_4 a_5)] - \ldots,$$

when expressed as a sum of compound perpetuant syzygies of the first kind, will be found to be zero.

The work closely corresponds to that with

$$(a_2 a_3 a_4 a_5) - (a_1 a_3 a_4 a_5) + (a_1 a_2 a_4 a_5) - \ldots .$$

Thus from the existence of a perpetuant syzygy of the third kind $(a_1 a_2 a_3 a_4 a_5)$ we infer that of another, which we call

$$J^{(3)}[(a_1 a_6)^\omega (a_1 a_2 a_3 a_4 a_5)].$$

This process, which we may call *symbolical multiplication*, evidently applies to the n-th kind also, starting from $(a_1 a_2 \ldots a_{n+2})$. It is clear that other syzygies can be formed by choosing more elaborate symbolical factors as multipliers, such as, for example,

$$(a_1 a_{n+3})^\lambda (a_1 a_{n+4})^\mu (a_2 a_{n+5})^\nu.$$

Primary Perpetuant Syzygies.

DEF.—We shall call a perpetuant syzygy of the n-th kind *primary* if it is (1) the simple one of degree $(n+2)$, *i.e.* $(a_1 a_2 \ldots a_{n+2})$, or (2) derived from this by *symbolical multiplication.*

NOTATION.—We shall use

$$S^{(n)}(\lambda_1 \lambda_2 \ldots \lambda_p),$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_p \ (\geqslant 2)$ and $p \leqslant (n+2)$,

to denote such a primary perpetuant syzygy of the n-th kind as that formed by the symbolical multiplication of $(a_1 a_2 \ldots a_{n+2})$ by the product of p perpetuants

$(a_1 a_{n+3})^{\rho_{1,1}} (a_1 a_{n+4})^{\rho_{1,2}} \ldots (a_1 a_{n+\lambda_1+1})^{\rho_{1,\lambda_1-1}}$ of degree λ_1,

$(a_2 a_{n+\lambda_1+2})^{\rho_{2,1}} (a_2 a_{n+\lambda_1+3})^{\rho_{2,2}} \ldots (a_2 a_{n+\lambda_1+\lambda_2})^{\rho_{2,\lambda_2-1}}$ of degree λ_2,

... ...

$(a_p a_{n+\sum_1^{p-1} \lambda_r - p+4})^{\rho_{p,1}} (a_p a_{n+\sum_1^{p-1} \lambda_r - p+5})^{\rho_{p,2}} \ldots (a_p a_{n+\sum_1^{p} \lambda_r - p+2})^{\rho_{p,\lambda_p-1}}$ of degree λ_p.

Each factor in this product contains *one* symbolical letter (and one only) of those contained in $(a_1 a_2 \dots a_{n+2})$.

We use the notation $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ to denote also any of the primary perpetuant syzygies of the n-th kind derived from the above by interchanging the suffixes of the symbolical letters involved.

Properties of Primary Perpetuant Syzygies.

(i) $S^{(n)}(\lambda_1 \dots \lambda_p)$ is clearly irreducible, for no two of the compound perpetuant syzygies of the $(n-1)$-th kind that compose it have a common perpetuant factor.

By multiplication by a perpetuant or product of perpetuants, we can form compound primary perpetuant syzygies of the n-th kind.

(ii) $S^{(n)}(\lambda_1 \dots \lambda_p)$ resolves $C_{\lambda_1} S^{(n-1)}(\lambda_2 \dots \lambda_p)$, a compound primary perpetuant syzygy of the $(n-1)$-th kind, which has only a single perpetuant factor, of degree not less than 2 [unless $S^{(n)}$ be simply $(a_1 \dots a_{n+2})$]. $S^{(n-1)}(\lambda_2 \dots \lambda_p)$ resolves, in its turn, $C_{\lambda_2} S^{(n-2)}(\lambda_3 \dots \lambda_p)$.

It follows that every irreducible primary perpetuant syzygy of the n-th kind comes from a doubly compound primary perpetuant syzygy of the $(n-2)$-th kind *with only two perpetuant factors.* It is proved further on* that the perpetuant syzygy of the n-th kind that arises from a primary doubly compound perpetuant syzygy of the $(n-2)$-th kind is itself necessarily primary. Therefore there is a complete correspondence between irreducible primary perpetuant syzygies of the n-th kind, and doubly compound primary perpetuant syzygies of the $(n-2)$-th kind *with only two perpetuant factors.* This is true whatever the degree.

VII. LEMMA 1.—*A perpetuant syzygy of the first kind (of degree greater than 3) that resolves a product with a factor C_1 is compound.*

By our arrangement scheme, a syzygy is never to be regarded as resolving a product with a factor C_1 if there is any term without such a factor. It follows that a syzygy resolving a C_1 product must consist entirely of C_1 products. Hence we wish to prove that a syzygy consisting entirely of C_1 products must be compound.

* In Section X.

This is shown by the following more general theorem :—

Let F be a rational integral function of the $\binom{\delta}{2}$ variables $x_{r,s}$,

$$r = 1, 2, \ldots, \delta, \quad s = 1, 2, \ldots, \delta;$$

but $$r \neq s \quad (x_{r,s} = -x_{s,r}).$$

Each term is to be of the same total degree in the x's taken all together.

Further, let F have the property of vanishing when all such relations as

$$x_{p,q} + x_{q,r} + x_{r,p} = 0$$

hold good. We shall call such a function *a null function*. Then we shall prove that, if F be such that each *term* involves at most $(\delta-1)$ suffixes, F can be expressed as the sum of a number of *null functions*, each of which likewise involves at most $(\delta-1)$ suffixes. *E.g.*,

$$F \equiv x_{2,4}(x_{1,2} + x_{2,3} + x_{3,4} - x_{1,4})$$

can be written

$$x_{2,4}(x_{2,3} + x_{3,4} - x_{2,4}) + x_{2,4}(x_{1,2} - x_{1,4} + x_{2,4}),$$

i.e., as the sum of two null functions, one involving only the suffixes 2, 3, and 4; the other involving only 1, 2, and 4.

PROOF.—Take any term of F, say

$$x_{p,q}^{\lambda} x_{r,s}^{\mu} \ldots,$$

where the suffix omitted is t.

If $t \neq \delta$, subtract

$$f \equiv x_{p,q}^{\lambda} x_{r,s}^{\mu} - \ldots - (x_{p,\delta} - x_{q,\delta})^{\lambda} (x_{r,\delta} - x_{s,\delta})^{\mu} \ldots,$$

f is a null function not containing the suffix t.

If $t = \delta$, subtract

$$f' \equiv x_{p,q}^{\lambda} x_{r,s}^{\mu} - \ldots - (x_{p,\delta-1} - x_{q,\delta-1})^{\lambda} (x_{r,\delta-1} - x_{s,\delta-1})^{\mu} \ldots.$$

Treating every term of F in this manner, we finally get a new null function

$$F - \Sigma f - \Sigma f'$$

$=$ a rational integral function of $x_{1,\delta} x_{2,\delta} \ldots x_{\delta-1,\delta}$ from each term of which at least one of the suffixes $1 \ldots \delta-1$ is absent

$+$ a rational integral function of $x_{1,\delta-1} x_{2,\delta-1} \ldots x_{\delta-2,\delta-1}$.

Write $$A \equiv x_{1,\delta-1}, \qquad a \equiv x_{1,\delta},$$
$$B \equiv x_{2,\delta-1}, \qquad b \equiv x_{2,\delta},$$
$$Z \equiv x_{\delta-2,\delta-1}, \qquad z \equiv x_{\delta-2,\delta},$$

and $$\epsilon \equiv -x_{\delta-1,\delta}.$$

Then $$\phi \equiv F - \Sigma f - \Sigma f' = \psi(a, b, \ldots, z, \epsilon) + \chi(A, B, \ldots, Z),$$

where at least one of $a, b, \ldots, z$ or ϵ is absent from each term of ψ. That is

$$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\psi = 0.$$

Now ϕ is a null function, *i.e.*, $\psi(a, b, \ldots, z, \epsilon) + \chi(A, B, \ldots, Z)$ vanishes in consequence of such relations as

$$A = a + \epsilon, \quad B = b + \epsilon, \quad \ldots.$$

Hence $$\psi(a, b, \ldots, z, \epsilon) = -\chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon);$$

therefore $$\phi = \chi(A, B, \ldots, Z) - \chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon).$$

But $$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\psi = 0;$$

therefore $$\frac{\partial}{\partial a}\frac{\partial}{\partial b}\cdots\frac{\partial}{\partial z}\frac{\partial}{\partial \epsilon}\chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon) = 0.$$

Put $$a + \epsilon = a', \quad b + \epsilon = b', \quad \ldots,$$

$$\frac{\partial}{\partial \epsilon} = \Sigma \frac{\partial a'}{\partial \epsilon}\frac{\partial}{\partial a'} = \Sigma \frac{\partial}{\partial a'},$$

while $$\frac{\partial}{\partial a} = \frac{\partial a'}{\partial a}\frac{\partial}{\partial a'} = \frac{\partial}{\partial a'}, \quad \ldots.$$

Hence $$\frac{\partial}{\partial a'}\frac{\partial}{\partial b'}\cdots\frac{\partial}{\partial z'}\left(\frac{\partial}{\partial a'} + \frac{\partial}{\partial b'} + \ldots + \frac{\partial}{\partial z'}\right)\chi(a', b', \ldots, z') = 0,$$

so $$\frac{\partial}{\partial A}\frac{\partial}{\partial B}\cdots\frac{\partial}{\partial Z}\left(\frac{\partial}{\partial A} + \frac{\partial}{\partial B} + \ldots + \frac{\partial}{\partial Z}\right)\chi(A, B, \ldots, Z) = 0.$$

The most general solution of this equation is

$$\chi = \Omega_1 + \Omega_2 + \ldots + \Omega_{\delta-2} + \Delta;$$

where Ω_1 does not involve A, Ω_2 does not involve B, and so on, while Δ is

a function of the *differences* of $A, B, \ldots, Z$. Hence

$$\begin{aligned}\phi = &\ \chi(A, B, \ldots, Z) - \chi(a+\epsilon, b+\epsilon, \ldots, z+\epsilon)\\ = &\ \Omega_1(B, C, \ldots, Z) - \Omega_1(b+\epsilon, c+\epsilon, \ldots, z+\epsilon)\\ &+\Omega_2(A, C, \ldots, Z) - \Omega_2(a+\epsilon, c+\epsilon, \ldots, z+\epsilon)\\ &+\ldots\\ &+\Delta(A, B. \ldots, Z) - \Delta(a, b, \ldots, z),\end{aligned}$$

for $$\Delta(a+\epsilon, b+\epsilon, \ldots z+\epsilon) = \Delta(a, b, \ldots, z).$$

Reverting to our x notation, we see that ϕ is thus proved to be expressible as the sum of $(\delta-1)$ null functions, the first of which does not contain the suffix 1, the second of which does not contain the suffix 2, and so on; the $(\delta-2)$-th of which does not contain the suffix $(\delta-2)$, while the last

$$\Delta(x_{1,\delta-1} \ldots x_{\delta-2,\delta-1}) - \Delta(x_{1,\delta} \ldots x_{\delta-2,\delta}),$$

say $$\Sigma(x_{p,\delta-1} - x_{q,\delta-1})^{\lambda}(x_{r,\delta-1} - x_{s,\delta-1})^{\mu} \ldots$$

$$-\Sigma(x_{p,\delta} - x_{q,\delta})^{\lambda}(x_{r,\delta} - x_{s,\delta})^{\mu} \ldots,$$

is equal to

$$\Sigma[(x_{p,\delta-1} - x_{q,\delta-1})^{\lambda}(x_{r,\delta-1} - x_{s,\delta-1})^{\mu} \ldots - x^{\lambda}_{p,q}\, x^{\mu}_{r,s} \ldots]$$

$$+\Sigma[x^{\lambda}_{p,q}\, x^{\mu}_{r,s} \ldots - (x_{p,\delta} - x_{q,\delta})^{\lambda}(x_{r,\delta} - x_{s,\delta})^{\mu} \ldots],$$

say $$N_{\delta} + N_{\delta-1},$$

where N_t is a null function not containing the suffix t. Hence

$$\phi \equiv F - \Sigma f - \Sigma f' = \sum_{1}^{\delta} N_t,$$

i.e., $$F = \Sigma f + \Sigma f' + \Sigma N_t$$

$$= \text{the sum of a number of null functions, each quite free from at least one suffix, which is our theorem.}$$

Example.—Take $F \equiv$ $x_{1,3}x_{2,3}x_{1,2}$ (suffix 4 omitted from this term)

$$-x_{1,4}x_{2,4}x_{1,2} \quad (\ ,,\ 3 \quad ,, \quad ,,\ )$$

$$+x_{1,4}x_{3,4}x_{1,3} \quad (\ ,,\ 2 \quad ,, \quad ,,\ )$$

$$-x_{2,4}x_{3,4}x_{2,3} \quad (\ ,,\ 1 \quad ,, \quad ,,\ ),$$

F is easily seen to be a null function, or the following work shows it.

(N.B.—No confusion will arise from writing x_{13} instead of $x_{1,3}$, so we will drop these commas.)

Subtract the null functions

$$f_1 \equiv -x_{24}x_{34}x_{23}+x_{24}x_{34}(x_{24}-x_{34}),$$
$$f_2 \equiv \quad x_{14}x_{34}x_{13}-x_{14}x_{34}(x_{14}-x_{34}),$$
$$f_3 \equiv -x_{14}x_{24}x_{12}+x_{14}x_{24}(x_{14}-x_{24}),$$
$$f' \equiv \quad x_{13}x_{23}x_{12}-x_{13}x_{23}(x_{13}-x_{23}).$$

We get
$$\begin{aligned}\phi &\equiv F-f_1-f_2-f_3-f' \\ &= -x_{24}x_{34}(x_{24}-x_{34}) = \quad b\epsilon(b+\epsilon) \\ &\quad +x_{14}x_{34}(x_{14}-x_{34}) \quad -a\epsilon(a+\epsilon) \\ &\quad -x_{14}x_{24}(x_{14}-x_{24}) \quad -ab(a-b) \\ &\quad +x_{13}x_{23}(x_{13}-x_{23}) \quad +AB(A-B),\end{aligned}$$

where
$$A \equiv x_{13}, \qquad a \equiv x_{14},$$
$$B \equiv x_{23}, \qquad b \equiv x_{24},$$
$$\epsilon \equiv -x_{34},$$

so
$$\begin{aligned}\phi &= \quad \psi(a, b, \epsilon)+\chi(A, B) \\ &= -\chi(a+\epsilon, b+\epsilon)+\chi(A, B),\end{aligned}$$

as is easily verified. Also $\chi(A, B)$, *i.e.*, $AB(A-B)$, is equal to

$$\Omega_1+\Omega_2+\Delta,$$

where $\Delta \equiv -\frac{1}{3}(A-B)^3, \quad \Omega_1 \equiv -\frac{1}{3}B^3, \quad \Omega_2 \equiv \frac{1}{3}A^3$

so $\phi = -\frac{1}{3}B^3+\frac{1}{3}A^3-\frac{1}{3}(A-B)^3+\frac{1}{3}(b+\epsilon)^3-\frac{1}{3}(a+\epsilon)^3+\frac{1}{3}(a-b)^3$

$$= \quad \tfrac{1}{3}[(x_{24}-x_{34})^3-x_{23}^3]+\tfrac{1}{3}[x_{13}^3-(x_{14}-x_{34})^3]$$
$$-\tfrac{1}{3}[(x_{13}-x_{23})^3-x_{12}^3]+\tfrac{1}{3}[(x_{14}-x_{24})^3-x_{12}^3]$$

(the two terms in x_{12}^3 cancelling)

of form
$$N_1+N_2+N_4+N_3.$$

So, finally,
$$F = f_1+f_2+f_3+f'+N_1+N_2+N_3+N_4,$$

i.e., a sum of null functions each with one suffix omitted.

VIII. LEMMA 2.—*To prove that a perpetuant syzygy of the n-th kind that resolves a primary compound perpetuant syzygy of the* $(n-1)$*-th kind must itself be primary.*

Since, by our scheme of arrangement, a non-primary syzygy is always

to precede a primary one, the theorem amounts to :—*A perpetuant syzygy of the n-th kind, made up entirely of primary compound perpetuant syzygies of the* $(n-1)$*-th kind, is itself primary.*

By definition, a perpetuant syzygy of the n-th kind is such a linear function of compound perpetuant syzygies of the $(n-1)$-th kind, that when these are replaced by the linear functions of compound perpetuant syzygies of the $(n-2)$-th kind that they respectively represent, the resulting expression, regarded as a linear function of compound perpetuant syzygies of the $(n-2)$-th kind, is identically zero.

In the present case, when the perpetuant syzygies of the $(n-1)$-th kind are all primary, they are all symbolical (or actual) products of perpetuant syzygies of the $(n-1)$-th kind of the form $(a_{r_1} a_{r_2} \dots a_{r_{n+1}})$.

Let the first term be* $a_1 C_1^m C_{\lambda_1} C_{\lambda_2} \dots C_{\lambda_p} S^{(n-1)}(\mu_1 \mu_2 \dots \mu_q)$ (where no λ or μ is < 2).

This resolves a perpetuant syzygy of the $(n-2)$-th kind that is the symbolical product of $(a_{r_2} \dots a_{r_{n+1}})$ by $C_1^m C_{\lambda_1} C_{\lambda_2} \dots C_{\lambda_p} C_{\mu_1} C_{\mu_2} \dots C_{\mu_q}$.

This must be cancelled by another perpetuant syzygy of the $(n-2)$-th kind, equal except in sign, and this can only arise from a perpetuant syzygy of the $(n-1)$-th kind, which is the symbolical product of $C_1^m C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$ and $(a_\kappa a_{r_2} \dots a_{r_{n+1}})$, a_κ being some other of the letters involved.

Consider any other perpetuant syzygy of the $(n-2)$-th kind included in either of these two perpetuant syzygies of the $(n-1)$-th kind. We see that there will be a third perpetuant syzygy of the $(n-1)$-th kind, that is, the symbolical product of $C_1^m C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$, and, say $(a_\kappa a_\rho a_{r_3} \dots a_{r_{n+1}})$. In the preceding, the factors $C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_q}$ are exactly the same in each case, but the C_1^m's are not.

Proceeding in this way, we see that, unless the syzygy is the sum of two or more others, every perpetuant syzygy of the $(n-1)$-th kind in it must have a symbolical factor $C_{\lambda_1} \dots C_{\lambda_p} C_{\mu_1} \dots C_{\mu_p}$.

If we divide this out (a symbolical operation, of course) we are left with a new perpetuant syzygy of the n-th kind of the form

$$\Sigma \beta_r C_1^m (a_{r_1} \dots a_{r_{n+1}}) = 0.$$

The next step is to show that this is primary. By our arrangement scheme $C_1^m (a_{r_1} a_{r_2} \dots a_{r_{n+1}})$ precedes $C_1^m (a_{r'_1} a_{r'_2} \dots a_{r'_{n+1}})$, where

$$r_1 < r_2 \dots < r_{n+1} \quad \text{and} \quad r'_1 < r'_2 \dots < r'_{n+1},$$

* In the following, the α's, β's, γ's and δ's are numerical constants.

if the first of the differences $r_1 - r'_1$, $r_2 - r'_2$, ..., $r_{n+1} - r'_{n+1}$, that does not vanish is *positive.*

Arrange the terms of $\Sigma\beta_r C_1^m(a_{r_1}a_{r_2}\ldots a_{r_{n+1}})$ in this way. Take the symbolical letters involved to be $a_1 a_2 \ldots a_{m+n+1}$. If the first term be $\beta_s C_1^m(a_{s_1}a_{s_2}\ldots a_{s_{n+1}})$, subtract $\beta_s C_1^{m-1}(a_{s_1-1}a_{s_1}a_{s_2}\ldots a_{s_{n+1}})$, a perpetuant syzygy of the n-th kind involving $a_1 \ldots a_{m+n+1}$.

The difference resolves a perpetuant syzygy of the $(n-1)$-th kind that follows $C_1^m(a_{s_1}a_{s_2}\ldots a_{s_{n+1}})$ in the sequence.

Remove this by a fresh subtraction of a suitable perpetuant syzygy of the n-th kind of the same general form as before.

Now there are only a finite number $\binom{m+n}{m-1}$ of perpetuant syzygies of the $(n-1)$-th kind, of the form of $C_1^m(a_{s_1}a_{s_2}\ldots a_{s_{n+1}})$ in which $s_1 > 1$; therefore, by the repetition of the above process, we shall, sooner or later, arrive at a stage where every term left is of the form

$$\gamma_r C_1^m(a_1 a_{r_2} a_{r_3}\ldots a_{r_{n+1}}).$$

Let $\gamma_t C_1^m(a_1 a_{t_2}\ldots a_{t_{n+1}})$ be the first (in the fixed sequence) of these terms. Then, in the expanded form of the syzygy as an identically vanishing linear function of perpetuant syzygies of the $(n-2)$-th kind, there is nothing to cancel the term $\gamma_t C_1^{m+1}(a_{t_2}\ldots a_{t_{n+1}})$. Therefore $\gamma_t = 0$, *i.e.*, all the γ's are zero. Therefore $\Sigma\beta_r C_1^m(a_{r_1}a_{r_2}\ldots a_{r_{n+1}})$ is expressible as a sum of perpetuant syzygies of the n-th kind, like

$$\Sigma\delta_s C_1^{m-1}(a_{s_1-1}a_{s_1}a_{s_2}\ldots a_{s_{n+1}}),$$

i.e., is *primary.* Therefore the syzygy from which we started, being derived from the last by symbolical multiplication, is also primary, *i.e., a perpetuant syzygy of the n-th kind that resolves a primary perpetuant syzygy of the* $(n-1)$*-th kind is itself primary.*

IX. LEMMA 3.—*No perpetuant syzygy of the n-th kind of degree* $> (n+2)$ *can be irreducible if it resolve a perpetuant syzygy of the* $(n-1)$*-th kind with* C_1 *as a factor.*

Let $S^{(n)}$ be an irreducible perpetuant syzygy of the n-th kind resolving $C_1^m C_{\kappa_1}\ldots C_{\kappa_p} S^{(n-1)}$ (where $m \neq 0$ and no κ is unity).

To prove that $m = 1$ and $\kappa_1 = \kappa_2 = \ldots = \kappa_p = 0$, and $S^{(n-1)}$ is of the form $(a_1 a_2 \ldots a_{n+1})$, so $S^{(n)}$ is simply of the form $(a_1 a_2 \ldots a_{n+2})$, of degree $(n+2)$.

Assume the theorem true for all kinds lower than n. We shall see it is then true for n.

Let $S^{(n-1)}$ resolve $C_{\lambda_1} \dots C_{\lambda_q} S^{(n-2)}$.

Unless $S^{(n-1)}$ be $(a_1 \dots a_{n+1})$, in which case it resolves $C_1(a_1 \dots a_n)$, no λ can be unity, by hypothesis.

Now the sum of compound perpetuant syzygies of the $(n-1)$-th kind which $S^{(n)}$ represents must contain one, at least, of the forms

$$C_{\mu_1} \dots C_{\mu_i} [C_{\mu_i+1} \dots C_{\mu_r} C_1 S^{(n-2)}],$$

where $C_{\mu_1} \dots C_{\mu_r}$ are simply the other $(m-1)$ C_1's, and $C_{\kappa_1} \dots C_{\kappa_p}$, $C_{\lambda_1} \dots C_{\lambda_q}$ in some order; for if this were not the case, every term of $S^{(n)}$ would be divisible by C_1^m, and so $S^{(n)}$ would be *compound*, not irreducible.

Now by our scheme of arrangement, of two compound perpetuant syzygies of the $(n-1)$-th kind, containing a power of C_1 as a factor, that whose power of C_1 is the lower stands first (provided both the $S^{(n-1)}$'s are primary or both non-primary).

The number of C_1's in $C_{\mu_1} \dots C_{\mu_i}$ is at most $(m-1)$, since no κ or λ is unity. Also, from Lemma 2, both of the $S^{(n-1)}$'s in the next line are primary, or both non-primary. Therefore

$$C_{\mu_1} \dots C_{\mu_i} [C_{\mu_i+1} \dots C_{\mu_r} C_1 S^{(n-2)}] \quad \text{precedes} \quad C_1^m C_{\kappa_1} \dots C_{\kappa_p} S^{(n-1)}.$$

Therefore $S^{(n)}$ does *not* resolve $C_1^m C_{\kappa_1} \dots C_{\kappa_p} S^{(n-1)}$.

This contradicts a fact with which we started. Therefore the assumption that no λ is unity is false; therefore, by our hypothesis for kind $(n-1)$, $S^{(n-1)}$ is of form $(a_1 \dots a_{n+1})$ and resolves $C_1 S^{(n-2)}$, where $S^{(n-2)}$ is of form $(a_1, \dots, a_n)$; therefore, from Lemma 2, $S^{(n)}$ is primary.

Therefore as $m \neq 0$, $S^{(n)}$ is the simple perpetuant syzygy of the n-th kind of degree $(n+2)$ and of form $(a_1 a_2 \dots a_{n+2})$ [for we can easily see that no other *primary irreducible* perpetuant syzygy of the n-th kind resolves a compound with a factor C_1], and

$$\kappa_1 = \kappa_2 = \dots = \kappa_p = 0.$$

The theorem is true, by Lemma 1, for kind unity, therefore we have proved it in general, by induction.

X. THEOREM.—*All perpetuant syzygies of the n-th kind of degree* $< (2n+2)$ *are primary.*

The proof is by induction.

Assume the theorem to be true for all kinds less than n. We shall deduce its truth for kind n.

For an irreducible perpetuant syzygy of the $(n-2)$-th kind to be *non-primary*, its degree, by hypothesis, must be at least $(2n-2)$. Therefore the total degree of the perpetuant factors of a *doubly compound* perpetuant syzygy of the $(n-2)$-th kind of total degree $< (2n+2)$, of which the irreducible part is *non-primary*, is $(2n+1)-(2n-2) = 3$, at most.

Therefore the perpetuant factors are $C_1 C_2$, C_1^3, or C_1^2.

Now, by Lemma 3, no irreducible perpetuant syzygy of the $(n-1)$-th kind of degree $> (n+1)$ resolves a compound perpetuant syzygy of the $(n-2)$-th kind with a factor C_1.

Therefore $C_1 C_2 S^{(n-2)}$ is never resolvable in more than one way at most (*i.e.*, by $C_1[C_2 S^{(n-2)}]$) by a *compound* perpetuant syzygy of the $(n-1)$-th kind, while $C_1^3 S^{(n-2)}$ and $C_1^2 S^{(n-2)}$ are never resolvable at all by a compound perpetuant syzygy of the $(n-1)$-th kind, if $S^{(n-2)}$ is non-primary [and hence of degree $\geqslant (2n-2)$, *i.e.*, $\geqslant n$, as $(n-2) \geqslant 0$]. Therefore no perpetuant syzygies of the n-th kind arise from these doubly compound perpetuant syzygies of the $(n-2)$-th kind.

In forming the generating function for perpetuant syzygies of the n-th kind of degree $< (2n+2)$, *we need only consider those doubly compound perpetuant syzygies of the* $(n-2)$*-th kind which are primary.*

Therefore a perpetuant syzygy of the n-th kind of degree $< (2n+2)$ connects two compound perpetuant syzygies of the $(n-1)$-th kind that themselves resolve primary perpetuant syzygies of the $(n-2)$-th kind, *i.e.*, from Lemma 2, a perpetuant syzygy of the n-th kind of degree $< (2n+2)$ connects two *primary* compound perpetuant syzygies of the $(n-1)$-th kind.

We now have only to prove that such a perpetuant syzygy of the n-th kind is necessarily primary.

The perpetuant syzygy of the n-th kind in question is formed as follows :—

Subtract the two compound primary perpetuant syzygies of the $(n-1)$-th kind [of course, as these resolve the same perpetuant syzygies of the $(n-2)$-th kind, not more than one of them can be canonical], so that their difference is free from the doubly compound perpetuant syzygies of the $(n-2)$-th kind from which we started.

This difference is a perpetuant syzygy of the $(n-1)$-th kind that resolves another doubly compound perpetuant syzygy of the $(n-2)$-th kind.

But this last must be resolvable by a member of the canonical set [for, if not, the difference of a canonical and an uncanonical syzygy is one linearly independent of the canonical set, which is absurd, as by definition

all syzygies of a given kind and degree are expressible linearly in terms of the canonical set].

We subtract this canonical perpetuant syzygy of the $(n-1)$-th kind, so that the resulting difference (if not zero) resolves yet another doubly compound primary perpetuant syzygy of the $(n-2)$-th kind.

Continue this process as far as possible.

Now since the number of perpetuant products of given degree and weight is necessarily finite, the same is true of the perpetuant syzygy of the first kind connecting them, and therefore of the second kind connecting these, and so on; therefore for perpetuant syzygies of the $(n-2)$-th kind.

Thus the above process of subtraction must stop in time. But we have shown that we cannot come to an irresolvable, and whenever a resolvable appears we go on.

Therefore finally the difference is zero.

Each perpetuant syzygy of the $(n-1)$-th kind involved must be *primary*, as it resolves a *primary* perpetuant syzygy of the $(n-2)$-th kind, and as this last is *doubly compound*, the perpetuant syzygy of the $(n-1)$-th kind must be *compound*, for an irreducible primary syzygy resolves only *singly compounds*.

Thus we have formed the perpetuant syzygy of the n-th kind in question and shown that it connects only *primary* compound perpetuant syzygies of the $(n-1)$-th kind.

Therefore, by Lemma 2, it is itself primary.

This completes the induction.

But the theorem is true for kind 1, for the only perpetuant syzygy of the first kind of degree < 4 is the Jacobian identity.

Therefore it is true universally that *all perpetuant syzygies of the n-th kind of degree* $< (2n+2)$ *are primary.*

XI. *Note.*—In what follows we shall use the symbol $\left(\begin{matrix}\delta\\ \lambda_1\lambda_2\ldots\lambda_p\end{matrix}\right)$ to denote the number of ways of picking, from δ things, a group of p subgroups, containing $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ things respectively. If of the p λ's, r are equal to one integer, s to another, t to yet another, &c., and the rest are all unequal, then clearly

$$\left(\begin{matrix}\delta\\ \lambda_1\lambda_2\ldots\lambda_p\end{matrix}\right) = \frac{1}{r!\,s!\,t!\ldots}\,\frac{\delta!}{\lambda_1!\,\lambda_2!\ldots\lambda_p!\left(\delta-\sum_1^p\lambda_\kappa\right)!}.$$

We shall also, to save space, denote the expression

$$\frac{x^N}{(1-x)^{\sum_1^p\lambda_p-p}}$$

where $$N = \sum_{1}^{p} 2^{\lambda_r - 1} - p + 1,$$

by the symbol $F(\lambda_1 \lambda_2 \dots \lambda_p)$; and make frequent use of the identity

$$\frac{x^{2^{\lambda_1 - 1} - 1}}{(1-x)^{\lambda_1 - 1}} F(\lambda_2 \dots \lambda_p) = F(\lambda_1 \lambda_2 \dots \lambda_p).$$

"Well Ordered" Syzygies. Their Generating Functions.

To distinguish one particular primary perpetuant syzygy of the n-th kind from others of the same general form $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$, we shall sometimes use a different notation. Thus

$$J^{(n)}[(a_1 a_{n+3})^{\rho_1} (a_2 a_{n+4})^{\rho_2} (a_1 a_2 a_3 \dots a_{n+2})]$$

will be used to denote the result of symbolical multiplication of the perpetuant syzygies of the n-th kind $(a_1 a_2 a_3 \dots a_{n+2})$ by $(a_1 a_{n+3})^{\rho_1} (a_2 a_{n+4})^{\rho_2}$.

This is of form $S^{(n)}(2, 2)$.

If in $$J^{(n)}[(a_r a_{r_1})^{\rho_{r_1}} (a_r a_{r_2})^{\rho_{r_2}} (a_r a_{r_3})^{\rho_{r_3}} \dots (a_s a_{s_1})^{\rho_{s_1}} \dots (a_t a_{t_1})^{\rho_{t_1}} \dots (a_r a_s a_t \dots)],$$

where the letters are $a_1 a_2 \dots a_\delta$ in some order

$$r < r_1 < r_2 < \dots,$$
$$s < s_1 < s_2 < \dots,$$
$$t < t_1 < t_2 < \dots,$$

we call the syzygy *well ordered.*

Consider $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ a *well ordered* primary perpetuant syzygy of the n-th kind. We shall prove by induction that, given

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_p \geqslant 2,$$

the *generating function* for all such well ordered ones is

$$\binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p} F(\lambda_1 \lambda_2 \dots \lambda_p),$$

provided $\delta < (2n+2)$.

Assume this for kinds $< n$.

(I) If $\lambda_1 >$ any other λ, $S^{(n)}$ resolves $C^{\lambda}_{\lambda_1} S^{(n-1)}(\lambda_2 \dots \lambda_p)$, where $S^{(n-1)}$ is also well ordered.

Conversely, $C_{\lambda_1} S^{(n-1)}(\lambda_2 \dots \lambda_p)$ is always resolvable by a well ordered perpetuant syzygy of the n-th kind if $S^{(n-1)}$ be well ordered, and $\lambda_1 >$ any

other λ; therefore the generating function for $S^{(n)}$ is the same as that for the compound perpetuant syzygy of the $(n-1)$-th kind, *i.e.*,

$$\binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2 \dots \lambda_p} F(\lambda_2 \dots \lambda_p) = \binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p} F(\lambda_1 \lambda_2 \dots \lambda_p).$$

(II) If $\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$, we have κ λ's equal to λ (say). $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ will have κ terms of form

$$C_\lambda S^{(n-1)}[\lambda\lambda \dots \text{to } (\kappa-1) \text{ terms} \dots \lambda_p].$$

Let us arrange the κ C_λ's according to our fixed sequence, and call the first C_{λ_1}, the second C_{λ_2}, and so on.

With this convention $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ still resolves $C_{\lambda_1} S^{(n-1)}(\lambda_2 \lambda_3 \dots \lambda_p)$, a well ordered compound perpetuant syzygy of the $(n-1)$-th kind, and conversely, if this last is well ordered, it is always resolvable by $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$, a well ordered perpetuant syzygy of the n-th kind.

However $C_{\lambda_r} S^{(n-1)}(\lambda_1 \lambda_2 \dots \lambda_{r-1} \lambda_{r+1} \dots \lambda_p)$ is clearly not so resolvable if $r > 1$.

Therefore the generating function for $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ is the same as that for $C_{\lambda_1} S^{(n-1)}(\lambda_2 \dots \lambda_p)$, *i.e.*,

$$\frac{1}{\kappa} \binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2 \dots \lambda_p} F(\lambda_2 \dots \lambda_p)$$

$$= \frac{1}{\kappa} \binom{\delta}{\lambda_1} \binom{\delta-\lambda_1}{\lambda_2 \dots \lambda_p} F(\lambda_1 \lambda_2 \dots \lambda_p)$$

$$= \frac{1}{\kappa} \frac{\delta!}{\lambda_1!(\delta-\lambda_1)!} \frac{1}{(\kappa-1)!\, s!\, t! \dots} \frac{(\delta-\lambda_1)!}{\lambda_2! \lambda_3! \dots \lambda_p! \left(\delta - \sum_1^p \lambda_r\right)!} F(\lambda_1 \lambda_2 \dots \lambda_p)$$

$$= \binom{\delta}{\lambda_1 \lambda_2 \dots \lambda_p} F(\lambda_1 \lambda_2 \dots \lambda_p).$$

Therefore in both cases the generating function for well ordered perpetuant syzygies of the n-th kind of form $S^{(n)}(\lambda_1 \lambda_2 \dots \lambda_p)$ is this expression.

So if this formula holds good for kind $(n-1)$, it holds for kind n.

Now the formula can, by hypothesis, only fail for kind $(n-1)$ when degree $\geqslant 2n$.

Therefore the formula can only fail for kind n when degree $\geqslant 2n+\lambda_1$, *i.e.*, when degree $\geqslant (2n+2)$ [for, if $\lambda_1 = 1$ degree of $S^{(n-1)}$ must, by

Lemma III, be $(n+1)$, and in this case formula does not fail, so for failure $\lambda_1 \geqslant 2$].

Finally the formula holds good for perpetuant syzygies of the first kind of degree 3 (*i.e.* < 4) ; and therefore the induction shows that the generating function for $S^{(n)}(\lambda_1\lambda_2\lambda_3 \dots \lambda_p)$, if well ordered and of degree $\delta < (2n+2)$, is always

$$\left(\begin{matrix}\delta \\ \lambda_1\lambda_2\lambda_3 \dots \lambda_p\end{matrix}\right) F(\lambda_1\lambda_2\lambda_3 \dots \lambda_p).$$

If $\delta \geqslant (2n+2)$, in some cases the syzygies of low weight degenerate. This is due ultimately to the fact that the perpetuant syzygy of the first kind $J^{(1)}[(a_1a_4)^\rho (a_1a_2a_3)]$ and $J^{(1)}[(a_2a_3)^\rho (a_2a_1a_4)]$ are equal when $\rho = 1$. In consequence the generating function, when expanded in powers of x, has some of the coefficients at the beginning less than those given by the above formula.

We can obtain our linearly independent sets, as linear functions of the members of which any other perpetuant syzygy of the n-th kind can be expressed, by taking all primary perpetuant syzygies of the n-th kind [*of degree* $< (2n+2)$] *to be well ordered.*

The proof is this :—

It is clear that any well ordered perpetuant syzygy of the n-th kind resolves a compound well ordered perpetuant syzygy of the $(n-1)$-th kind, which in turn resolves a well ordered doubly compound perpetuant syzygy of the $(n-2)$-th kind. Thus every well ordered perpetuant syzygy of the n-th kind arises from a well ordered doubly compound perpetuant syzygy of the $(n-2)$-th kind. We must prove the converse of this to justify the statement at the head of the paragraph. To do this, we find the generating function for perpetuant syzygies of the n-th kind arising from well ordered doubly compound perpetuant syzygies of the $(n-2)$-th kind, and notice that they are the same as those for well ordered perpetuant syzygies of the n-th kind.

Unless $S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$ is of degree n, for perpetuant syzygies of the n-th kind to arise from $C_{\lambda_1}C_{\lambda_2}S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$, λ_1 and λ_2 must, save in one case, be $\geqslant 2$. Therefore if degree of $S^{(n)}$ be $< (2n+2)$, that of $S^{(n-2)}$ is $< (2n-2)$. The one exception is when $S^{(n-2)}$ is of degree n, which is also $< (2n-2)$. Therefore in every case we may use the generating function found above.

Consider then $C_{\lambda_1}C_{\lambda_2}S^{(n-2)}(\lambda_3\lambda_4 \dots \lambda_p)$, where $\lambda_3 \geqslant \lambda_4 \geqslant \dots \geqslant \lambda_p \geqslant 2$ and $S^{(n-2)}$ is well ordered.

The following cases must be distinguished:—

(I) $\lambda_1 > \lambda_2 >$ any other λ.

(II) $\lambda_1 = \lambda_2 >$ any other λ.

(III) $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3 = \lambda_4 = \ldots = \lambda_\kappa > \lambda_{\kappa+1}$.

(IV) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \ldots = \lambda_\kappa > \lambda_{\kappa+1}$.

(V) One (or both) of λ_1 and λ_2 less than the greatest of the other λ's.

(I) $\lambda_1 > \lambda_2 >$ any other λ.

$C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \ldots \lambda_p)$ is *always* resolvable in two ways, *i.e.*, by $C_{\lambda_1} S^{(n-1)}(\lambda_2 \lambda_3 \ldots \lambda_p)$ and by $C_{\lambda_2} S^{(n-1)}(\lambda_1 \lambda_3 \ldots \lambda_p)$. Therefore the generating function for perpetuant syzygies of the n-th kind is the same as that for $C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \ldots \lambda_p)$, *i.e.*,

$$\binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1}-1}-1}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2} \frac{x^{2^{\lambda_2}-1}-1}{(1-x)^{\lambda_2-1}} \binom{\delta-\lambda_1-\lambda_2}{\lambda_3 \ldots \lambda_p} F(\lambda_3 \ldots \lambda_p)$$
$$= \binom{\delta}{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p),$$

which is exactly that for well ordered perpetuant syzygies of the n-th kind of form $S^{(n)}(\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p)$.

(II) $\lambda_1 = \lambda_2 >$ any other λ.

Again, $C_\lambda^2 S^{(n-2)}(\lambda_3 \ldots \lambda_p)$ is always resolvable in two ways. Therefore the generating function for perpetuant syzygies of the n-th kind is

$$\frac{1}{2} \binom{\delta}{\lambda_1} \binom{\delta-\lambda_1}{\lambda_2} \binom{\delta-\lambda_1-\lambda_2}{\lambda_3 \ldots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p) \quad \left[\frac{1}{2}, \text{ since } \lambda_1 = \lambda_2\right]$$
$$= \frac{1}{2!} \frac{\delta!}{\lambda_1! \lambda_2! \lambda_3! \ldots \lambda_p! \left(\delta - \sum_1^p \lambda_\kappa\right)!} F(\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p)$$
$$= \binom{\delta}{\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p} F(\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_p), \quad \text{as } \lambda_1 = \lambda_2.$$

This is also the generating function for well ordered perpetuant syzygies of the n-th kind.

(III) $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3 = \lambda_4 = \ldots = \lambda_\kappa > \lambda_{\kappa+1}$.

$C_{\lambda_1} C_{\lambda_2} S^{(n-2)}(\lambda_3 \ldots \lambda_p)$ is always resolvable by $C_{\lambda_2} S^{(n-1)}(\lambda_1 \lambda_3 \ldots \lambda_p)$, but by $C_{\lambda_1} S^{(n-1)}(\lambda_2 \lambda_3 \ldots \lambda_p)$ only if C_{λ_2} happens to precede C_{λ_3}, C_{λ_4}, ..., C_{λ_p}, in our fixed sequence.

Therefore the generating function for perpetuant syzygies of the n-th kind equals the generating function for these extra resolutions

$$= \binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2\lambda_3\dots\lambda_p} F(\lambda_2\lambda_3\dots\lambda_p)$$

$$= \binom{\delta}{\lambda_1\lambda_2\lambda_3\dots\lambda_p} F(\lambda_1\lambda_2\lambda_3\dots\lambda_p) \text{ (as } \lambda_1 > \text{any other } \lambda)$$

= generating function for the well ordered ones.

(IV) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \dots = \lambda_\kappa > \lambda_{\kappa+1}$.

$S^{(n)}(\lambda_1\lambda_2\dots\lambda_p)$ has κ terms of form $C_\lambda S^{(n-1)}[\lambda\lambda\dots$ to $(\kappa-1)$ terms $\dots\lambda_p]$.

Arrange these in accordance with our sequence, and call the first $C_{\lambda_1}S^{(n-1)}$, the second $C_{\lambda_2}S^{(n-2)}$, and so on.

With this convention, i and j being both $< \kappa$,

$$C_{\lambda_i}C_{\lambda_j}S^{(n-2)}[\lambda \dots \text{ to } (\kappa-2) \text{ terms } \dots \lambda_p]$$

is, in general, irresolvable.

If $i = 1$, it is resolvable by $C_{\lambda_j}S^{(n-1)}(\lambda_1\lambda_2\dots\lambda_{j-1}\lambda_{j+1}\dots\lambda_p)$, but in no other way, unless $j = 2$, when it is resolvable also by $C_{\lambda_1}S^{(n-1)}(\lambda_2\lambda_3\dots\lambda_p)$.

The generating function for perpetuant syzygies of the n-th kind is that for these extra resolutions

$$= \frac{1}{\kappa}\binom{\delta}{\lambda_1} \frac{x^{2^{\lambda_1-1}-1}}{(1-x)^{\lambda_1-1}} \binom{\delta-\lambda_1}{\lambda_2\lambda_3\dots\lambda_p} F(\lambda_2\lambda_3\dots\lambda_p)$$

$$= \frac{1}{\kappa} \frac{\delta!}{\lambda_1!\,(\delta-\lambda_1)!} \frac{1}{(\kappa-1)!\,s!\,t!\dots} \frac{(\delta-\lambda_1)!}{\lambda_2!\,\lambda_3!\dots\lambda_p!\left(\delta-\sum_1^p \lambda_r\right)!} F(\lambda_1\lambda_2\dots\lambda_p)$$

$$= \frac{1}{\kappa!\,s!\,t!\dots} \frac{\delta!}{\lambda_1!\,\lambda_2!\dots\lambda_p!\left(\delta-\sum_1^p \lambda_r\right)!} F(\lambda_1\lambda_2\dots\lambda_p)$$

$$= \binom{\delta}{\lambda_1\lambda_2\dots\lambda_p} F(\lambda_1\lambda_2\dots\lambda_p),$$

the same as for well ordered ones.

(V) If one or both of λ_1 and λ_2 be less than the greatest of the remaining λ's, it is clear that $C_{\lambda_1}C_{\lambda_2}S^{(n-2)}(\lambda_3\dots\lambda_p)$ can never be resolvable in more than one way.

Therefore there are no perpetuant syzygies of the n-th kind arising here.

There are also no well ordered perpetuant syzygies of the n-th kind of

the form $S^{(n)}(\lambda_1\lambda_2\lambda_3 \dots \lambda_p)$, where λ_1 or λ_2 is less than a following λ, by the convention with which we started.

To sum up, the canonical set of perpetuant syzygies of the n-th kind of any degree $< (2n+2)$ *may be taken to be composed solely of well ordered primary ones (irreducible and compound).*

Of course not *all* the compounds of degree δ of well ordered primary perpetuant syzygies of the n-th kind of degrees $< \delta$ belong to the canonical set. Those that do may be distinguished thus :—

Let $C_{\lambda_{i+1}}C_{\lambda_{i+2}} \dots C_{\lambda_p}$ be the perpetuants that are symbolical factors of a well ordered primary perpetuant syzygy of the n-th kind $S^{(n)}(\lambda_{i+1} \dots \lambda_p)$.

Then $C_{\lambda_1} \dots C_{\lambda_i} S^{(n)}(\lambda_{i+1} \dots \lambda_p)$, of degree δ $[< (2n+2)]$, belongs to the canonical set for kind n and degree δ if, in the fixed sequence, no one of $C_{\lambda_1} \dots C_{\lambda_i}$ precede the first of $C^{\lambda}{}_{i+1} \dots C_{\lambda_p}$.

XII. *Formation of Non-Primary Perpetuant Syzygies of the n-th Kind of Degree* $(2n+2)$.

Let large letters $S_1 S_2 \dots S_n$ denote perpetuant syzygies of the first kind, and small letters $s_1 s_2 \dots s_n$ the corresponding sums of perpetuant products.

The S's are such that no one has more than one symbolical letter in common with all the others put together.

Denote by $s_1 S_2$ the result of symbolical multiplication of S_2 by s_1. This will be a sum of *compound* perpetuant syzygies of the first kind, as s_1 and S_2 have only one symbol in common.

Then it is clear that $(s_1 S_2 - s_2 S_1)$ is a perpetuant syzygy of the second kind.

This may be written $\begin{vmatrix} s_1 & S_1 \\ s_2 & S_2 \end{vmatrix}$.

Similarly $s_1(s_2 S_3 - s_3 S_2) + s_2(s_3 S_1 - s_1 S_3) + s_3(s_1 S_2 - s_2 S_1)$

is a perpetuant syzygy of the third kind, since it is a linear function of compound perpetuant syzygies of the second kind that vanishes when expressed as a linear function of compound perpetuant syzygies of the first kind. This may be written

$$\begin{vmatrix} s_1 & s_1 & S_1 \\ s_2 & s_2 & S_2 \\ s_3 & s_3 & S_3 \end{vmatrix}.$$

In general, if we denote the vanishing determinant

$$\begin{vmatrix} s_1 & s_1 & \dots & s_1 & S_1 \\ s_2 & & & & \\ \vdots & & & & \\ s_n & & & & S_n \end{vmatrix}$$

by $$[(s_1 s_2 \dots s_{n-1} S_n)],$$

then expanding it in terms of the constituents of the first column and their first minors, we get

$$\Sigma (-1)^{p+1} s_p [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)],$$

which must vanish identically if we replace the first minors by *their* expansions in terms of the constituents of *their* first columns and the corresponding first minors, since the original determinant will be now expanded in terms of the constituents of its first two columns, which are identical.

Assume that $[(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{q-1} s_{q+1} \dots s_{n-1} S_n)]$ is a perpetuant syzygy of the $(n-2)$-th kind, and $[(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)]$ is a perpetuant syzygy of the $(n-1)$-th kind.

Then, since all the s's are sums of perpetuant products, and no s has more than one letter common with all the others put together, it follows that $s_p s_q [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{q-1} s_{q+1} \dots s_{n-1} S_n)]$ is a sum of doubly compound perpetuant syzygies of the $(n-1)$-th kind, and

$$s_p [(s_1 s_2 \dots s_{p-1} s_{p+1} \dots s_{n-1} S_n)]$$

is a sum of compound perpetuant syzygies of the $(n-1)$-th kind.

The vanishing of the sum of expansions of each term of the expansion of $[(s_1 \dots s_{n-1} S_n)]$ shows that this last is a perpetuant syzygy of the n-th kind.

Therefore the device gives a perpetuant syzygy of the n-th kind for kind n, if it does so for kinds $(n-1)$ and $(n-2)$.

But we have seen that it does so for kinds 2 and 3; therefore it does so in general.

We can take S_1 to be the Stroh syzygy $(a_1 a_2 a_4 a_3)_w$, and $S_r (r > 1)$ the Jacobian identity $(a_4 a_{2r+1} a_{2r+2})$.

In this case, since a perpetuant syzygy of the n-th kind that contains a non-primary compound perpetuant syzygy of the $(n-1)$-th kind is itself non-primary, $[(s_1 s_2 \dots S_n)]$ is non-primary if $[(s_1 s_2 \dots S_{n-1})]$ is.

But $[(s_1 S_2)]$ is clearly non-primary, as it contains a compound Stroh syzygy.

Therefore $[(s_1 s_2 \ldots S_n)]$ is a non-primary perpetuant syzygy of the n-th kind, of degree $(2n+2)$.

Thus non-primary perpetuant syzygies of the n-th kind do exist for degree $(2n+2)$, *although not for lower degrees.*

In conclusion, I wish to express my best thanks to Dr. A. Young for his kindness in suggesting the subject of syzygies to me, and also to him and to Mr. J. H. Grace for much helpful interest and encouragement in the course of the work.

THE TRANSFORMATIONS OF COORDINATES WHICH CAN BE USED TO TRANSFORM ONE PHYSICAL PROBLEM INTO ANOTHER

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1. *Introduction.*

In a previous communication* the invariance of the fundamental equations of the theory of electrons under a transformation of the space and time coordinates was investigated on the assumption that the transformation and the electromagnetic field to which it is applied are independent of one another. The transformations that are relevant are characterised by the invariance of the differential equation

$$(dx)^2+(dy)^2+(dz)^2-(dt)^2 = 0. \qquad (1)$$

This equation indicates that a disturbance originating at the point (x, y, z) at time t is propagated *in all directions* round this point with the velocity of light.† Now recent theories of radiation suggest that the radiation from a source is not emitted equally in all directions, but is concentrated along a finite number of narrow tubes issuing from the source.‡ It is natural then to enquire whether there are any transformations for which the fundamental equations are invariant, and for which the property that disturbances are propagated in *particular* directions with the velocity of light is also invariant. The present investigations show that this is the case, and that any transformation which is characterised by an equation of the type

$$\lambda(dx^2+dy^2+dz^2-dt^2)+\mu\,[(s_x dx+s_y dy+s_z dz)^2-s^2 dt^2]$$
$$= \lambda'(dx'^2+dy'^2+dz'^2-dt'^2)+\mu'[(s'_x dx'+s'_y dy'+s'_z dz')-s'^2 dt'^2], \qquad (2)$$

where $\qquad s_x^2+s_y^2+s_z^2 = s^2, \qquad s_x'^2+s_y'^2+s_z'^2 = s'^2,$

* These *Proceedings*, Ser. 2, Vol. 8, p. 223.

† The units of length and time have been chosen so that the velocity of radiation is represented by unity.

‡ J. J. Thomson, *Electricity and Matter*, London, 1904, p. 63; *Proc. Camb. Phil. Soc.*, Vol. XIV (1907), p. 417; *Phil. Mag.* (6), Vol. 19, p. 301; N. R. Campbell, *The New Quarterly* (1909).

and is such that the system of differential equations

$$\frac{dx}{s_x} = \frac{dy}{s_y} = \frac{dz}{s_z}$$

is transformed into

$$\frac{dx'}{s'_x} = \frac{dy'}{s'_y} = \frac{dz'}{s'_z},$$

can be used to transform any electrodynamical field, in which (s_x, s_y, s_z) are proportional to the components of Poynting's vector at the point (x, y, z) at time t. Since

$$s^2(dx^2+dy^2+dz^2)-(s_x dx+s_y dy+s_z dz)^2$$
$$= (s_y dz-s_z dy)^2+(s_z dx-s_x dz)^2+(s_x dy-s_y dx)^2,$$

the second condition may be replaced by an equation, such as

$$s^2(dx^2+dy^2+dz^2)-(s_x dx+s_y dy+s_z dz)^2$$
$$= \theta\,[s'^2(dx'^2+dy'^2+dz'^2)-(s'_x dx'+s'_y dy'+s'_z dz')^2],$$

and equation (2) reduced to the form

$$f^2(dx^2+dy^2+dz^2-dt^2)+g\,[(s_x dx+s_y dy+s_z dz)^2-s^2dt^2]$$
$$= dx'^2+dy'^2+dz'^2-dt'^2. \quad (2')$$

It is clear that if energy is propagated with the velocity of light in directions given by

$$\frac{dx}{s_x} = \frac{dy}{s_y} = \frac{dz}{s_z},$$

then the differential form on the left-hand side of (2′) vanishes, and it follows that there is a corresponding flow of energy with the velocity of light in the transformed field.

If we suppose that the only available means of measuring the distance between two consecutive points is the flow of energy, then it follows that we can only measure distances from the point (x, y, z) at time t in the direction of Poynting's vector.* The existence of large classes of transformations of the type (2) indicates, moreover, that we are quite unable to give an absolute description of any state of motion by means of electromagnetic measurements, because there are certain states of motion which are completely obscured by the observer's choice of a system of units. The interpretation to be given to a series of measurements is indeterminate, by reason of the fact that an infinite number of other possible interpretations may be obtained from a given one by applying any transformation of a certain group. This group of transformations is really a

* It is assumed throughout this paper that the flow of energy in an electromagnetic field takes place along Poynting's vector, and that free æther is the medium through which the energy is propagated.

composite one, for it is built up of transformations of the type (2), and the spherical wave transformations which change $dx'^2+dy'^2+dz'^2-dt'^2$ into $\lambda'[d\xi^2+d\eta^2+d\zeta^2-d\tau^2]$. The second group of transformations is independent of the existing electromagnetic field, but the first is determined by it. This suggests the very interesting problem of determining the relations which must be satisfied (if any) by the quantities (s_x, s_y, s_z) in order that a certain state of motion may be completely obscured.

It appears that the effect of a uniform motion of rotation on electromagnetic phenomena can only remain undetected on account of the existence of a transformation of type (2), if the electromagnetic field satisfies certain conditions, one of which is that the direction of the Poynting vector at any point should meet the axis of rotation. This question is considered in § 3.

Some simple examples of transformations of type (2) are given in §§ 2, 3, and the method of application is illustrated in the case of the reflection of plane waves of light at a mirror whose shape is a paraboloid of revolution.

The transformations of type (2) are not the only ones which possess the property that a flow of energy with the velocity of light in the original field corresponds to a flow of energy with the velocity of light in the transformed field.

When the electrodynamical field consists of a series of separate pulses travelling through the æther, the electric and magnetic forces (E_x, E_y, E_z), (H_x, H_y, H_z) in the pulses are connected by the relations

$$E_xH_x+E_yH_y+E_zH_z = 0,$$

$$E_x^2+E_y^2+E_z^2 = H_x^2+H_y^2+H_z^2,$$

and the transformations which can be used are of a more general character, the equation analogous to (2) being now of the form

$$f^2(dx^2+dy^2+dz^2-dt^2)+2(a_x dx+a_y dy+a_z dz+a\,dt) \times (s_x dx+s_y dy+s_z dz-s\,dt) = dx'^2+dy'^2+dz'^2-dt'^2. \quad (3)$$

In this case the propagation of energy is in the one direction given by

$$\frac{dx}{s_x} = \frac{dy}{s_y} = \frac{dz}{s_z} = \frac{dt}{s},$$

where, as before, $$s_x^2+s_y^2+s_z^2 = s^2.$$

The transformation of Laplace's equation and the equation of wave motion depend upon somewhat similar considerations. It may be shown

that any transformation which is such that

$$dx'^2+dy'^2+dz'^2 = \lambda(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2, \qquad (4)$$

where $$\kappa^4 = \lambda-p^2-q^2-r^2$$

can be used to transform any solution of

$$\frac{\partial}{\partial x}\left(\kappa^2\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa^2\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa^2\frac{\partial V}{\partial z}\right)=0,$$

which satisfies the condition

$$p\frac{\partial V}{\partial x}+q\frac{\partial V}{\partial y}+r\frac{\partial V}{\partial z}=0, \qquad (5)$$

into a solution of $$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=0.$$

If κ is a solution of Laplace's equation it is clear that κV is also a solution of Laplace's equation. The case when $\kappa = 1$ is of special interest, because then any solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which satisfies the condition (5), is transformed into a solution of

$$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=0.$$

The condition (5) implies that the direction (p, q, r) is perpendicular to the direction of the electric force, and it is possible to regard (p, q, r) as the direction in which energy flows from and to the point (x, y, z) if the electrostatic state corresponds to a state of steady motion instead of a state of equilibrium.

In some cases it appears that the relation

$$\frac{\partial^2\kappa}{\partial x^2}+\frac{\partial^2\kappa}{\partial y^2}+\frac{\partial^2\kappa}{\partial z^2}=0$$

is a consequence of the identical relation (4). This result may be established by writing down the conditions that the quadratic differential form on the right-hand side of (4) can be expressed in the form on the left.*

* In the notation used in the theory of the invariants of quadratic differential forms the conditions are of the type $(rk, ih) = 0$ for all combinations of the indices i, k, h, r. Cf. J. E. Wright's *Invariants of Quadratic Differential Forms*, p. 10.

The author has succeeded in verifying the result in the case when

$$p^2+q^2+r^2=0 \quad \text{and} \quad p\,dx+q\,dy+r\,dz$$

is a perfect differential.

In the case of the equation of wave motion

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial t^2},$$

the transformations which are specified by an identical relation of the type

$$f^2(dx^2+dy^2+dz^2-dt^2)+(a_x dx+a_y dy+a_z dz+a\,dt)(s_x dx+s_y dy+s_z dz-s\,dt) = dx'^2+dy'^2+dz'^2-dt'^2, \quad (6)$$

can be used to transform a function V which satisfies the equations

$$\left.\begin{aligned} 0&=\frac{\partial}{\partial x}\left(\kappa\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa\frac{\partial V}{\partial z}\right)-\frac{\partial}{\partial t}\left(\kappa\frac{\partial V}{\partial t}\right)\\ 0&=a_x\frac{\partial V}{\partial x}+a_y\frac{\partial V}{\partial y}+a_z\frac{\partial V}{\partial z}+a\frac{\partial V}{\partial t}\\ 0&=s_x\frac{\partial V}{\partial x}+s_y\frac{\partial V}{\partial y}+s_z\frac{\partial V}{\partial z}-s\frac{\partial V}{\partial t}\end{aligned}\right\} \quad (7)$$

into a solution of $\dfrac{\partial^2 V}{\partial x'^2}+\dfrac{\partial^2 V}{\partial y'^2}+\dfrac{\partial^2 V}{\partial z'^2}-\dfrac{\partial^2 V}{\partial t'^2}=0.$

The quantity κ is given by the equation

$$\kappa=\frac{\sqrt{(-\Delta)}}{f^2},$$

where $\sqrt{\Delta}$ is the determinant of the quadratic differential form on the left-hand side of (6).

If $$s_x^2+s_y^2+s_z^2=s^2,$$

we have $$\Delta=-f^4\left[as+a_x s_x+a_y s_y+a_z s_z+f^2\right]^2,$$

and so $$\kappa=f^2+a_x s_x+a_y s_y+a_z s_z+as.$$

If κ is a constant the function V is a solution of the equation of wave motion in both systems of coordinates. If $\sqrt{\kappa}$ is a solution of the equation of wave motion, then equation (7) implies that $\sqrt{\kappa}\,V$ is a solution of the equation of wave motion.

An interesting case occurs when

$$\frac{a_x}{s_x}=\frac{a_y}{s_y}=\frac{a_z}{s_z}=-\frac{a}{s},$$

so that the fundamental equation is of the form

$$f^2(dx^2+dy^2+dz^2-dt^2)+(s_x dx+s_y dy+s_z dz-s\,dt)^2 = dx'^2+dy'^2+dz'^2-dt'^2. \quad (8)$$

The function V is then subject to only one condition of the type

$$s_x\frac{\partial V}{\partial x}+s_y\frac{\partial V}{\partial y}+s_z\frac{\partial V}{\partial z}-s\frac{\partial V}{\partial t}=0,$$

and

$$\kappa = f^2+s_x^2+s_y^2+s_z^2-s^2.$$

The case in which κ is a constant is again of special interest.

[*August 17th*, 1910.—The case in which $s_x^2+s_y^2+s_z^2=s^2$ is also of considerable interest, for then the transformation can be used to transform an electromagnetic field into another, provided the components $(E_xE_yE_z)$, $(H_xH_yH_z)$ of the electric and magnetic vectors at any point are connected with s_x, s_y, s_z by relations of the form

$$s\,(sE_x+s_yH_z-s_zH_y) = s_x(s_xE_x+s_yE_y+s_zE_z),$$
$$s\,(sE_y+s_zH_x-s_xH_z) = s_y(s_xE_x+s_yE_y+s_zE_z),$$
$$s\,(sE_z+s_xH_y-s_yH_x) = s_z(s_xE_x+s_yE_y+s_zE_z).$$

There are really only two independent relations, and it is easy to see that we have also

$$s\,(sH_x-s_yE_z+s_zE_y) = s_x(s_xE_x+s_yE_y+s_zE_z).$$

If

$$s_x dx+s_y dy+s_z dz-s\,dt = s'_x dx'+s'_y dy'+s'_z dz'-s'dt,$$

it can be shown that

$$s'^2_x+s'^2_y+s'^2_z = s'^2,$$

and that the vector (s'_x, s'_y, s'_z) plays the same part in the transformed field as (s_x, s_y, s_z) does in the original one.

It is interesting to notice that, if s_x, s_y, s_z are connected with the electric and magnetic vectors by the above relations, there is an integral invariant of the form

$$V(s_x dx+s_y dy+s_z dz-s\,dt),$$

for the whole group of spherical wave transformations.]

These transformations appear to be the only types of point transformations for which the equation of wave motion is invariant. There is, however, another class of point transformations which enable us to derive

solutions of the equation of wave motion from solutions of Laplace's equation; these transformations are specified by an identical relation of the type

$$(p^2+q^2+r^2-s^2)(dx^2+dy^2+dz^2-dt^2)-(p\,dx+q\,dy+r\,dz+s\,dt)^2 = dx'^2+dy'^2+dz'^2,$$

and have been discussed elsewhere.*

In the case of the transformations of type (6), it may be shown that any function U which satisfies the equations

$$\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2=\left(\frac{\partial U}{\partial t}\right)^2,$$

$$a_x\frac{\partial U}{\partial x}+a_y\frac{\partial U}{\partial y}+a_z\frac{\partial U}{\partial z}+a\frac{\partial U}{\partial t}=0,$$

$$s_x\frac{\partial U}{\partial x}+s_y\frac{\partial U}{\partial y}+s_z\frac{\partial U}{\partial z}-s\frac{\partial U}{\partial t}=0,$$

is also a solution of

$$\left(\frac{\partial U}{\partial x'}\right)^2+\left(\frac{\partial U}{\partial y'}\right)^2+\left(\frac{\partial U}{\partial z'}\right)^2=\left(\frac{\partial U}{\partial t'}\right)^2.$$

These transformations appear to be the general ones for which the last equation is invariant. The contact transformations for which the equation is invariant have been considered by E. O. Lovett.†

In the present paper the analysis is given only for the transformations connected with the electrodynamical equations, and it has been thought sufficient to merely state the results for the transformation of Laplace's equation and the equation of wave motion. Simple examples of the transformations have been found in the former case, but not in the latter, and we have preferred to give prominence to the transformations which can be illustrated by the discussion of a simple case.

In order to discover other transformations it will be necessary to make a profound study of the theory of the transformation of quadratic differential forms. This theory has already been developed to a considerable extent, but the present investigations indicate that there are many problems still to be solved.

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* *Camb. Phil. Trans.*, Vol. 21, pp. 257–280 (1910).

† *Camb. Phil. Trans.*, Vol. 10, 1899, p. 256.

In order to distinguish between an integral form of the second order and a quadratic differential form, it has been thought advisable to change the notation of the former and write

$$H_x d(y, z)+H_y d(z, x)+H_z d(x, y)+E_x d(x, t)+E_y d(y, t)+E_z d(z, t)$$

in place of

$$H_x dy\, dz+H_y dz\, dx+H_z dx\, dy+E_x dx\, dt+E_y dy\, dt+E_z dz\, dt,$$

and similarly for integral forms of a higher order.

The symbol $d(y, z)$ possesses the properties

$$d(y, z) = -d(z, y), \qquad d(x, x) = 0,$$

and according to the method of multiplication explained in my paper on "The Transformation of the Electrodynamical Equations"

$$d(y, z)\, d(x, t) = d(y, z, x, t),$$

$$d(y, z)\, d(z, t) = 0,$$

$$d(y, z)\, dx = d(y, z, x) = d(z, x, y) = d(x, y, z).$$

These rules of multiplication apply to all the symbols specified by the same kind of d. The coefficients of these symbols are multiplied together by the ordinary rules of multiplication, unless they are symbols of a like nature specified by an independent variational operator δ; in this case the coefficients are multiplied together by rules analogous to those given above.

2. *Transformations for which the Fundamental Equations of the Theory of Electrons are Invariant.*

It was pointed out in my former paper that the nature of the transformations for which the equations

$$\left.\begin{aligned} \operatorname{curl} H &= \frac{\partial D}{\partial t}+\rho w, \quad & \operatorname{div} D &= \rho \\ \operatorname{curl} E &= -\frac{\partial B}{\partial t}, & \operatorname{div} B &= 0 \end{aligned}\right\} \qquad \text{(I)}$$

are invariant, depends upon the form of the relations connecting the components of the vectors E, B, D, H. It was shown that if these rela-

tions are of the type

$$\left.\begin{aligned} \pm B_x\sqrt{(-\Delta)} = {} & (hw-vg)\,D_x+(bw-vf)\,D_y+(fw-cv)\,D_z \\ & -(bc-f^2)\,H_x-(fg-ch)\,H_y-(hf-bg)\,H_z \\ \pm E_x\sqrt{(-\Delta)} = {} & (ad-u^2)\,D_x+(hd-uv)\,D_y+(gd-uw)\,D_z \\ & -(hw-vg)\,H_x-(gu-aw)\,H_y-(av-hu)\,H_z \end{aligned}\right\}, \qquad \text{(II)}$$

where the coefficients are minors of the determinant

$$\Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix},$$

then a possible transformation is obtained by assuming the invariance of the integral forms

$$\left.\begin{aligned} & B_x d(y,z)+B_y d(z,x)+B_z d(x,y)+E_x d(x,t)+E_y d(y,t)+E_z d(z,t) \\ & D_x d(y,z)+D_y d(z,x)+D_z d(x,y)-H_x d(x,t)-H_y d(y,t)-H_z d(z,t) \\ & \rho\, d(x,y,z)-\rho w_x d(y,z,t)-\rho w_y d(z,x,t)-\rho w_z d(x,y,t) \end{aligned}\right\}, \qquad \text{(III)}$$

and the bilinear form

$$\begin{aligned} & a\,dx\,\delta x+b\,dy\,\delta y+c\,dz\,\delta z+d\,dt\,\delta t+f(dy\,\delta z+dz\,\delta y) \\ & \quad +g\,(dz\,\delta x+dx\,\delta z)+h(dx\,\delta y+dy\,\delta x)+u(dx\,\delta t+dt\,\delta x) \\ & \quad +v(dy\,\delta t+dt\,\delta y)+w(dz\,\delta t+dt\,\delta z). \end{aligned} \qquad \text{(IV)}$$

We shall now consider the question whether it is possible to choose the coefficients in this bilinear form in such a way that the equations (II) reduce to the form

$$B_x = H_x, \quad D_x = E_x, \quad \ldots .$$

This question may be solved in a general way by making these substitutions in equations (II), and choosing the coefficients $a, h, g, \ldots$, so that the resulting equations are satisfied. Instead of doing this we shall discuss a case of a simple nature in which the conditions are satisfied. This case does not correspond to the most general solution of the question, but it seems to be the one which admits of the simplest interpretation.

Let (s_x, s_y, s_z) be the components of Poynting's vector; these are connected with the components of the electric and magnetic forces (E_x, E_y, E_z),

(H_x, H_y, H_z) by the relations

$$\left.\begin{aligned} s_x E_x + s_y E_y + s_z E_z &= 0 \\ s_x H_x + s_y H_y + s_z H_z &= 0 \end{aligned}\right\}. \qquad (V)$$

We shall assume that the bilinear form which remains invariant is

$$\lambda(dx\,\delta x + dy\,\delta y + dz\,\delta z - dt\,\delta t) + \mu\left[(s_x dx + s_y dy + s_z dz)(s_x \delta x + s_y \delta y + s_z \delta z) - s^2 dt\,\delta t\right], \qquad (VI)$$

where $$s^2 = s_x^2 + s_y^2 + s_z^2.$$

Evaluating the determinant

$$\Delta = \begin{vmatrix} \lambda+\mu s_x^2 & \mu s_x s_y & \mu s_x s_z & 0 \\ \mu s_x s_y & \lambda+\mu s_y^2 & \mu s_y s_z & 0 \\ \mu s_x s_z & \mu s_y s_z & \lambda+\mu s_z^2 & 0 \\ 0 & 0 & 0 & -(\lambda+\mu s^2) \end{vmatrix},$$

we find that $$\Delta = -\lambda^2(\lambda+\mu s^2)^2.$$

Now
$$bc - f^2 = \lambda(\lambda+\mu s_y^2+\mu s_z^2), \quad hw - vg = 0,$$
$$fg - ch = -\lambda\mu s_x s_y, \quad bw - vf = 0,$$
$$hf - bg = -\lambda\mu s_x s_z, \quad fw - cv = 0;$$

hence the first of equations (II) becomes

$$\pm H_x\lambda(\lambda+\mu s^2) = -\lambda(\lambda+\mu s_y^2+\mu s_z^2)H_x + \lambda\mu s_x s_y H_y + \lambda\nu s_x s_z H_z,$$

and if the negative sign be taken, the equation is satisfied on account of the identities

$$s_x H_x + s_y H_y + s_z H_z = 0,$$
$$s_x^2 + s_y^2 + s_z^2 = s^2.$$

Again, if we substitute the values of $(ad-u^2)$, ..., in the second equation (II), and give the negative sign to the left-hand side, it can be shown in the same way that this equation is satisfied.

It follows from the invariance of the bilinear form (VI) that the quadratic form

$$\lambda(dx^2+dy^2+dz^2-dt^2)+\mu\left[(s_x dx + s_y dy + s_z dz)^2 - s^2 dt^2\right]$$

is an invariant. We may obtain a suitable transformation by assuming

that

$$\lambda(dx^2+dy^2+dz^2-dt^2)+\mu\,[(s_x dx+s_y dy+s_z dz)^2-s^2 dt^2]$$
$$=\lambda'(dx'^2+dy'^2+dz'^2-dt'^2)+\mu'\,[(s'_x dx'+s'_y dy'+s'_z dz')^2-s'^2 dt'^2].$$

Since the quantities λ, μ do not enter into the fundamental equations and relations, there may possibly be two relations of the above type with different sets of values of λ, μ, λ', μ', it is convenient to consider the case in which the two relations may be replaced by two of the type

$$f^2(dx^2+dy^2+dz^2-dt^2)+g\,[(s_x dx+s_y dy+s_z dz)^2-s^2 dt^2]$$
$$=dx'^2+dy'^2+dz'^2-dt'^2,$$

$$s^2(dx^2+dy^2+dz^2)-(s_x dx+s_y dy+s_z dz)^2$$
$$=\theta\,[s'^2(dx'^2+dy'^2+dz'^2)-(s'_x dx'+s'_y dy'+s'_z dz')^2].$$

The last equation then implies that a velocity in the direction given by

$$\frac{dx}{s_x}=\frac{dy}{s_y}=\frac{dz}{s_z}$$

corresponds to a velocity in the direction given by

$$\frac{dx'}{s'_x}=\frac{dy'}{s'_y}=\frac{dz'}{s'_z}.$$

As a first example, we shall suppose that the flow of energy is always parallel to the axis of x; an appropriate transformation is then obtained by putting

$$\lambda\,[dx^2+dy^2+dz^2-dt^2]+\mu[dx^2-dt^2]$$
$$=\lambda'[dx'^2+dy'^2+dz'^2-dt'^2]+\mu'[dx'^2-dt'^2].$$

This equation may be satisfied by taking

$$\left.\begin{aligned} x' &= f(x+t)+g(x-t)\\ t' &= f(x+t)-g(x-t)\\ y'+iz' &= \phi(y+iz)\\ y'-iz' &= \phi(y-iz)\end{aligned}\right\}, \qquad (A)$$

and it is clear that two independent sets of values of λ, μ, λ', μ' may be obtained.

The above transformation may be applied in particular to the large

class of electromagnetic fields specified by the potentials

$$\Phi = [G(x-t)-F(x+t)]\,P(y,\,z),$$

$$A_x = [F(x+t)+G(x-t)]\,P(y,\,z),$$

$$A_y = [Q(y+iz)+R(y-iz)]\,H(x,\,t),$$

$$-iA_z = [Q(y+iz)-R(y-iz)]H(x,\,t).$$

These satisfy the relation

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{\partial \Phi}{\partial t}=0,$$

and if $$E_x = -\frac{\partial A_x}{\partial t}-\frac{\partial \Phi}{\partial x}; \quad H_x = \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}, \quad \ldots,$$

the Poynting's vector is parallel to the axis of x.

The values of ρ, ρw_x, ρw_y, ρw_z are given by

$$-4\pi\rho = [G(x-t)-F(x+t)]\left(\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right),$$

$$-4\pi\rho w_x = [F(x+t)+G(x-t)]\left(\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right),$$

$$-4\pi\rho w_y = [Q(y+iz)+R(y-iz)]\left(\frac{\partial^2 H}{\partial x^2}-\frac{\partial^2 H}{\partial t^2}\right),$$

$$-4\pi i\rho w_z = [Q(y+iz)-R(y-iz)]\left(\frac{\partial^2 H}{\partial x^2}-\frac{\partial^2 H}{\partial t^2}\right).$$

In order that these formulæ may represent a real electromagnetic field which is due solely to negative electrons at rest and in motion, the inequalities

$$\rho^2 < \rho^2[w_x^2+w_y^2+w_z^2],$$

$$\Phi^2 < A_x^2+A_y^2+A_z^2,$$

must be satisfied, and $Q \equiv R$. Lorentz's transformation

$$x' = \frac{x-vt}{\sqrt{(1-v^2)}}, \qquad t' = \frac{t-vx}{\sqrt{(1-v^2)}}, \qquad y' = y, \qquad z' = z,$$

is obtained by making f, g and ϕ linear functions of their arguments.

A second transformation of considerable interest is given by

$$\left.\begin{aligned} x' &= \frac{x}{r}[f(t+r)+\psi(t-r)] \\ y' &= \frac{y}{r}[f(t+r)+\psi(t-r)] \\ z' &= \frac{z}{r}[f(t+r)+\psi(t-r)] \\ \pm t' &= f(t+r)-\psi(t-r)+\text{const.} \end{aligned}\right\}. \qquad \text{(B)}$$

In this case

$$dx'^2+dy'^2+dz'^2-dt'^2 = \lambda[dx^2+dy^2+dz^2-dt^2]+\mu[dr^2-dt^2],$$

where

$$\lambda = \frac{1}{r^2}[f(t+r)+\psi(t-r)]^2,$$

$$\mu = -4f'(t+r)\psi'(t-r)-\lambda,$$

and there is also a relation of the type

$$dx^2+dy^2+dz^2-dt^2 = \lambda'[dx'^2+dy'^2+dz'^2-dt'^2]+\mu'[dr'^2-dt'^2].$$

The formulæ connecting the electric and magnetic vectors in the two systems are obtained by putting

$$\begin{aligned} H_x = {} & H'_x\frac{\partial(y',z')}{\partial(y,z)}+H'_z\frac{\partial(z',x')}{\partial(y,z)}+H'_z\frac{\partial(x',y')}{\partial(y,z)} \\ & +E'_x\frac{\partial(x',t')}{\partial(y,z)}+E'_y\frac{\partial(y',t')}{\partial(y,z)}+E'_z\frac{\partial(z',t')}{\partial(y,z)}, \\ E_x = {} & H'_x\frac{\partial(y',z')}{\partial(x,t)}+H'_y\frac{\partial(z',x')}{\partial(x,t)}+H'_z\frac{\partial(x',y')}{\partial(x,t)} \\ & +E'_x\frac{\partial(x',t')}{\partial(x,t)}+E'_y\frac{\partial(y',t')}{\partial(x,t)}+E'_z\frac{\partial(z',t')}{\partial(x,t)}. \end{aligned}$$

Substituting the values of the Jacobians, we have

$$\begin{aligned} H_x = {} & \left(\lambda+\frac{y^2+z^2}{2r}\frac{\partial\lambda}{\partial r}\right)H'_x-\frac{xy}{2r}\frac{\partial\lambda}{\partial r}H'_y-\frac{xz}{2r}\frac{\partial\lambda}{\partial r}H'_z \\ & +\sqrt{\lambda}\frac{z}{r}\frac{\partial t'}{\partial r}E'_y-\sqrt{\lambda}\frac{y}{r}\frac{\partial t'}{\partial r}E'_z, \\ E_x = {} & \frac{1}{\sqrt{\lambda}}\left[\lambda\frac{\partial t'}{\partial t}+\frac{x^2}{2r}\frac{\partial(\lambda,t')}{\partial(r,t)}\right]E'_x+\frac{xy}{2r\sqrt{\lambda}}\frac{\partial(\lambda,t')}{\partial(r,t)}E'_y \\ & +\frac{xz}{2r\sqrt{\lambda}}\frac{\partial(\lambda,t')}{\partial(r,t)}E'_z-\tfrac{1}{2}z\frac{\partial\lambda}{\partial t}H'_y+\tfrac{1}{2}y\frac{\partial\lambda}{\partial t}H'_z. \end{aligned}$$

These give

$$xH_x+yH_y+zH_z = \sqrt{\lambda}\,[x'H'_x+y'H'_y+z'H'_z],$$

$$xE_x+yE_y+zE_z = \frac{1}{\sqrt{\lambda}}\left[\left(\frac{\partial t'}{\partial t}\right)^2-\left(\frac{\partial t'}{\partial r}\right)^2\right][x'E'_x+y'E'_y+z'E'_z],$$

and we have a verification that if

$$\frac{s_x}{x} = \frac{s_y}{y} = \frac{s_z}{z},$$

then

$$\frac{s'_x}{x'} = \frac{s'_y}{y'} = \frac{s'_z}{z'}.$$

Again,

$$dx' = \sqrt{\lambda}\,dx+x\,d(\sqrt{\lambda});$$

therefore

$$E_x dx+E_y dy+E_z dz$$

$$= \frac{1}{\sqrt{\lambda}}\,(E_x dx'+E_y dy'+E_z dz')$$

$$= \frac{\partial t'}{\partial t}\,(E'_x dx'+E_y dy'+E'_z dz')$$

$$+\frac{1}{2\sqrt{\lambda}}\,\frac{\partial \lambda}{\partial t}\,[dx'\,(yH'_z-zH'_y)+dy\,(zH'_x-xH'_z)+dz\,(xH'_y-yH'_x)]. \quad \text{(VII)}$$

Now let us suppose that the conditions

$$E'_xH'_x+E'_yH'_y+E'_zH'_z = 0,$$

$$x'E'_x+y'E'_y+z'E'_z = 0,$$

are satisfied, then

$$\frac{E'_x}{y'H'_z-z'H'_y} = \frac{E'_y}{z'H'_x-x''H'_z} = \frac{E'_z}{x'H'_y-y'H'_x},$$

or, since

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z},$$

$$\frac{E'_x}{yH'_z-zH'_y} = \frac{E'_y}{zH'_x-xH'_z} = \frac{E'}{xH'_y-yH'_x}.$$

The relation (VII) thus becomes

$$E_x dx+E_y dy+E_z dz = \theta\,[E'_x dx'+E'_y dy'+E'_z dz'],$$

where θ is some factor.

This relation indicates that if, in one system, the direction of the tube

of force issuing from an electron at the point (x, y, z) is at right angles to the direction of motion of the electron, and if the electric and magnetic vectors are at right angles, then the same holds true in the transformed system. The invariance of the relation

$$E'_x H'_x + E'_y H'_y + E'_z H'_z = 0$$

is a consequence of the invariance of the relation

$$E'_x d(y', z') + E'_y d(z', x') + E'_z d(x', y') - H'_x d(x', t') - H'_y d(y', t') - H'_z d(z', t'),$$

for if we multiply this invariant by itself by Grassmann's rule, we obtain the invariant

$$(E'_x H'_x + E'_y H'_y + E'_z H'_z)\, d(x', y', z', t').$$

J. J. Thomson* has shown that when an electron is moving with uniform velocity in a straight line the tubes of force tend to set themselves at right angles to the direction of motion. The invariance of the condition that this should occur for our group of transformations seems to indicate that the tube of force is at right angles to the direction of motion when the motion is of a more general character.

Two particular transformations of the type (B) deserve special mention,

If $$2f(t+r) = \frac{k^2}{t+r}, \qquad 2\psi(t-r) = \frac{k^2}{r-t},$$

the formulæ become

$$x' = \frac{k^2 x}{r^2 - t^2}, \qquad y' = \frac{k^2 y}{r^2 - t^2},$$

$$z' = \frac{k^2 z}{r^2 - t^2}, \qquad t' = \frac{k^2 t}{r^2 - t^2},$$

and we find that $\mu = 0$. Cunningham† has shown that any electrodynamical field may be transformed into another by means of this transformation.

Secondly, if

$$f(t+r) = -\frac{1}{2a} e^{-a(t+r)}, \qquad \psi(t-r) = \frac{1}{2a} e^{-a(t-r)},$$

* *Phil. Mag.*, February, 1910.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, 1910, p. 77.

the formulæ become

$$x' = xe^{-at}\frac{\sinh ra}{ra}, \quad y' = ye^{-at}\frac{\sinh ra}{ra}, \quad z' = ze^{-at}\frac{\sinh ra}{ra},$$

$$t' = \frac{1}{a}(1-e^{-at}) - \frac{1}{a}e^{-at}(\cosh ra - 1).$$

An approximate form of this transformation, viz.,

$$x' = xe^{-at}, \quad y' = ye^{-at}, \quad z' = ze^{-at},$$

$$at' = 1-e^{-at}\left(1+\frac{r^2a^2}{2}\right),$$

where a is small, has been used by Lorentz* to represent a uniform dilatation round the origin.

3. *Transformations of Various Types.*

We shall now endeavour to obtain an equation of the form

$$\lambda[dx^2+dy^2+dz^2-dt^2]+\mu[(s_x dx+s_y dy+s_z dz)^2-s^2dt^2] = dx'^2+dy'^2+dz'^2-dt'^2, \quad (1)$$

associated with a transformation which makes a system at rest correspond with a system having a uniform velocity of rotation.

Consider the transformation

$$x = \xi\cos\omega t+\eta\sin\omega t,$$

$$y = \xi\sin\omega t-\eta\cos\omega t,$$

$$z = \zeta,$$

where (ξ, η, ζ) are functions of (x', y', z') and not of t'.

If we substitute the values of dx, dy, dz, and express the condition that the resulting expression for dt'^2 should be a perfect square, we obtain an equation which can only exist if

$$\frac{s_x}{x} = \frac{s_y}{y}.$$

This means that the direction of the line of flow at any point must

* *K. Akad. van Wetenschappen te Amsterdam*, May 18th, 1901.

intersect the axis. It is also found that

$$dt' = \frac{[\lambda+\mu s^2-\lambda\omega^2(x^2+y^2)]\,dt-\mu\omega[s_x dx+s_y dy+s_z dz]}{[\lambda+\mu s^2-\lambda\omega^2(x^2+y^2)]^{\frac{1}{2}}};$$

hence this expression must be a perfect differential, also

$$dx'^2+dy'^2+dz'^2 = \lambda(dx^2+dy^2+dz^2)+\mu\,(s_x dx+s_y dy+s_z dz)^2.$$

It is only when λ and μ can be chosen so that these conditions are satisfied that the motion of rotation about the axis can be obscured on account of the existence of a transformation of the type (1).

A transformation in which the lines of flow in the two systems are normals to a series of parallel wave surfaces is obtained by putting

$$dx'^2+dy'^2+dz'^2-dV'^2 = \mu[dx^2+dy^2+dz^2-dV^2],$$

$$dV'^2-dt'^2 = \lambda\,[dV^2-dt^2],$$

where

$$\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2 = 1,$$

$$\left(\frac{\partial V}{\partial x'}\right)^2+\left(\frac{\partial V}{\partial y'}\right)^2+\left(\frac{\partial V}{\partial z'}\right)^2 = 1;$$

we then have

$$dx'^2+dy'^2+dz'^2-dt'^2 = \mu[dx^2+dy^2+dz^2-dt^2]+(\lambda-\mu)(dV^2-dt^2).$$

It should be noticed that the first equation may be written

$$\left(\frac{\partial V}{\partial y'}dz'-\frac{\partial V}{\partial z'}dy'\right)^2+\left(\frac{\partial V}{\partial z'}dx'-\frac{\partial V}{\partial x'}dz'\right)^2+\left(\frac{\partial V}{\partial x'}dy'-\frac{\partial V}{\partial y'}dx'\right)^2$$

$$= \mu\left[\left(\frac{\partial V}{\partial y}dz-\frac{\partial V}{\partial z}dy\right)^2+\left(\frac{\partial V}{\partial z}dx-\frac{\partial V}{\partial x}dz\right)^2+\left(\frac{\partial V}{\partial x}dy-\frac{\partial V}{\partial y}dx\right)^2\right]$$

and this shows that the system of differential equations

$$\frac{dx}{\frac{\partial V}{\partial x}} = \frac{dy}{\frac{\partial V}{\partial y}} = \frac{dz}{\frac{\partial V}{\partial z}}$$

is transformed into the system

$$\frac{dx'}{\frac{\partial V}{\partial x'}} = \frac{dy'}{\frac{\partial V}{\partial y'}} = \frac{dz'}{\frac{\partial V}{\partial z'}}.$$

The transformation can be used to solve problems in the reflection and

refraction of light when the orthotomic surfaces before and after incidence are known. The arbitrary functions occurring in the transformation must be chosen so that the two sets of values (x, y, z, t), (x', y', z', t') agree at points on the reflecting or refracting surface.

As an example, we shall consider the problem of the reflection of plane waves of light at a mirror whose reflecting surface is a paraboloid of revolution. We shall suppose that the waves are propagated in a direction parallel to the axis of the paraboloid, and that this direction is chosen for the axis of x.

Taking the focus of the mirror for origin, it is clear that a transformation for which

$$dy'^2+dz'^2 = \lambda[dx^2+dy^2+dz^2-dr^2],$$

$$dx'^2-dt'^2 = \mu[dr^2-dt^2],$$

is suitable for the problem in hand, since the plane waves are transformed into spherical waves having their centre at the focus.

These equations may be satisfied by taking

$$\pm x' = f(r+t)+g(r-t),$$

$$t' = f(r+t)-g(r-t),$$

$$y'+iz' = F\left[\log\tan\frac{\theta}{2}+i\phi\right],$$

$$y'-iz' = F\left[\log\tan\frac{\theta}{2}-i\phi\right],$$

where $x = r\cos\theta$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi$.

We may satisfy the conditions that $x = x'$, $y = y'$, $z = z'$, $t = t'$, at the surface of the mirror by taking

$$F(\mu) = ae^{\mu}, \quad \phi = \phi', \quad r'\sin\theta' = a\tan\frac{\theta}{2}.$$

The last equation gives

$$2r\cos^2\frac{\theta}{2} = a \quad \text{or} \quad r+x = a,$$

and this is the equation of the mirror.

The conditions $x' = x$, $t' = t$ give

$$r-a = f(r+t)+g(r-t),$$

$$t = f(r+t)-g(r-t),$$

whence
$$f(r+t) = \tfrac{1}{2}(r+t-a),$$
$$g(r-t) = \tfrac{1}{2}(r-t-a).$$

The transformation is thus
$$x' = a-r, \qquad t' = t,$$
$$y' = \frac{ay}{r+x}, \qquad z' = \frac{az}{r+x}.$$

Taking a plane wave in which the electric and magnetic forces are given by
$$E'_x = 0, \qquad H'_x = 0,$$
$$E'_y = \psi(x'-t'), \qquad H'_y = -\chi(x'-t'),$$
$$E'_z = \chi(x'-t'), \qquad H'_z = \psi(x'-t'),$$

we may calculate the corresponding quantities in the transformed system by using the general formulæ for a transformation which transforms a right-handed system of axes into a left-handed system. These formulæ are
$$-H_x = H'_x\frac{\partial(y', z')}{\partial(y, z)} + H'_y\frac{\partial(z', x')}{\partial(y, z)} + H'_z\frac{\partial(x', y')}{\partial(y, z)}$$
$$+E'_x\frac{\partial(x', t')}{\partial(y, z)} + E'_y\frac{\partial(y', t')}{\partial(y, z)} + E'_z\frac{\partial(z', t')}{\partial(y, z)},$$
$$-E_x = H'_x\frac{\partial(y', z')}{\partial(x, t)} + H'_y\frac{\partial(z', x')}{\partial(x, t)} + H'_z\frac{\partial(x', y')}{\partial(x, t)}$$
$$+E'_x\frac{\partial(x', t')}{\partial(x, t)} + E'_y\frac{\partial(y', t')}{\partial(x, t)} + E'_z\frac{\partial(z', t')}{\partial(x, t)},$$

and applying them to the case in hand, we find that
$$-H_x = \frac{a}{r(r+x)}[z\psi(a-r-t) - y\chi(a-r-t)],$$
$$-H_y = \frac{ayz}{r(r+x)^2}\psi(a-r-t) + \left[\frac{ax}{r(r+x)} + \frac{az^2}{r(r+x)^2}\right]\chi(a-r-t),$$
$$+H_z = \frac{ayz}{r(r+x)^2}\chi(a-r-t) + \frac{ax}{r(r+x)}\psi(a-r-t) + \frac{ay^2}{r(r+x)^2}\psi(a-r-t),$$
$$E_x = \frac{a}{r(r+x)}[z\chi(a-r-t) + y\psi(a-r-t)],$$
$$E_y = \frac{ayz}{r(r+x)^2}\chi(a-r-t) - \left[\frac{ax}{r(r+x)} + \frac{az^2}{r(r+x)^2}\right]\psi(a-r-t),$$
$$E_z = \frac{ayz}{r(r+x)^2}\psi(a-r-t) - \left[\frac{ax}{r(r+x)} + \frac{ay^2}{r(r+x)^2}\right]\chi(a-r-t).$$

At points of the surface $r+x=a$, we have

$$E_x+E'_x=\frac{z}{r}\chi+\frac{y}{r}\psi, \qquad H_x+H'_x=-\left(\frac{z}{r}\psi-\frac{y}{r}\chi\right),$$

$$E_y+E'_y=\frac{y}{a}\left(\frac{z}{r}\chi+\frac{y}{r}\psi\right), \qquad H_y+H'_y=-\left(\frac{yz}{ar}\psi+\frac{z^2+a^2}{ar}\chi\right),$$

$$E_z+E'_z=\frac{z}{a}\left(\frac{z}{r}\chi+\frac{y}{r}\psi\right), \qquad H_z+H'_z=\frac{yz}{ar}\chi+\frac{y^2+a^2}{ar}\psi,$$

and since the direction cosines of the normal to the surface are proportional to $\left(1, \frac{y}{a}, \frac{z}{a}\right)$, the resultant electric force is along the normal and the resultant magnetic force perpendicular thereto. The conditions for a perfect reflector are thus satisfied.

The above analysis gives the solution for the case in which the light is reflected from the concave side of the paraboloid, the case in which it is reflected from the convex side may be worked out in a similar way by making E'_x, E'_y, ... functions of $x'+t'$, instead of $x'-t'$.

[*August 26th.*—It should be remarked that the present method cannot be used to solve *any* problem in reflection; so far transformations have only been found in cases where a single plane or spherical wave is transformed into a single plane or spherical wave. In the general case an arbitrary spherical wave is split up on reflection into an infinite number of spherical waves enveloping a wave surface; consequently, if a transformation exists it should make a space-time point at the centre of the primary wave correspond to space-time points at the centres of the infinite set of spherical waves enveloping the reflected wave. The theory of transformations of this type has yet to be worked out.]

Correction.—In the inequalities on p. 480, *read* $\not<$ *instead of* $<$.

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It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of fifteen (or in certain cases ten) guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the second Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County and Westminster Bank, 1 St. James's Square, London, S.W., or to the Treasurer, Sir J. Larmor, Sec. R.S., at St. John's College, Cambridge.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the Session 1909–1910 are the *second* Thursdays in November, December, January, February, March, May, June, and the *fourth* Thursday in *April*. Cards giving the Programmes of the meetings as they occur can be had by Members a few days beforehand on application to C. F. Hodgson & Son, 2 Newton Street, Kingsway, London, W.C.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

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